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CONTRACTION TYPE MAPPING ON GENERAL 2-METRIC SPACE

by

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The notion of 2-metric was investigated by Gähler [1]. Antonovskii et al. [2] defined topological semifield in 1960. Sharma and Sharma [4] proved a fixed point theorem in 2-metric space over topological semifield. In [illegible] we wish to further study fixed point theorem in 2-metric space over topological semifield. For this we shall require some definitions.

Definition 1. We shall call a commutative associative topological ring E, a *topological semifield* if there is isolated in some set K satisfying

1. $K + \bar{K} \subset K$, $K . K \subset K$
2. $K - K = E$.
3. The least upper bound and greatest lower bound exist.
4. For a, $b \in K$ the equation $ax = b$ has at least one solution in K.
5. The intersection $\bar{K} \cap (- \bar{K})$ contains only the zero element of the ring.

Remark 1. The axioms for a topological semifield are so chosen that its properties recall those of the fields of real numbers. Infact, it was proved by Antonovskii et al. [2] that any topological semifield contains a topological field isomorphic with the real line.

Remark 2. We shall call elements of the set K positive elements of the semifield E and elements of the set $\bar{K} - K$ will be called boundary elements of the semifield E.

Remark 3. We agree to write the relation $x-y \in K$, $x-y \in \bar{K}$ also in the form $x > y$, $x \geqslant y$ (or in the form $y < x$, $y \leqslant x$). In particular, the inequality $x > 0$ means that $x \in K$ and $x \geqslant 0$ means that $x \in \bar{K}$.

Remark 4. The set K contains elements which are different from zero.

Defintion 2. Let E be a semifield and K be the set of all its positive elements. The set X is called a 2–metric space over the semifield E if there exists a mapping (called the metric),

$$d: X \times X \times X \longrightarrow \bar{K} \text{ for each triple of points } x, y, z \in X$$

such that to each pair of points x, y; $x \neq y$ from X, there exists $z \in X$ satisfying:

(Ia) $d(x, y, z) \neq 0.$

(Ib) $d(x, y, z) = 0$ only when atleast two of the three elements are equal.

(II) $d(x, y, z) = d(x, z, y) = d(y, z, x).$

(III) $d(x, y, z) \leqslant d(x, y, w) + d(x, w, z)$
$+ d(w, y, z).$

Remark. *A* 2– metric space over topological semifield is called ***bounded*** if there exists a constant M such that

$$d(x, y, z) \leqslant M \text{ for all } x, y, z \in X.$$

If $E = R$, the field of real numbers, then we arrive at the definition of 2–metric space Gähler [1]. Also, if X consists of two points, then we get the definition of metric space over topological semifield, Antonovskii et al. [2, 3].

Definition 3. A sequence $\{ x_n \}$ in a 2–metric space over a topological semifield X is called a Cauchy sequence if

$$\lim_{m, n \to \infty} d(x_m, x_n, y) \in U \text{ for all } y \in X \text{ where } U \in E$$

is the neighbourhood of the origin.

Definition 4. A sequence $\{x_n\}$ in a 2– metric space over topological semifield X is called a convergent sequence if there is an $x \in X$ such that $\lim_n d(x_n, x, y) \in U$ for all $y \in X$.

Definition 5. A 2–metric space over a topological semifield, X, in which every Cauchy seqnence converges is called a complete 2–metric space.

THEOREM. *Let X be a bounded complete 2–metric space over a topological semifield E, T_1 and T_2 be two mappings on X such that*

$$(1) \quad d(T_1 x, T_2 y\ a) \leqslant k \max \{ d(x.y, a), \tfrac{1}{2} [d(x, T_1 x, a) + d(y, T_2 x, a)], \tfrac{1}{2} [d(x, T_2 y, a) + d(y, T_1 x, a)],$$

$$\frac{d(x, T_2 y, a) [1 + d(x, T_1 x, a) + d(y, T_1 x, a)]}{2 [1 + d(x, y, a)]},$$

$$\frac{d(y, T_1 x, a) [1 + d(y, T_2 y, a) + d(x, T_2 y, a)]}{2 [1 + d(x, y, a)]} \}$$

for all x, y in X and some point $a \in X$ where $k \in (0, 1)$. then T_1 and T_2 have a unique common fixed point in X.

