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This volume of

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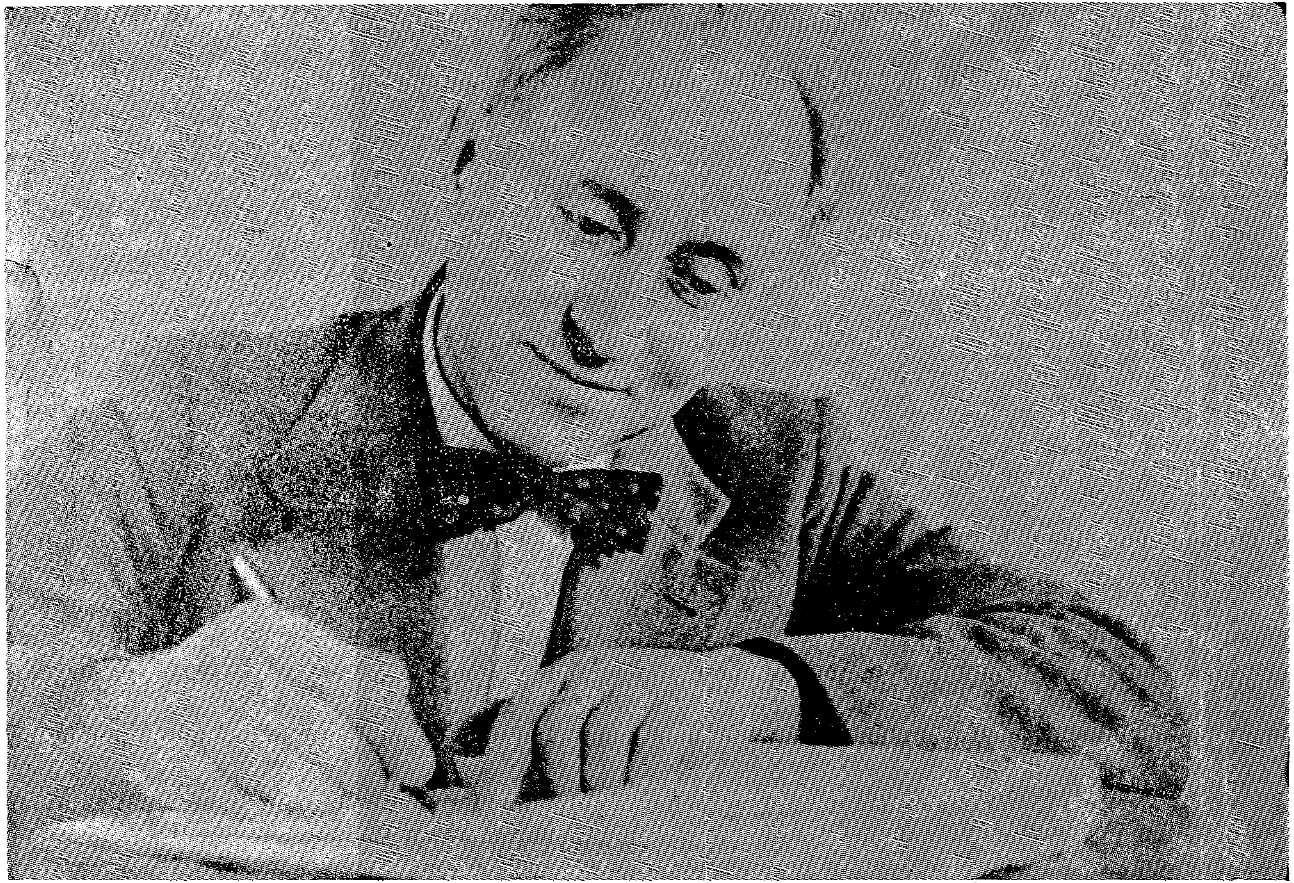
is being dedicated to

the memory of

Professor Arthur Erdélyi

(1908–1977)



Professor Arthur Erdélyi
(October 2, 1908–December 12, 1977)

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ARTHUR ERDÉLYI

(1908—1977)

By

H. M. SRIVASTAVA

Department of Mathematics, University of Victoria
Victoria, British Columbia, Canada V8W 2Y2

The world of mathematics suffered an irreparable loss with the passing away of Professor Arthur Erdélyi, F. R. S., F. R. S. E., who died suddenly at his home in Edinburgh on Monday, December 12, 1977, at the age of 69, leaving behind a widow, Eva, and a stepson, David. He had apparently made a marvellous recovery after his surgery of 1973, though in private he often had to rest, and had been his usual self in the Department of Mathematics (University of Edinburgh) on the day of his death. He had his evening meal at home, and later was taken ill and died about 9 : 00 P. M.

Arthur Erdélyi was born on October 2, 1908 in Budapest, Hungary, the eldest of five children of Ignác and Frieda (*née* Roth) Diamant. After his father's death he was adopted by his mother's second husband, Paul Erdélyi. He received his primary and secondary education in Budapest from 1914 to 1926.

Upon completion of the school curriculum in Budapest, Erdélyi wanted to become a mathematician and was indeed offered a place in Budapest University to study mathematics. {This was not a small tribute to his ability in view of the quota on the admission of Jews imposed by the University under the *numerus clausus*.} Due to circumstances then obtaining in Hungary (regarding, for example, university appointment of Jews), however, he planned to become an engineer and then set up in private practice in Hungary. But the quota in engineering for Jews in Hungarian universities was nil. Consequently, after consulting L. Fejér at Budapest, Erdélyi enrolled at the Deutche Technische Hochschule in Brno, Czechoslovakia, to study electrical engineering. Two years leter, in 1928, he passed with distinction the first of the two "state examinations" (necessary in order to obtain a degree at the Technische Hochschule; one in mathematics, physics, and related scientific subjects, and the other in professional subjects), but never completed work for the second. Such a situation was not at all uncommon in Central Europe in those days, especially because a university degree was not a prerequisite for a successful career then.

During his first year at Brno, Erdélyi was awarded both the first and second prizes in a mathematics competition organized by the Professor of Algebra and Geometry at the Technische Hochschule. At this stage in his life, Erdélyi realized that he could obtain employment as a mathematician if he stayed in Czechoslovakia and, following the advice of the Professor of Mathematical Analysis at the Technische Hochschule, he decided to devote himself to mathematics. He began active research in mathematics about 1930 and published his first paper in 1934.

In 1937 he matriculated at the German University of Prague, and in 1938 he submitted a collection of his published papers to the German University of Prague for the award of the degree of *Doctor rerum naturalium*. Under normal circumstances, a successful career as a professional mathematician in Czechoslovakia would have been assured.

Unfortunately, circumstances in Czechoslovakia were far from being normal and the German occupation of Czechoslovakia in 1938 put Erdéyi, as a Jew, in mortal danger. {As a matter of fact, the convocation ceremony of 1938 was the last *Promotion* which the German University of Prague held before it was taken over by the Nazis} Erdélyi was ordered to leave Czechoslovakia by the end of 1938 or else he was to risk internment in a concentration camp. Erdélyi was already in active correspondence with Professor Francesco G. Tricomi of the University of Torino (Italy) because of their related interests in research. Tricomi, however, discouraged Erdélyi's possible move to Italy because he felt that the surge of anti-Semitism in Italy militated against such a move. At the suggestion of Tricomi, Erdélyi then wrote to Professor E. T. (later Sir Edmund) Whittaker of the University of Edinburgh keeping in view the fact that he had by that time published over twenty papers, mostly devoted to the confluent hypergeometric function which Whittaker had discovered in 1904. Through the dedicated efforts of Whittaker and Professor S. Brodetsky of the University of Leeds, funds (sufficient to satisfy the British government regarding its policy of not issuing a visa to a refugee unless £ 400 per annum could be guaranteed for his/her support) were finally collected, and

Whittaker wrote about it to Erdélyi in December 1938. The offer came just in time, as evidenced by Erdélyi's letter of January 26, 1939 to Whittaker : "... *Necessity and danger compel me to trouble you once more* ... *You know, perhaps, what it means today if a few is to be put on the German or Hungarian frontier*. .." In the last days of January 1939. Erdélyi was finally able to leave Czechoslovakia, and in February 1939 he duly appeared at the Mathe matical Institute of the University of Edinburgh, At Waverly railway station in Edinburgh, carrying a battered suitcase and very little money, Erdélyi was received by Barry Spain who had been deputed by Whittaker to give Erdélyi two pounds and arrange for his lodgings. At the same time Walter Ledermann was called in to act as an interpreter since Erdélyi spoke practically no English though his German was fluent. These welcoming signs marked the beginning of Erdélyi association with Edinburgh which was to flower into a major love of his life, sufficient to warrant his return from U. S. A. many years later at a considerable personal sacrifice.

Undoubtedly, the invitation to Edinburgh saved Erdélyi's life for the entire Jewish community in Brno was wiped out during the war. His two brothors and a sister died in concentration camps, but it is pleasing to record that his mother survived and lived in Israel for many years after the war and that Erdélyi was always a most devoteds on.

For the next few years, Erdélyi continued his work under a research grant from the University of Edinburgh and some financial support from the Society for the Protection of Science of Learing. His research publications continued unabated, and the University of Edinburgh recognized his widening international reputation by awarding him the degree of Doctor of Science in 1940. The following year Whittaker managed to persuade the University authorities to make Erdélyi an Assistant Lecturer in Mathematics. In 1942 Erdélyi was appointed Lecturer and, with his future reasonably secured, he married Eva Neuburg (daughter of Frederic and Helen Neuburg and second cousin of Max Perutz, F. R. S.) in Glasgow on November 4, 1942. The marriage brought him great joy and proved to be very happy and harmonious.

Erdélyi's association with Edinburgh was soon to be interrupted. On January 21, 1946, Professor Harry Bateman (1882—1946) of the California Institute of Technology died suddenly of coronary thrombosis, leaving behind him a mass of notes which he had intended to edit into a monumental work on special functions. Since Whittaker was at that time the world's senior authority on special functions, and had known Bateman as a student at Trinity College, Cambridge, he was naturally asked by Caltech if he could recommend someone to supervise the editing and publication of the Bateman manuscripts. Without hesitation he recommended Erdélyi (who had just been elected a Fellow of the Royal Society of Edinburgh in 1945) as the most highly qualified expert for the proposed work. On July 1, 1947, Erdélyi took up a visiting professorship at Caltech, thus beginning his sixteen-year association with one of the outstanding centres of scientific learning in the world.

Erdélyi's initial appointment at Caltech was for one year, and he was assigned the task of evaluating the contents of the Bateman manuscripts and of determining the time needed to prepare them for publication. Upon completion his careful study, he reported that the job might take for him as many as fifteen years unless it could possibly be done by four highly trained mathematicians working over a period of four years. The alternative proposal was accepted by Caltech and followed up by an offer to Erdélyi of a premanent Full Professorship at Caltech, with directing the Bateman Manuscript Project as part of his duties. The University of Edinburgh responded by promoting him to Senior Lecturer, but the attraction of a Professorship at Caltech at double his Edinburgh salary proved to be irresistible, and after returining to Edinburgh for the academic session 1948–49, he resigned his position at Edinburgh and moved to U. S. A. He was joined at Caltech by W. Magnus from the University of Göttingen, Fritz Oberhettinger from the University of Mainz, and Francesco G. Tricomi from the University of Torino. Thus in 1949 the collaboration of the team of Erdélyi, Magnus, Oberhettinger and Tricomi on the Bateman Manuscript Project started, the famous team which produced the three volumes of *Higher Transcendental Functions* and the two volumes of *Tables of Integral Transforms.* These volumes were destined to be among the most widely cited mathematical works of all time and a basic source of reference for generations of mathematicians, physicists and engineers throughout the world.

The Bateman Manuscript Project marked a turning point in Erdélyi's development as a mathematician. Until this time most of his work was in the area of special functions, and although his results were often striking and elegant, his investigations were undertaken mainly for their own sake and not, in general, to illuminate other areas of mathematics. However, as the Bateman Manuscript Project neared completion around 1951, Erdélyi became involved with investigations into various other areas, such as the analytical theory of singular partial differential equations (*cf.* [90] and [91]), diffraction theory ([104] and [109]), and the asymptotic expansions of solutions of certain classes of differential equations ([107] and [110]). The most important of these investigations was his work on asymptotics, and around 1950 Erdélyi and his co-workers began publishing a long series of papers on the asymptotic expansions of integrals and solutions to differential equations. Much of this work was summarized in a short paperback *Asymptotic Expansions* which Erdélyi published in 1956 and which soon became the standard work on the subject.

As the years progressed and Erdélyi's international statute as a mathematician grew, he continued broadening and deepening his knowledge of mathematics, laying the foundation for the rest of his life's work. His last book *Operational Claulus and Generalized Functions* appeared in 1962, and he published his work on singular perturbation theory (*of.* [131], [134], [137], [143] and [144] and his fundamental papers on the asymptotic evaluation of certain classes of integrals ([130] and [149]) in the early 1960's.

During the years that Erdélyi spent at Caltech, Whittaker had retired and had been succeded by Aitken. In 1963 the University of Edinburgh created a second Chair of Mathematics and, on account of Aitken's extremely poor health, the Department was in desperate need of a strong and effective leader. The position was advertised twice, but for various reasons, Erdélyi did not apply. Fortunately for the University of Edinburgh a catalyst, in the person of Professor Ian N. Sneddon of the University *of* Glasgow, was visiting Caltech while the second advertisement was running. In conversation with Sneddon, Erdélyi once remarked that he thought of the University of Edinburgh as his real *alma mater* because of how well he was reeived there as a refugee in 1939 and that, if the University invited him to occupy the chair, he would think about it very seriously. In spite of his failure to persuade Erdélyi to apply, Sneddon passed on the information by telephone to Edinburgh as soon as he returned to Glasgow three days after the conversation. Shortly thereafter Erdélyi was officially invited to occupy the chair, and after three months of correspondence, he finally accepted the invitation on October 1, 1963, the day before his fifty-fifth birthday. It was in July 1964 when, to the delight of his many friends in Edinburgh and elsewhere in Scotland, Erdélyi returned to the University of Edinburgh where he eventually occupied (until his sudden death on December 12, 1977) the celebrated Chair of Mathematics held once by Mclaurin and, more recently, by Whittaker and Aitken.

Many academic honours were accorded to Arthur Erdélyi. In addition to those that have already been mentioned, he was elected a Foreign Member of the Academy of Sciences of Torino (Italy) in 1953 and a Fellow of the Royal Society in 1975. In 1977 he was awarded the Gunning Victoria Jubilee Prize of the Royal Society of Edinburgh, the only mathematician to receive this coveted prize in recent years other than Sir William Hodge. His service to the mathematical community included the Presidency of the Edinburgh Mathematical Society, Council Member of the American Mathematical Society, membership on various advisory bodies appointed by the National Academy of Sciences of U. S. A., and his active association in an editorial capacity with many journals such as :

Proceedings of the Edinburgh Mathematical Society,
Journal of the Indian Mathematical Society,
Mathematical Tables and Other Aids to Computation,
Journal of Mathematics and Mechanics,
Archive for Rational Mechanics and Analysis,
Journal of the Society for Industrial and Applied Mathematics,
Journal of Mathematical Physics,
Canadian Journal of Mathematics,
Proceedings of the Royal Society of Edinburgh.

He also found time to contribute innumerable reviews to *Mathematical Reviews.* His seventieth birhday was to have been celebrated by a special issue of *Applicable Analysis* and by the *Proceedings of the* 1978 *Dundee Conference on Differential Equations*. Both of these have now been dedicated to him as a lasting memorial to his work and the great influence which it continues to exert on current mathematical research.

The British citizenship (to which Erdélyi had become entitled by naturalization in (1947) did not diminish his interest in Jewish affairs. He paid several visits to Israel and spent the academic year 1956-57 on sabbatical leave as Visiting Professor of Applied Mathematics at the Hebrew University, Jerusalem. He was a member of the *Friends of the Hebrew Univrsity* and, while in U. S. A., served on its Academic Council. For a period of time he also was Vice-President of the Edinburgh section of the British Zionist Association.

Erdélyi attracted many invitations to visit mathematical centres elsewhere. He travelled widely and lectured frequently in Europe, North America and Australia. These include his Visiting Professorship at the University of Melbourne in 1970, and the invited lectures at the 1954 International Congress of Mathematicians in Amsterdam, the 1961 Summer Research Institute of the Canadian Mathematical Congress in Montreal, and the 1974 International Conference on Fractional Calculus and Its Applications in New Haven. It was at the New Haven conference in June 1974 when I first met Erdélyi personally. I am fortunate in having been in correspondence with him for over a decade prior to our first meeting and having remained in contact with him afterwards. During the academic year 1975–76, while on sabbatical leave at the University of Glasgow, I had many opportunities of seeing Arthur Erdélyi during my frequent visits to Edinburgh and elsewhere in Scotland, especially at the meetings of the Edinburgh Mathematical Society. During the last few weeks in his life, Erdléyi and I were involved, as external referees, in the evaluation af a certain case for promotion to Full Professor at a Canadian univrsity.

One cannot leave the subject of Erdélyi's contributions without mentioning that the work, which, he initiated and encouraged, has been carried on by his Ph. D. students, R.H. Owens (1952), P G. Rooney (1952), C. A. Swanson (1957), J. Rice (1959), T. Beohme (1960), D. W. Willett (1963), J. W. Macki (1964), D. L Colton (1967), J. Wimp (1968), J. Searl (1969), and A. McBride (1971). On the other hand, one just cannot overemphasize the importance of Erdélyi's classic paper of 1956 ([120]) in which he used operators of fractional integration. He continued using there operators in all his future work on singular equations of the GASPE type. In 1958 Erdélyi and Copson [123] made use of fractional integration operators and the Mellin transform to study a singular hyperbolic equation with intersecting singular lines, and in 1965 Erdélyi ([154] and [155]) returned to his study of fractional integration and the generalized axially symmetric potential equation. His last paper on this subject appeared in

1970 ([163]) in which he applied fractional integration operators to study the Euler-Poisson-Darboux equation.