Proof. Take an element $x_0 \in X$ and define $x_{2n+1} = T_1 x_{2n}$, $x_{2n+2} = T_2 x_{2n+1}$, $n = 0, 1, 2 \ldots$.

Then by (1)

for $x = x_0$, and $y = x_1$, after simplification,

$$(2) \quad d(x_1, x_2, a) \leqslant k \max \{ d(x_0, x_1. a), \tfrac{1}{2} [d(x_0, x_1, a) + d(x_1, x_2, a)], \tfrac{1}{2} [d(x_0, x_2, a)], \tfrac{1}{2} d(x_0, x_2, a), 0 \}.$$

If

$$(3) \quad \max \{ d(x_0, x_1, a), \tfrac{1}{2} [d(x_0, x_1, a) + d(x_1, x_2, a)],$$

$\frac{1}{2} d(x_0, x_2, a)\} = d(x_0, x_1, a)$,

then

(4) $d(x_1, x_2, a) \leqslant k\, d(x_0, x_1, a)$.

If in (3), the maximum of three numbers is

$\frac{1}{2}[d(x_0, x_1, a) + d(x_1, x_2, a)]$

then $d(x_1, x_2, a) \leqslant k \frac{1}{2}[d(x_0, x_1, a) + d(x_1, x_2, a)]$

which implies

(5) $d(x_1, x_2, a) \leqslant \frac{k}{2-k} d(x_0, x_1, a) \leqslant k\, d(x_0, x_1, a)$.

If maximum of these three numbers in (3) is

$\frac{1}{2} d(x_0, x_2, a)$,

then

$d(x_1, x_2, a)\ k \frac{1}{2} d(x_0, x_2, a)$

$\leqslant k \frac{1}{2}[d(x_0, x_2, x_1) + d(x_0, x_1, a)$

$+ d(x_1, x_2, a)]$,

and from (2), we see that $d(x_1, x_2, x_0) = 0$.

Thus

(6) $d(x_1, x_2, a) \leqslant \frac{k}{2-k} d(x_0, x_1, a) \leqslant k\, d(x_0, x_1, a)$.

From (4)–(6) we have

(7) $d(x_1, x_2, a) \leqslant k\, d(x_0, x_1, a)$.

Similarly by (1) and (2) we have

$d(x_2, x_3, a) \leqslant k \max\{d(x_1, x_2, a), \frac{1}{2}[d(x_2, x_3, a)$
$+ d(x_1, x_2, a)], \frac{1}{2} d(x_1, x_3, a), 0, \frac{1}{2} d(x_2, x_3, a)\}$.

Now by following the similar argument as above we have

(8) $d(x_2, x_3, a) \leqslant k\, d(x_1, x_2, a) \leqslant k^2 d(x_0, x_1, a)$.

In general repeating the argument n times we obtain

(9) $d(x_n, x_{n+1}, a) \leqslant k^n d(x_0, x_1, a)$.

Therefore by axioms (II) and (III) for $n < m$ we have

$$d(x_n, x_m, a) \leqslant d(x_n, x_m, x_{n+1}) + d(x_n, x_{n+1}, a)$$
$$+ d(x_{n+1}, x_m, a)$$
$$\leqslant d(x_n, x_{n+1}, x_m) + d(x_n, x_{n+1}, a) + d(x_{n+1}, x_m, a)$$
$$\leqslant d(x_n, x_{n+1}, x_m) + d(x_n, x_{n+1}, a)$$
$$+ d(x_{n+1}, x_m, x_{n+2}) + d(x_{n+1}, x_{n+2}, a)$$
$$+ d(x_{n+2}, x_m, a).$$
$$\leqslant d(x_n, x_{n+1}, x_m) + d(x_{n+1}, x_{n+2}, x_m) + d(x_n, x_{n+1}, a)$$
$$+ d(x_{n+1}, x_{n+2}, a) + d(x_{n+2}, x_m, a)$$
$$\leqslant \ldots\ldots\ldots d(x_n, x_{n+1}, x_m) +$$
$$+ d(x_{n+1}, x_{n+2}, x_m) + \ldots\ldots\ldots + d(x_{m-2}, x_{m-1}, x_m) +$$
$$+ d(x_n, x_{n+1}, a) + d(x_{n+1}, x_{n+2}, a) + \ldots\ldots\ldots$$
$$+ d(x_{m-1}, x_m, a).$$