Although Erdélyi used fractional integration in various papers on special functions published in 1939 and 1940 ([50], [57] and [58], his first major contribution in this area appeared in [59] and [60], partly in collaboration with H. Kober. In these papers Erdélyi and Kober introduced and studied certain *homogeneous* modifications of the Riemann-Liouville and Weyl fractional integrals and discussed their connections with the Hankel transform. Their results involving these generalized fractional integration operators, usually called Erdélyi-Kober operators, lay dormant for over twenty years, until in 1961 Erdélyi and Ian Sneddon came together as lecturers at the Summer Research Institute of the Canadian Mathematical Congress in Montreal. While delivering a series of lectures on mixed boundary value problems, Sneddon remarked that a unified treatment of the dual integral equations that arose in such problems depended on the development of certain relationships between Hankel transforms and fractional integration operators. This, of course, was precisely the topic discussed by Erdélyi and Kober in [59], and hence the chance remark by Sneddon resulted in a joint paper [139] in which Erdélyi and Sneddon solved, in a systematic and unified way, a rather general class of dual integral equations which occur frequently in a number of branches of mathematical physics. In his later years, Erdélyi (influenced by Zemanian's book *Generalized Integral Transformations*) began to extend fractional calculus to generalized functions (*of.* [166], [167] and [170]).This work on fractional integrals of generalized functions was subsequently extended further to include the Stieltjes transform (*cf.* [175]), and during his final days Erdélyi was actively developing and refining his investigations in this area.

Arthur Erdélyi leaves behind him the memory of a mathematician with the astonishing range of interests, one who made no compromise with the highest standards of his profession, one who served his discipline with devotion and distinction, one who made significant and lasting contributions to his fields of expertise, and one who inspired many of his colleagues and students to carry out independent researches for themselves. He will indeed be remembered for his great intellectual gifts and research contributions, for his courtesy and kindliness and, above all, for his intensely human personality with a keen sense of humour and a great love of art, of music, of the countryside and of children.

Acknowledgements

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The photograph of Arthur Erdélyi was taken in 1957 in Jerusalem when he was spending his sabbatical year 1956-57 at the Hebrew University there; it has been reproduced with the technical assistance by Mr. Elisabeth Grambart of Media and Technical Services, University of Victoria.

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(Dedicated to the memory of Professor Arthur Erdélyi)

ON LAPLACE TRANSFORMS AND THEIR APPLICATIONS

By

R. K. RAINA

Department of Mathematics,

*S. K. N. Agriculture College, Jobner-***303328**, *Rajasthan, India*

and

C. L. KOUL

Department of Mathematics,

M. R. Engineering College, Jaipur - **302004**, *India*

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ABSTRACT

In this paper a theorem concerning the Laplace transform of the product of the H-function of Fox and a suitably chosen function F(x,t) is established. This theorem is then used to establish anothor theorem concerning certain fractional integrals. Applications of these theorems are capable of yielding a number of known and new results by choosing the various functions and parameters suitably.

1. Introduction

The Laplace transform of a function f(z) is defined as

$$(1.1)\qquad L\left[f(z)\,;\, p \right] = \int_0^\infty e^{-pz}\, f(z)dz, \quad Re\,(p) > 0.$$

The Riemann–Liouville fractional integral of order q is given by

$$(1.2)\qquad \underset{y}{\overset{q}{I}}\left[f(z) \right] \frac{1}{\Gamma(q)} \int_0^y (y-z)^{q-1} f(z)\, dz,\ Re\,(q) > 0.$$

The H-function of Fox [4, p. 408] is defined and represented in the following form :

$$(1.3)\qquad H_{P,Q}^{M,N}\left[z \,\middle|\, \begin{matrix} (a_j, \alpha_j)_{1,P} \\ (b_j, \beta_j)_{1,Q} \end{matrix} \right] = \frac{1}{2\pi\omega} \int_L \theta\,(s)\, z^s ds,$$

with $\omega = \sqrt{-1}$ and

$$(1.4)\qquad \theta(s) = \frac{\prod_{j=1}^{M} \Gamma(b_j - \beta_j s) \prod_{j=1}^{N} \Gamma(1 - a_j + \alpha_j s)}{\prod_{j=M+1}^{Q} \Gamma(1 - b_j + \beta_j s) \prod_{j=N+1}^{P} \Gamma(a_j - \alpha_j s)}$$

where an empty product is is interpreted as unity, L is a suitable contour and the integers M,N,P,Q satisfy $0 \leqslant M \leqslant Q$, $0 \leqslant N \leqslant P$. The coefficient α_j (j=1,..., P) and β_j (j=1, . , Q) are all positive.

In the sequel, we also require the *H*–function of two variables which was defined earlier [8, p. 117]. The notation used here for representing this function is a slight variant of the one given by Srivastava and Panda [13, p 266, Eq. (1.5)] :

$$(1.5)\quad H^{0,n_1 : m_2, n_2 ; m_3, m_3}_{p_1,q_1 : p_2,q_2 ; p_3,q_3}\left[\begin{matrix} x \\ y \end{matrix}\middle| \begin{matrix} (a_j ; \alpha_j, A_j)_{1,p_1} : (c_j, \gamma_j)_{1,p_2} ; (e_j, E_j)_{1,p_3} \\ (b_j ; \beta_j, B_j)_{1,q_1} : (d_j ; \delta_j)_{1,q_2} ; (f_j, F_j)_{1,q_3} \end{matrix}\right]$$

$$= \frac{-1}{4\pi^2} \int_{L_1} \int_{L_2} \theta_1(s)\, \theta_2(t)\, \phi(s,t)\, x^s y^t ds\, dt,$$

where

$$(1.6)\qquad \phi(s,t) = \frac{\prod_{j=1}^{n_1} \Gamma(1 - a_j + \alpha_j s + A_j t)}{\prod_{j=n_1+1}^{p_1} \Gamma(a_j - \alpha_j s - A_j t) \prod_{j=1}^{q_1} \Gamma(1 - b_j + \beta_j s + B_j t)}$$

and $\theta_1(s)$ and $\theta_2(t)$ are the Γ—quotients anologous to the quotient $\theta(s)$ in the equation (1.4) involving appropriately the parameters of the second and third sets, respectively. The contours L_1 and L_2 are suitably defined and the integers n_j, p_i. q_i (i=1,2,3), m_j (j=2,3) are such that $0 \leqslant n_i \leqslant p_i$, $q_1 \geqslant 0$, $0 \leqslant m_j \leqslant q_j$.

The coefficients A_j, B_j, E_j, F_j, α_j, β_j, γ_j, δ_j, are all positive. The conditions of existence of the functions defined in (1.3) and (1.5) are given in [5, p. 597, Eqns. (3.1) to (3.6)] and [8, p. 119, (i) to (vi)]. We assume that these conditions are satisfied by the different H–functions occurring in this paper.

2. Main Results

Theorem 1. Let

$$(2.1) \qquad F(x,t) = \sum_{n=0}^{\infty} C_n f_n(x)\, t^n,$$

where $f_n(x)$ *is a polynomial of degree* n *in* x.

Then

$$(2.2) \quad L\left[z^{\lambda-1} (1-z)^{\mu-1} H_{P,Q}^{M,N}\left[yz^{\sigma} \middle| \begin{matrix} (a_j, \alpha_j)_{1,P} \\ (b_j, \beta_j)_{1,Q} \end{matrix} \right] F(x, z^{\alpha} (1-z)^{\beta} t; p \right]$$

$$= p^{-\lambda} \sum_{n=0}^{\infty} C_n f_n(x) \frac{(t p^{-\alpha})^n}{\Gamma(1-\mu-\beta n)}$$

$$H_{1,0\,:\,1,1\,;\,P,Q}^{0,1\,:\,1,1\,;\,M,N}\left[\begin{matrix} -p^{-1} \\ yp^{-\sigma} \end{matrix} \middle| \begin{matrix} (1-\lambda-\alpha n\,;\,1,\sigma) : (\mu+\beta n, 1)\,;\,(a_j, \alpha_j)_{1,P} \\ \text{———} \quad ;\,(0,1) \quad ;\,(b_j, \beta_j)_{1,Q} \end{matrix} \right],$$

provided that

$Re\,(p) > 0$, $Re\,[\lambda + \sigma\,(b_i/\beta_i)] > 0\ (i = 1, \ldots, M)$,

$\sigma > 0$, $\alpha, \beta, \mu \geqslant 0$ *(α, β being not zero simultaneously)*

and $F(x,t)$ *is such that the left-hand side of (2.2) exists.*

Theorem 2. *Let $F(x,t)$ be as given by (2.1).*

Then

$$(2.3) \quad I_y^q\left[z^{\lambda-1} (1-z)^{\mu-1} H_{P,Q}^{M,N}\left[az^{\sigma} \middle| \begin{matrix} (a_j, \alpha_j)_{1,P} \\ (b_j, \beta_j)_{1,Q} \end{matrix} \right] F(x, z^{\alpha} (1-z)^{\beta}\, t) \right]$$

$$= y^{q+\lambda-1} \sum_{n=0}^{\infty} C_n f_n(x) \frac{(ty^{\alpha})^n}{\Gamma(1-\mu-\beta n)}$$

$$H^{0,1\,:\,1,1\,;\,M,N}_{1,1\,:\,1,1\,;\,P,Q}\left[\begin{matrix} -y \\ ay^{\sigma} \end{matrix}\middle|\begin{matrix} (1-\lambda-\alpha n;\ 1,\sigma) : (\mu+\beta n,1)\ ;\ (a_j,\ \alpha_j)_{1,P} \\ (1-\lambda-q-\alpha n;\ 1,\sigma) : (0,1)\ ;\ (b_j,\ \beta_j)_{1,Q} \end{matrix}\right]$$

provided that $Re(q) > 0$, $Re[\lambda+\sigma(b_i/\beta_i)] > 0$ $(i=1,..\ M)$, $\sigma > 0$, $\alpha,\beta,\mu \geqslant 0)$ (α *and* β *being not zero simultaneously*) *and* F(x,t) *is such that left-hand side of* (2.3) *exists.*

Proofs : To prove (2.2), we substitute for F(x,t) from (2.1) into the left-hand side of (2.2), use [5, p. 600, Eqn. (4.7)], take the Laplace transform term-by-term, and apply [8, p. 122, Eqn. (2.2)]. To prove (2.3), we use the convolution property of the Laplace transforms [3, p. 131, Eqn. (20)], take the inverse Laplace transform of the right-hand side of (2.2), term-by-term with the help of [8, p. 122, Eqn. (2.2)], and invoke (1.2).

3, Applications

(*a*) *Special case of Theorem 1* :

If in Theorem 1, we take $\alpha=1$, $\beta=0$, $C_n=(n!)^{-1}$,

$$f_n(x) = {}_{u+1}F_v\left[\begin{matrix} -n,(a'_u) \\ (b'_v) \end{matrix}\ ;\ x\right],$$

so that from [12, p. 233, Eqn. (13)], we have

$$F(x,t) = e^t\ {}_uF_v\left[\begin{matrix} (a'_u) \\ (b'_v) \end{matrix}\ ;\ -xt\right] \tag{3.1}$$

then (2.2), with these substitutions, gives

$$L\left[\ z^{\lambda-1}(1-z)^{\mu-1} H^{M,N}_{P,Q}\left[yz^{\sigma}\middle|\begin{matrix}(a_j,\ \alpha_j)_{1,P} \\ (b_j,\ \beta_j)_{1,Q}\end{matrix}\right] e^{tx} \right. \tag{3 2}$$

$$\left. {}_uF_v\left[\begin{matrix}(a'_u) \\ (b'_v)\end{matrix}\ ;\ -xtz\right]\ ;\ p\ \right]$$

$$\frac{p^{-\lambda}}{\Gamma(1-\mu)} \sum_{n=0}^{\infty} {}_{u+1}F_v\left[\begin{matrix}-n, & (a'_u) \\ & (b'_v)\end{matrix}; x\right] \frac{(tp^{-1})^n}{n!} \times$$

$$H^{0,1:1,1;M,N}_{1,0:1,1;P,Q}\left[\begin{matrix}-p^{-1} \\ yp^{-\sigma}\end{matrix}\middle| \begin{matrix}(1-\lambda-n;1,\sigma):(\mu,1);(a_j,\alpha_j)_{1,P} \\ :(0,1);(b_j,\beta_j)_{1,Q}\end{matrix}\right].$$

On applying [5, p. 600, Eqn. (4.6)] and [6, p. 124, Eqn. (3.6)], putting $\mu=1$, using [8, p. 122, Eqn. (2.2)], replacing t by pt, and y by yp^{σ}, we are led to special cases of [13, p. 267, Eqn. (2.1)] and also of [11, p. 302, Eqn. (2.2)].

(b) Special cases of Theorem 2 :

1. On letting $y \to 1$, using [1, p. 215, Eqn. (1)], putting $\alpha=\mu=1$, $\beta=0$ in Theorem 2, it reduces to [9, p. 187, Theorem (2.1)].

2. On setting $\alpha=0$, $\beta=1$, in Theorem 2 and choosing $f_n(x)$ and C_n as in (a) above, expressing the ${}_uF_v$ function and the exponential function as power series, and putting $\sigma=1$, $M=N=P=Q=1$, $a_1=1-v$, $\alpha_1=1$, $b_1=0$, $\beta_1=1$, therein, it gives on replacing a by a/y.

$$(3.3) \quad \sum_{n=0}^{\infty} {}_{u+1}F_v\left[\begin{matrix}-n, & (a'_u) \\ & (b'_v)\end{matrix}; x\right] F_1(\lambda, 1-\mu-n, \nu; \lambda+q; y, -a)\frac{t^n}{n!}$$

$$= e^t \sum_{n,r=0}^{\infty} \frac{(a'_u)_n(-xt)^n(-yt)^r(\lambda)_r}{(b'_v)_n(\lambda+q)_r\, r!\, n!} F_1(\lambda+r, 1-\mu-n, \nu; \lambda+r+q; y, -a)$$

If we put $\lambda=\beta$, $\lambda+q=\delta$, $u=\nu=1$, $a'_1=\alpha$, $b'_1=\gamma$, $\mu=1$. $v=0$ and let $a \to -y$, then (3.3) yields [10, p. 473, Eqn. (20)].

3. Lastly, if in Theorem 2, we take $f_n(x)=L_n^{(\nu-n)}(x)$, $C_n=1$, so that from [2, p. 189, Eqn. (19)], $F(x,t)=(1-t)^{\nu}e^{-xt}$, and put $\alpha=1$, $\beta=0$, then on proceeding as before, we get

$$(3.4) \quad \sum_{n=0}^{\infty} \frac{(-xt)^n}{n!\,\Gamma(-\nu)} H^{0,1:1,1;M,N}_{1,1:1,1;P,Q}\left[\begin{matrix}t \\ a\end{matrix}\middle| \begin{matrix}(1-\lambda-n;1,\sigma):(1+\nu,1);(a_j,\alpha_j)_{1,P} \\ 1-\lambda-n-q;1,\sigma):(0,1);(b_j,\beta_j)_{1,Q}\end{matrix}\right]$$

$$= \sum_{n=0}^{\infty} L_n^{(\nu-n)}(x)\; H^{M,\,N+1}_{P+1,\,Q+1}\left[a \,\middle|\, \begin{matrix} (1-\lambda-n,\,\sigma),\,(a_j,\,\alpha_j)_{1,P} \\ (b_j,\,\beta_j)_{1,Q} \end{matrix} \right] t^n.$$

For $\sigma=1$, $M=N=P=Q=1$, $a_1=1-\gamma$, $\alpha_1=1$, $\lambda=\beta$, $\lambda+q=\delta$.

$b_1=0$, $\beta_1=1$, (3.4) reduces to

$$(3.5)\quad \sum_{n=0}^{\infty} \frac{(\beta)_n}{(\delta)_n} F_1(\beta+n,-\nu,\gamma;\,\delta+n;\,t,\,-a)\,\frac{(-xt)^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{(\beta)_n}{(\delta)_n} L_n^{(v-n)}(x)\;{}_2F_1(\beta+n,\,\gamma;\,\delta+n;\,-a)\,t^n.$$

if we let $a \to 0$, (3.5) corresponds to [7, *p.* 57, Eqn (4.11)].

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(*Dedicated to the memory of Professor Arthur Erdélyi*)

SOME FINITE SERIES CONCERNING THE H-FUNCTION OF n VARIABLES

By

O. P. TANDON

U. G. C. Teacher-Fellow, 1-KH-8 Vaishali Nagar
Ajmer–**305001**, *Rajasthan, India*

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ABSTRACT

In this paper, we shall first prove some three-term recurrence relations for the H–function of n variables and then use them to generate some finite series involving the H–function of n variables.

The recurrence relations established and the series generated are believed to be new.

1. INTRODUCTION

Srivastava and Panda [3, p. 271, Eq. (4.1)] introduced the H–function of n variables by means of multiple Mellin-Barnes type contour integral in the following manner (with slight change of parameters) :

$$H_{A,\,B\,:\,[C',\,D']\,;\,\ldots\,;\,[C^{(n)},\,D^{(n)}]}^{0,\,\lambda\,:\,(\mu',\,v')\,;\,\ldots\,;\,(\mu^{(n)},\,v^{(n)})}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(n)}]:[(c'):\phi']\,;\,\ldots\,;\,[(c^{(n)}):\phi^{(n)}]\,;\\ [(b):\epsilon',\ldots,\epsilon^{(n)}]:[(d'):\delta']\,;\,\ldots\,;\,[(d^{(n)}):\delta^{(n)}]\,;\end{matrix}\; z_1,\ldots,z_n\right)$$

$$=\frac{1}{(2\pi\omega)^n}\int_{L_1}\!\!\ldots\int_{L_n} U_1(s_1)\ldots U_n(s_n)\, V(s_1,\ldots,s_n)\, z_1^{s_1}\ldots z_n^{s_n}\cdot ds_1\ldots ds_n,$$

$$\omega=\sqrt{-1} \qquad \ldots(1.1)$$

where

$$V(s_1, \ldots, s_n) = \frac{\prod_{j=1}^{\lambda} \Gamma\left(1-a_j+\sum_{i=1}^{n} \theta_j^{(1)} s_i\right)}{\prod_{j=\lambda+1}^{A} \Gamma\left(a_j-\sum_{i=1}^{n} \theta_j^{(1)} s_i\right) \prod_{j=1}^{B} \Gamma\left(1-b_j+\sum_{i=1}^{n} \epsilon_j^{(1)} s_i\right)} \tag{1.2}$$

and

$$U_i(s_i) = \frac{\prod_{j=1}^{\mu^{(1)}} \Gamma\left(d_j^{(1)}-\delta_j^{(1)} s_i\right) \prod_{j=1}^{\nu^{(1)}} \Gamma\left(1-c_j^{(1)}+\emptyset_j^{(1)} s_i\right)}{\prod_{j=\mu^{(1)}+1}^{D^{(1)}} \Gamma\left(1-d_j^{(1)}+\delta_j^{(1)} s_i\right) \prod_{j=\nu^{(1)}+1}^{C^{(1)}} \Gamma\left(c_j^{(1)}-\emptyset_j^{(1)} s_i\right)}$$

$$i = 1, \ldots, n. \tag{1.3}$$

For a detailed account of conditions under which the definition (1.1) makes sense, see [3], [4] and [5]. Here we assume, following Srivastava and Panda [3], that

(a) stands for the sequence of parametrs $a_1, \ldots, a_A$;

(b) for $b_1, \ldots, b_B$; $(c^{(1)})$ for $c_1^{(1)}, \ldots, c^{(1)}_{C^{(1)}}$

$(d^{(1)})$ for $d_1^{(1)}, \ldots, d^{(1)}_{D^{(1)}}$, etc., $i=1, \ldots, n$, it being understood, for example that $c^{(1)}=c'$, $c^{(2)} = c''$, and so on.