By (9) the above inequality implies

$$d(x_n, x_m, a) \leqslant [k^n + k^{n+1} + \ldots k^{m-2}]\, d(x_0, x_1, x_m) +$$
$$+ [k^n + k^{n+1} \ldots k^{m-1}]\, d(x_0, x_1, a).$$

Since X is bounded, we have

$$d(x_n, x_m, a) \leqslant 2\,[k^n + k^{n+1} + \ldots + k^{m-1}]\, M.$$

By hypothesis $k < 1$, then we have

$$\lim_{n,m \to \infty} d(x_n, x_m, a) \in U.$$

Hence $\{x_n\}$ is a Cauchy sequence. Therefore $\{x_n\}$ has a limit u.

We shall show that u is a fixed point of T_1 and T_2.

By axiom (III) we have

(10) $d(T_1u, u, a) + d(T_1 u, u, x_{2n+2}) +$

$+ d(T_1u, x_{2n+2}, a) + d(x_{2n+2}, u, a)$.

From (1),

(11) $d(T_1u, x_{2n+2}, a) = d(T_1u, T_2 x_{2n+1}, a)$

$\leqslant k \max \{ d(u, x_{2n+1}, a), \frac{1}{2} [d(u, T_1 u, a) +$

$+ d(x_{2n+1}, T_2 x_{2n+1}, a)]$,

$\frac{1}{2} [d(u, T_2 x_{2n+1}, a) + d(x_{2n+1}, T_1 u, a)]$,

$$\frac{d(u\ T_2 x_{2n+1}, a)[(1 + d(u, T_1 u, a) + d(x_{2n+1}, T_1 u, a)]}{2[1 + d(u, x_{2n+1}, a)]}$$

$$\left.\frac{d(x_{2n+1}, T_1 u, a)[1 + d(x_{2n+1}, T_2 x_{2n+1}, a) + d(u, T_2 x_{2n+1}, a)]}{2[1 + d(u, x_{2n+1}, a)]}\right\}$$

$\leqslant k \max \{ d(u, x_{2n+1}, a), \frac{1}{2} [d(u, T_1 u, a) +$

$d(x_{2n+1}, x_{2n+2}, a)], \frac{1}{2} [d(u, x_{2n+2}, a) +$

$d(x_{2n+1}, T_1 u, a)]$,

$$\frac{d(u, x_{2n+2}, a)[1 + d(u, T_1 u, a) + d(x_{2n+1}, T_1 u, a)]}{2(1 + d(u, x_{2n+1}, a)]},$$

$$\left.\frac{d(x_{2n+1}, T_1 u, a)[1 + d(x_{2n+1}, x_{2n+2}, a) + d(u, x_{2n+2}, a)]}{2[1 + d(u, x_{2n+1}, a)]}\right\}.$$

Substituting (11) in (10) and taking limit as $n \to \infty$ we obtain $d(T_1 u, u, a) \in U$ for all $a \in X$. This implies $u = T_1 u$. Similarly, it follows that $T_2 u = u$, so that u is a fixed point of T_1 and T_2.

Now suppose v is another fixed point of T_1 and T_2.

Then

$d(u, v, a) d = (T_1 u, T_2 v, a)$,

so that by (1) we have

$$d(u, v, a) = k \max\{ d(u, v, a), 0, d(u, v, a), \tfrac{1}{2} d(u, v, a), \tfrac{1}{2} (u, v, a) \},$$

which implies $u = v$.

This completes the proof.

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REFERENCES

[1] S. Gāhler, 2- metrische Raume Und Ichre Topologische Struktur, *Math. Nachr.* **26** (1963/64), 115-148.

[2] M. Ya. Antonovskii, V. G. Boltyanskii and T. A. Sarymaskov, *Topological Semifields*, Tashkent, 1960.

[3] M. Ya. Antonovskii, V. G. Boltyanskii and T. A. Sarymaskov, *Metric Space Over Semifields*, Tashkent State Univ. of Samarkand, 1960-61.

[4] B. K. Sharma and P. L. Sharma, Contraction Type Mapping on General 2-metric Space, *Tamkang J. Math.* **7** (1976), 219-222.