2. NOTATIONS

Let us assume that :

$$\theta^{(1)}_{A-1} = \alpha_{A-1} \; h_i, \; \forall_i \in \{1, \ldots, n\} \tag{2.1}$$

$$\theta^{(1)}_{A} = \alpha_{A} \; h_i, \; \forall_i \in \{1, \ldots, n\} \tag{2.2}$$

$$\epsilon^{(1)}_{1} = \beta_1 \; h_i, \; \forall i \in \{1, \ldots, n\} \tag{2.3}$$

$$\in_{2}^{(1)} = \beta_2 \; h_i \, , \; \forall i \; \in \; \{1, .., n\} \qquad \ldots(2.4)$$

and $$\theta_{1}^{(1)} = \alpha_1 \, h_i \, , \forall i \; \in \; \{1, \ldots, n\} \qquad \ldots(2.5)$$

Under the restrictions (2.2) and (2.3), let the H-function of n variables defined by (1.1) be briefly represented by $H^+ \, [a_A \, , b_1 \,]$ and furthermore by $H_0^+ \, [a_A \, , b_1 \,]$ in case $\theta_A^{(1)} = \in_1^{(1)}$. And under the same restrictions and replacing a_A by $a_A - 1$, we shall denote the H-function of n variables defined by (1.1) as $H^+ \, [a_A - 1, b_1 \,]$ and, furthermore, as $H_0^+ \, [a_A - 1, b_1$] provided $\theta_A^{(1)} = \in_1^{(1)}$, and so on. Likewise, $H^+ \, [\, b_1, \, b_2 \,]$ will mean the function of n variables defined by (1.1) under the restrictions (2.3) and (2.4), and so on.

Also, for the sake of notational convenience, the

determinants $\begin{vmatrix} a_A - r & b_1 \\ \alpha_A & \beta_1 \end{vmatrix}$, $\begin{vmatrix} a_A - r & b_1 \\ 1 & 1 \end{vmatrix}$ will be

represented by $\triangle \, (a_A - r, \, b_1 \,)$, $\triangle' \, (\, a_A - r, \, b_1 \,)$, respectively, and so on.

3. The Recurrence Relations

(1) $$\beta_1 \, H^+ \, [\, a_A - 1, b_1 \,] + \alpha_A \, H^+ \, [\, a_A \, , \, b_1 + 1 \,] = \triangle \, (a_A - 1, \, b_1) \, H^+ \, [\, a_A \, , \, b_1 \,] \qquad \ldots(3.1),$$

(2) $$\beta_1 \, H^+ \, [\, a_1, \, b_2 + 1 \,] - \beta_2 \, H^+ \, [\, b_1 + 1, \, b_2 \,] = -\triangle \, (b_2, \, b_1) \, H^+ \, [b_1, \, b_2] \qquad \ldots(3.2),$$

(3) $$\alpha_{A-1} \; H^+ \, [a_{A-1} \, , a_A - 1 \,] - \alpha_A \, H^+ \, [a_{A-1} - 1, \, a_A \,] = \triangle \, (a_A - 1, \, a_{A-1} - 1 \,) \, H^+ \, [\, a_{A-1} \, , \, a_A \,] \qquad \ldots(3.\cdot),$$

$$(4) \quad a_A H^+ [a_1-1, a_A] + a_1 H^+ [a_1, a_A - 1] = -\Delta (a_1-1, a_A - 1) \quad H^+ [a_1, a_A] \qquad \text{...(3.4)}$$

To develop the proof of the recurrence relation (3.1), we shall extend the method of Buschman, see [1, p. 41] and [2, p. 1419]. We note from the definition of the H-function of n veriables, with the restrictions (2.2) and (2.3), that the replacement of a_A by $a_A - 1$ introduces the multiplier $a_A - 1 - a_A \sum_{i=1}^{n} h_i s_i$ into the contour integral for $H^+ [a_A, b_1]$ Similarly replacement of b_1 by $b_1 + 1$ introduces the factor $-b_1 + \beta_1 \sum_{i=1}^{n} h_i s_i$. Consequently, we can form the three-term recurrence formula involving the undetermined coefficients L, M, N:

$$L\, H^+ [a_A - 1, b_1] + M\, H^+ [a_A, b_1 + 1] = N\, H^+ [a_A, b_1], \qquad \text{...(3.5)}$$

and then require that

$$L \left(a_A - 1 - a_A \sum_{i=1}^{n} h_i s_i\right) + M \left(-b_1 + \beta_1 \sum_{i=1}^{n} h_i s_i\right) = N \qquad \text{..(3.6)}$$

be an identy in $s_1, \ldots, s_n$. Hence L, M, N can be evaluated. Substituting for L, M, N in (3.3) and after a little simplification we obtain the desired result (3.1). The proofs of recurrence relations (3.2) — (3.4) can be developed in a similar manner.

4. Finite Series

$$(1) \quad \frac{\alpha_A}{\beta_1} \sum_{k=1}^{m} \frac{(-1)^k H^+ [a_A - k + 1, b_1 + 1]}{\Gamma\left[\frac{\alpha_A}{\beta_1} b_1 - (a_A - 1) + k\right]}$$

$$= \frac{(-1)^{m+1} H^+ [a_A - m, b_1]}{\Gamma\left[\frac{\alpha_A}{\beta_1} b_1 - (a_A - 1) + \right]} + \frac{H^+ [a_A, b_1]}{\Gamma\left[\frac{\alpha_A}{\beta_1} b_1 - (a_A - 1\right]} \qquad (4.1),$$

(2) $$\frac{\beta_1}{\alpha_A}\sum_{k=1}^{m}\frac{(-1)^k H^+[a_A-1,\ b_1+k-1]}{\Gamma\left[b_1+k-\frac{\beta_1}{\alpha_A}(a_A-1)\right]}$$

$$=\frac{(-1)^{m+1} H^+[a_A,\ b_1+m]}{\Gamma\left[b_1+m-\frac{\beta_1}{\alpha_A}(a_A-1)\right]}+\frac{H^+[a_A,\ b_1]}{\Gamma\left[b_1-\frac{\beta_1}{\alpha_A}(a_A-1)\right]} \quad \text{.. (4.2),}$$

(3) $$\sum_{k=1}^{m}(-1)^k \alpha_A^{k-1}\beta_1^{m-k}\Delta(a_A+k-1,\ b_1+k)\ H^+[a_A+k,\ b_1+k]$$

$$=(-1)^m \alpha_A^m H^+[a_A+m,\ b_1+m+1]-\beta_1^m H^+[a_A,\ b_1+1]$$

$$\text{....(4.3),}$$

(4) $$\sum_{k=0}^{m}\binom{m}{k} \overset{+}{H}_0[a_A-k,\ b_1+m-k]$$

$$=\frac{(-1)^m\Gamma(b_1-a_A+m+1)}{\Gamma(b_1-a_A+1)}\overset{+}{H}_0[a_A,\ b_1] \quad \text{.. (4.4),}$$

(5) $$\frac{\beta_1}{\beta_2}\sum_{k=1}^{m}\frac{(-1)^k H^+[b_1+k-1,\ b_2+1]}{\Gamma\left[b_1+k-\frac{\beta_1}{\beta_2}b_2\right]}$$

$$=\frac{(-1)^m H^+[b_1+m,\ b_2]}{\Gamma\left[b_1+m-\frac{\beta_1}{\beta_2}b_2\right]}-\frac{H^+[b_1,\ b_2]}{\Gamma\left[b_1-\frac{\beta_1}{\beta_2}b_2\right]} \quad \text{...(4.5),}$$

(6) $$\frac{\beta_2}{\beta_1}\sum_{k=1}^{m}\frac{(-1)^k H^+[b_1+1,\ b_2+k+1]}{\Gamma\left[b_2+k-\frac{\beta_2}{\beta_1}b_1\right]}$$

$$=\frac{(-1)^m H^+[b_1,\ b_2+m]}{\Gamma\left[b_2+m-\frac{\beta_2}{\beta_1}b_1\right]}-\frac{H^+[b_1,\ b_2]}{\Gamma\left[b_2-\frac{\beta_2}{\beta_1}b_1\right]} \quad \text{...(4.6)}$$

$$(7)\quad \frac{a_{A-1}}{a_A}\sum_{k=1}^{m}\frac{(-1)^k H^+(a_{A-1}-k+1,\, a_A-1]}{\Gamma\left[\frac{a_{A-1}}{a_A}(a_A-1)-(a_{A-1}-1)+k\right]}$$

$$=\frac{(-1)^m [a_{A-1}-m,\, a_A]}{\Gamma\left[\frac{a_{A-1}}{a_A}(a_A-1)-(a_{A-1}-1)+m\right]}-\frac{H^+[a_{A-1},\, a_A]}{\Gamma\left[\frac{a_{A-1}}{a_A}(a_A-1)-a_{A-1}-1)\right]}$$

...(4.7)

$$(8)\quad \frac{a_A}{a_{A-1}}\sum_{k=1}^{m}\frac{(-1)^k H^+[a_{A-1}-1,\, a_A-k+1]}{\Gamma\left[\frac{a_A}{a_{A-1}}(a_{A-1}-1)-(a_A-1)+k\right]}$$

$$=\frac{(-1)^m H^+[a_{A-1},\, a_A-m]}{\Gamma\left[\frac{a_A}{a_{A-1}}(a_{A-1}-1)-(a_A-1)+m\right]}-\frac{H^+[a_{A-1},\, a_A]}{\Gamma\left[\frac{a_A}{a_{A-1}}(a_{A-1}-1)-(a_A-1)\right]}$$

...(4.8),

$$(9)\quad \frac{a_1}{a_A}\sum_{k=1}^{m}\frac{H^+[a_1-k+1,\, a_A-1]}{\Gamma\left[\frac{a_1}{a_A}(a_A-1)-(a_1-1)+k\right]}$$

$$=\frac{H^+[a_1,\, a_A]}{\Gamma\left[\frac{a_1}{a_A}(a_A-1)-(a_1-1)\right]}-\frac{H^+[a_1-m,\, a_A]}{\Gamma\left[\frac{a_1}{a_A}(a_A-1)-(a_1-1)+m\right]}$$

.....(4.9),

and

$$(10)\quad \frac{a_A}{a_1}\sum_{k=1}^{m}\frac{(-1)^k H^+[a_1-1,\, a_A-k+1]}{\Gamma\left[\frac{a_A}{a_1}(a_1-1)-(a_A-1)+k\right]}$$

$$= \frac{(-1)^{m+1} H^{+} [a_1, a_A - m]}{\Gamma\left[\frac{a_A}{a_1}(a_1-1)-(a_A-1)+m\right]} + \frac{H^{+}[a_1, a_A]}{\Gamma\left[\frac{a_A}{a_1}(a_1-1)-(a_A-1)\right]} \quad ...(4.10)$$

Formulas (4.1) to (4.3) will be derived from recurrence relation (3.1) by the formulation of collapsing series whereas (4.4) by iteration of (3.1) in case $\theta_A^{(1)} = \in_1^{(1)}$.

Similarly formulas (4.5) and (4.6), (4.7) and (4.8), and (4.9) and (4.10), can be obtained from recurrences (3.2), (3.3) and (3.4), respectively, by the formation of collapsing series.

(i) Pairing terms involving b_1 and replacing a_A by $a_A - k+1$, (3.1) can be rewritten as :

$$a_A H^{+} [a_A - k+1, b_1+1] = -\beta_1 H^{+} (a_A - k, b_1] + \triangle(a_A - k, b_1) H^{+} [a_A - k+1, b_1]$$

and then noting that

$$\triangle(a_A - k, b_1) = -\beta_1 \frac{\Gamma\left[\frac{a_A}{\beta_1} b_1 - (a_A - 1) + k\right]}{\Gamma\left[\frac{a_A}{\beta_1} b_1 - (a_A - 1) + k - 1\right]},$$

We can sum on k and collapse the resulting series on the the right. Consequently, we obtain (4.1).

(ii) If however, we pair terms involving a , replace b_1 by b_1+k-1, rewrite (3.1), sum on k and collapse the resulting series on the right, we arrive at (4.2).

(iii) If we replace a_A by $a_A + k$, b_1 by $b_1 + k$ and rewrite (3.1) after multiplying both sides by $(-1)^k a_A^{k-1} \beta_1^{m-k}$, we can sum on k and collapse the resulting se ies on rhe right. Consequently we get (4.3).

(iv) If we take $\theta_A^{(i)} = \theta_1^{(1)}$, (3.1) becomes

$$\Delta'(a_A - 1, b_1)\, \overset{+}{H}_0 [a_A, b_1] = \overset{+}{H}_0 [a_A, b_1 + 1] + 1\, \overset{+}{H}_0 [a_A - 1, b_1].$$

Now multiply both sides by $\Delta'(a_A - 2, b_1)\, \Delta'(a_A - 1, b_1 + 1)$ and consequently expend each term on right hand side by iteration of this same relation, we get, after a little simplification :

$$(-1)^2 \frac{\Gamma(b_1 - a_A + 3)}{\Gamma(b_1 - a_A + 1)} \overset{+}{H}_0 [a_A, b_1]$$

$$\sum_{k=0}^{2} \binom{2}{k} \overset{+}{H}_0 [a_A - k, b_1 + 2 - k]$$

Further iterations will finaly yield :

$$(-1)^m \frac{\Gamma(b_1 - a_A + m + 1)}{\Gamma(b_1 - a_A + 1)} \overset{+}{H}_0 [a_A ; b_1]$$

$$= \sum_{k=0}^{m} \binom{m}{k} \overset{+}{H}_0 [a_A - k, b_1 + m - k].$$

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A MUTIPLE INTEGRAL INVOLVING THE PRODUCT OF THE *H*-FUNCTIONS OF ONE AND TWO VARIABLES

By

U. C. JAIN

Department of Mathematics, University of Udaipur, Udaipur - 313001, Rajasthan, India

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ABSTRACT

In this paper, a multiple integral involving the product of the *H*-functions of one and two variable has been evaluated. This integral is new and very general in nature. Some interesting integrals have been obtained as special cases of the main integral.

1. Introduction

The *H*-function of two variables occurring in this paper is a special case of the general *H*-function of two variables studied earlier by Mittal and Gupta [5]. The parameters of this function will be displayed in the following contracted notation (due essentially to Srivastava and Panda [7, p. 266, Eq. (1.5)]) which is a direct extension of that of Srivastava and Joshi [6] :

$$H^{o,\,o\,:\,m_2,\,n_2\,;\,m_3,\,n_3}_{p_1,\,q_1\,:\,p_2,\,q_2\,;\,p_3,\,q_3}\left[\begin{matrix} x \\ y \end{matrix}\middle| \begin{matrix} (a_j\,;\,\alpha_j,\,A_j)_{1,\,p_1} : (c_j,\,\epsilon_j)_{1,\,p_2}\,;\,(e_j,\,E_j)_{1,\,p_3} \\ (b_j\,;\,\beta_j,\,B_j)_{1,\,q_1} : (d_j,\,\delta_j)_{1,\,q_2}\,;\,(f_j,\,F_j)_{1,\,q_3} \end{matrix}\right]$$

$$=H_1\,[x, y] = -\frac{1}{4\,\pi^2}\int_{L_1}\int_{L_2} \phi\,(s,\,t)\;\theta_1\,(s)\;\theta_2\,(t)\;x^s\,y^t\;ds\,dt \qquad \text{.. (1.1)}$$

where

$$\phi\,(s,\,t)=\left[\prod_{j=1}^{p_1}\Gamma\,(a_j-\alpha_j\,s-A_j\,t)\prod_{j=1}^{q_1}\Gamma\,(1-b_j+\beta_j\,s+B_j\,t)\right]^{-1} \qquad \text{..(1.2)}$$

$$\theta_1(s) = \prod_{j=1}^{n_2} \Gamma(1-c_j+\epsilon_j s) \prod_{j=1}^{m_2} \Gamma(d_j-\delta_j s)$$

$$\left[\prod_{j=m_2+1}^{q_2} \Gamma(1-d_j+\delta_j s) \prod_{j=n_2+1}^{p_2} \Gamma(c_j-\epsilon_j s) \right]^{-1} \quad ...(1.3)$$

and with $\theta_2(t)$ defined analogously to $\theta_1(s)$ in terms of the parameters sets $(e_j, E_j)_{1, p_3}$ and $(f_j, F_j)_{1, q_3}$. Also $(a_j ; \alpha_j, A_j)_{1, p_1}$ abbreviates the parameter sequence $(a_1 ; \alpha_1, A_1), \ldots, (a_{p_1} ; \alpha_{p_1}, A_{p_1})$; $(c_j, \epsilon_j)_{1, p_2}$ abbreviates the parameter sequence $(c_1, \epsilon_1), \ldots, (c_{p_2}, \epsilon_{p_2})$, and so on.

The conditions on the parameters of the H-function af two variables for the convergence of the integral (1.1), the nature of contours L_1 and L_2, some of the properties, special cases and asymptotic expansions of $H_1[x, y]$ can be referred to in the paper by Goyal [2]. It will be assumed that the conditions (i)—(vi), modified appropriately, given on p. 119 in the paper by Goyal [2], are always satisfied by the various H-functions of two variables occurring in this paper.

To save space, three dots "..." appearing at a particular place in in any H-function of two variables indicate that the parameters in that position are exactly the same as that of the H-function of two variables defined by (1.1).

We shall require the following special case of a result given by Goyal [2, p. 120, Eq. (1.2)] :

$$H^{o, o : 1, N_2 ; 1, N_3}_{P_1, Q_1 : P_2, Q_2 ; P_3, Q_3} \left[\begin{matrix} x \\ y \end{matrix} \middle| \begin{matrix} (a'_j ; \alpha'_j, A'_j)_{1, p_1} : (c'_j, \epsilon_j)_{1, p_2} ; (e'_j, E'_j)_{1, p_3} \\ (b'_j ; \beta'_j, B')_{1, Q_1} : (d'_o, \delta'_o), (d'_j, \delta'_j)_{2, Q_2} ; (f'_o, F'_o), (f'_j, F'_j)_{2, Q_3} \end{matrix} \right] = H^{*}_1[x, y]$$

$$= \sum_{u,\,v=0}^{\infty} \frac{(-1)^{u+v}\; x^{\rho_u}\; y^{\sigma_v}}{u\,!\; v\,!\; \delta_0'\; F_0'}\; \emptyset'(\rho_u, \sigma_v)\; \theta_1'(\rho_u)\; \theta_2'(\sigma_v) \tag{1.4}$$

where $\rho_u = \dfrac{d_0' + u}{\delta_0'}$, $\sigma_v = \dfrac{f_0' + v}{F_0'}$ ..(1.5)

$$\emptyset'(\rho_u, \sigma_v) = \left[\prod_{j=1}^{p_1} \Gamma\left(a_j' - \alpha_j' \rho_v - A_j' \sigma_v\right) \prod_{j=1}^{Q_1} \Gamma\left(1 - b_j' + \beta_j \rho_u + B_j' \sigma_v\right) \right]^{-1}$$

...(1.6)

$$\theta_1'(\rho_u) = \prod_{j=1}^{N_2} \Gamma\left(1 - c_j' + \epsilon_j' \rho_u\right) \left[\prod_{j=N_2+1}^{P_2} \Gamma\left(c_j' - \epsilon_j' \rho_u\right) \prod_{j=2}^{Q_2} \Gamma\left(1 - d_j' + \delta_j' \rho_u\right) \right]^{-1}$$

(...1.7)

and with $\theta_2'(\sigma_v)$ defined analogously to $\theta_1'(\rho_u)$ in terms of the parameter sets $(e_j', E_j')_{1,P_3}$ and $(f_j', F_j')_{2,Q_3}$.

2. Main Integral—The following multiple integral has been established in this paper :

$$\int_0^{\infty} \cdots\cdots \int_0^{\infty} \prod_{i=1}^{r} \left\{ (x_i)^{k_i - 1}\, H_{w_i,\, v_i}^{u_i,\, o} \left[\xi_i\, (x_i)^{t_i} \left| \begin{matrix} (g_j^i,\, G_j^i)_{1,\, w_i} \\ (h_j^i,\, H_j^i)_{1,\, v_i} \end{matrix} \right. \right] \right\}$$

$$H_1^{*} \left[a \prod_{i=1}^{r} (x_i)^{\mu_i},\; b \prod_{i=1}^{r} (x_i)^{\nu_i} \right] H_1 \left[y \prod_{i=1}^{r} (x_i)^{\eta_i},\; z \prod_{i=1}^{r} (x_i)^{\lambda_i} \right]$$

$$\prod_{i=1}^{r} (dx_i)$$

$$= \sum_{u,v=o}^{\infty} \frac{(-1)^{u+v}\, \emptyset'(\rho_u, \sigma_v)\, \theta_1'(\rho_u)\, \theta_2'(\sigma_v)\, a^{\rho_u} b^{\sigma_v}}{u\,!\, v\,!\, \delta_o' F_o' \sum_{i=1}^{r} \left\{ t_i (\xi_i)^{(k_i+\mu_i \rho_u+\nu_i \sigma_v)/t_i} \right\}}$$

$$H^{o,\, U :\, \ldots\, ;\, \ldots}_{p_1+V,\, q_1+W\, :\, \ldots\, ;\, \ldots} \left[\begin{array}{c|c} y \prod_{i=1}^{r} (\xi_i)^{-\eta/t_i} & A(u,v) : \ldots ; \ldots \\ z \prod_{i=1}^{r} (\xi_i)^{-\lambda_i/t_i} & B(u,v) : \ldots ; \end{array} \right] \quad \ldots(2.1)$$

where $U = \sum_{i=1}^{r} u_i$, $V = \sum_{i=1}^{r} v_i$, $W = \sum_{i=1}^{r} w_i$,

$A(u, v)$ stands for $(a_j ; \alpha_j , A_j)_{1, p_1}$,

$$\left\{ \overline{(1-h_j^1 H_j^1 (k_i+\mu_i \rho_u+\nu_i \sigma_v)/t_i} ; H_j^1 \eta_i/t_i, H_j^1 \lambda_i/t_i)_{1, v_r} \right\}_{i=1}^{r}$$

$B(u, v)$ for $(b_j ; \beta_j , B_j)_{1, q_1}$,

$$\left\{ (1 - g_j^1 G_j^1 (k_i+ \mu_i \rho_u + \nu_i \sigma_v) / t_i ; G_j^1 \eta_i / t_i, G_j^1 \lambda_i/ t_i)_{1, w_r} \right\}_{i=1}^{r}$$

and the various symbols $\emptyset'(\rho_u, \sigma_v)$, $\theta_1'(\rho_u)$, $\theta_2'(\sigma_v)$, ρ_u and σ_v are explained in Section 1.

The Integral (2.1) is valid under the following conditions ;

$$\text{Re}\left\{ k_i + \mu_i (d_o'/\delta_o') + \nu_i (f_o'/F_o') \right\} + t_i \min_{1 \leqslant j \leqslant u_i} \left[\text{Re} (h_j^1/H_j^1) \right]$$

$$+ \eta_i \min_{1 \leqslant j \leqslant m_2} \left[Re (d_j/\delta_j) \right] + \lambda_i \min_{1 \leqslant j \leqslant m_3} \left[Re (f_j/F_j) \right] > o ,$$

$$A = -\sum_{j=1}^{u_i} G_j^i + \sum_{j=1}^{v_i} H_j^i - \sum_{j=v_i+1}^{w_i} H_j^i > o\,,\ |\arg \xi_i| < (1/2)\, A_i\, \pi\,,$$

$t_i\,, \mu_i\,, \nu_i\,, \eta_i\,, \lambda_i > o\,,\ i = 1, .\,, r.$

and the double series on the right-hand side of (2.1) is absolutely convergent.

Also, $(g_j^i, G_j^i)_{1, w_i}$ abbreviates the w_i—parameter sequence $(g_1^i, G_1^i), \ldots,$ $(g_{w_i}^i, G_{w_i}^i)$, and $\left\{ (a_j^i; \alpha_j^i, A_j^i)_{1, p_i} \right\}_{i=1}^{r}$ the $(p_1+p_2+\ldots+p_r)$—parameter sequence $(a_j^1; \alpha_j^1, A_j^1)_{1,p_1}, (a_j^2; \alpha_j^2, A_j^2)_{1,p_2}, \ldots, (a_j^r; \alpha_j^r, A_j^r)_{1,p_r}$.

Proof – To prove (2.1), we first use the equation (1.4) for H_1^* – function occurring in the integrand, change the order of integration and summation (which is assumed to be justified under the conditions mentioned with (2.1)), and then apply a known result of Vasishta and Goyal [8, p. 43, Eq. (2.1)].

3. Particular Cases – On account of the general nature of the integral established in (2.1), a number of other multiple integrals can be obtained by specializing the parameters of the various functions involved. To illustrate, we mention below some interesting particular cases.

(a) Taking $P_1 = Q_1 = o$, $N_2 = P_2 = Q_2 = N_3 = P_3 = 1$, $Q_3 = 2$, $\delta_0' = d_0' = E_1'$ $= F_0' = F_2' = c_1' = \epsilon_1' = 1$, $u_i = v_i = \mu_i = \nu_j = t_i = H_1^i = h_1^i = 1$, $w_i = o$, $i = 1, \ldots, r$, $e_1' = 1 - k$, $f_0' = (1/2) + n$, $f_2' = (1/2) - n$ in (2.1), and using the known results [2, p. 123, Eq. (3.4)], [3, p. 598, Eq. (4.1)], and [4, § 2.6] therein, we get the following interesting multiple integral :

$$\int_0^\infty \ldots \int_0^\infty \prod_{i=1}^{r} \left\{ (x_i)^{k_i} \exp(-\xi_i\, x_i) \right\} (1 + a \prod_{i=1}^{r} x_i)^{-1} \exp\left(-b/2 \sum_{i=1}^{r} x_i\right)$$

$$\cdot M_{k,n}\left(b\prod_{i=1}^{r}x_i\right) H_1\left[\, y\prod_{i=1}^{r}(x_i)^{\eta_i},\; z\prod_{i=1}^{r}(x_i)^{\lambda_i}\right]\prod_{i=1}^{r}(dx_i)$$

$$=\sum_{u,v=o}^{\infty}\frac{(-1^{u+v}(\frac{1}{2}+k+n)_v\; a^u\; b^{v+n+\frac{1}{2}}}{v!\;(2n+1)_v}\prod_{i=1}^{r}\left\{(\xi_i)^{-(k_i+v+u+n+\frac{5}{2})}\right\}$$

$$H^{o,r\,:\,\ldots\ldots;\,\ldots}_{p_1+r,\,q_1\,:\ldots;\,\ldots}\left[\begin{array}{c|c} y\prod_{i=1}^{r}(\xi_i)^{-\eta_i} & (-k_i-u-v-n-\frac{1}{2};\eta_i,\lambda_i)_{1,r}:\ldots;\ldots \\ & \ldots\qquad :\ldots;\ldots \\ z\prod_{i=1}^{r}(\xi_i)^{-\lambda_i} & \end{array}\right] \qquad \ldots(3.1)$$

where $M_{k,n}(x)$ is the whittaker function [4, p. 44, Eq. (2.3.4a)] and $\text{Re}\,(k_i+n+3/2)+\eta_i \min_{1\leqslant j\leqslant m_2}[\text{Re}\,(d_j/\delta_j)]+\lambda_i \min_{1\leqslant j\leqslant m_3}[\text{Re}\,(f_j/F_j)]>o,$ and $\eta_i.\ \lambda_i>o,\ i=1,\ldots,r.$

(b) Putting $p_1=q_1=n_3=p_3=o,\ m_3=q_3=1, f_1=o,\ F_1=1$ and letting $z\to o$, $\lambda_i=1,\ i=1,\ \ , r$ in (3.1), using the result [2, p. 123, Eq. (3.5)] therein, we get

$$\int_0^{\infty}\int_0^{\infty}\prod_{i=1}^{r}\left\{(x_i)^{k_i}\exp(-\xi_i x_i)\right\}\exp\left(-b/2\sum_{i=1}^{r}x_i\right) M_{k,n}\left(b\prod_{i=1}^{r}x_i\right)$$

$$\cdot H^{m_2,n_2}_{p_2,q_2}\left[y\prod_{i=1}^{r}(x_i)^{\eta_i}\,\middle|\,\begin{array}{c}(c_j,\varepsilon_j)_{1,p_2}\\(d_j,\delta_j)_{1,q_2}\end{array}\right]\prod_{i=1}^{r}(dx_i)$$

$$=\sum_{u,v=o}^{\infty}\frac{(-1)^{u+v}(\frac{1}{2}+k+n)_v a^u b^{v+n+\frac{1}{2}}}{v!\;(2n+1)_v}\prod_{i=1}^{r}\left\{(\xi_i)^{-(k_i+u+v+n+\frac{5}{2})}\right\}$$

$$. H^{m_2, n_2 + r}_{p_2 + r, q_2} \left[y \prod_{i=1}^{r} (\xi_i)^{-\eta_i} \middle| \begin{matrix} (-k_i - u - v - n - \frac{1}{2}, \eta_i)_{1,r}, (c_j, e_j)_{1,p_2} \\ (d_j, \delta_j)_{1,q_2} \end{matrix} \right] \quad ...(3.2)$$

where $\operatorname{Re}(k_i + n + 3/2) + \eta_i \min_{1 \leq j \leq m_2} [\operatorname{Re}(d_j/\delta_i)] > o$, $\eta_i > o$, $i = 1, \ldots, r$,

$$A = \sum_{j=1}^{m_2} \delta_j - \sum_{j=m_2+1}^{q_2} \delta_j + \sum_{j=1}^{n_2} \epsilon_j - \sum_{j=n_2+1}^{p_2} \epsilon_j > o, \text{ and}$$

$$|\arg y| < (1/2) A \pi,$$

(c) Again taking $p_1 = q_1 = n_3 = p_2 = n_3 = o$, $m_2 = q_2 = m_3 = q_3 = 2$, $p_3 = 1$, $\delta_1 = \delta_2 = E_1 = F_1 = F_2 = 1$, $d_1 = \nu/2$, $d_2 = -\nu/2$, $e_1 = 1 - k'$, $f_1 = n' + 1/2$, $f_2 = -n' + 1/2$, $\lambda_i = \eta_i = 1$, $i = 1, \ldots r$, replacing y by $y^2/4$ in (3.1), using the known results [2, p. 123, Eq. (3.5)], [3, p. 598, Eq. (4.1)] and [4, p. 53, § 2.6] therein, we get the following integral :

$$\int_o^\infty \cdots \int^\infty \prod_{i=1}^{r} \left\{ (x_i)^{k_i} \exp(-\xi_i x_i) \right\} \exp\left(-\frac{b+z}{2} \sum_{i=1}^{r} x_i\right)$$

$$. M_{k,n}\left(b \prod_{i=1}^{r} x_i\right) W_{k',n'}\left(z \prod_{i=1}^{r} x_i\right) K_\nu\left(y \prod_{i=1}^{r} x_i\right) \prod_{i=1}^{r} (dx_i)$$

$$= \sum_{u, v=o}^{\infty} \frac{(-1)^{u+v} (\frac{1}{2}+k+n)_v (a^u\, b^{v+n+\frac{1}{2}}}{2\; v!\; (2n+1)_v} \prod_{i=1}^{r} \left\{ (\xi_i)^{-(k_i+u+v+n+\frac{5}{2})} \right.$$

$$H^{o, r : 2, o ; 2, o}_{r, o : o, 2 ; 1, 2} \left[\begin{matrix} \frac{y^2}{4} \prod_{i=1}^{r} (\xi_i)^{-1} \\ z \prod_{i=1}^{r} (\xi_i)^{-1} \end{matrix} \middle| \begin{matrix} (-k_i - u - v - n - \frac{1}{2})_{1,r} : \; - \; ; \; 1 - k' \\ : \nu/2, \; \nu/2; n' + \frac{1}{2}, -n' + \frac{1}{2} \end{matrix} \right] \quad (3.3)$$

where $\operatorname{Re}(k_i + n \pm n' \pm \frac{\nu}{2} + 2) > o$, $i=1,\ldots, r$.

In (3.3), $G\begin{bmatrix} x \\ y \end{bmatrix}$ stands for the G-function of two variables introduced by Agarwal [1], $K_\nu(x)$ is the modified Bessel function, and $W_{k,n}(x)$ is the well-known Whittaker function.

Lastly, if we put $N_2=P_2=N_3=P_3=P_1=Q_1=o$, $Q_2=Q_3=1$, $\mu_i=\nu_i=1$. $i=1,\ldots, r$ in (2.1) and let a, $b \to o$ in it, we get a recent result of Vasishta and Goyal [8, p. 43, Eq. (2.1)] which is turn reduces to the result by Mittal and Gupta [5, p. 121] by taking $r=1$ in the result of Vasishta and Goyal [8].

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(*Dedicated to the memory of Professor Arthur Erdélyi*)

CHEMICAL EXAMINATION OF STEMS OF PLUMBAGO ZEYLANICA

by

G. S. NIRANJAN and B. C. SIKRORIA

Department of Chemistry, D. V. (Postgraduate) College, Orai-285001, U. P., India

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Plumbago zeylanica (F. Plumbagoinaceae) is distributed throughout India. The roots of the plant revealed a new plumbagin and 3-chloroplumbagin [4] Later on 33' biplumbagin and five other constituents ([2], [3]) were also reported. From the literature it seems that no chemical examination was made from the stems of the plant

The stems of the Plumbago zeylanica were collected from the interior of the Bundelkhand region and botanically identified. The stems of the plant were successively extracted with petroleum ether (60-80) and benzene. The benzene extract yields a compound 'A' m. p. 77°C; benzoyal derivative ; m. p. 145°C. The compound 'A' was identified as plumbagin by chemical degradations and special measurements. Finally, the structure of compound 'A' was confirmed bp spectral and degradative reactions.

EXPERIMENTAL

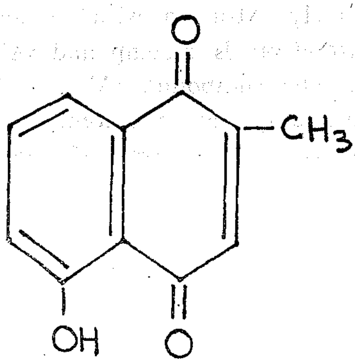
Isolation and purification of compound :

The concentrated benzene extract of Plumbago zeylanica revealed the presence of many constituents on TLC. The extract was placed for chromatography on silica gel column. A pigment 'A' was isolated in a state of reasonable purity. The compound was purified by two dimensional thin layer chromatography using silica gel G and solvent system, chloroform : methanol (9 : 1). The compound 'A' was crystalised in chloroform as orange yellow needles (m. p. 77°C).

Study of Compound 'A' :

Componnd 'A' Orange - yellow crystalline solid (m. p. 77°C), is very soluble in organic solvents such as ether, benzene, chloroform acetone and hot alcohal. In acidic medium it gave pink colour. The combustion and molecular weight determination indicates an aromatic ring compound. The deep voilet colour with neutral ferric chloride solution indicates the presence of phenolic (—OH) group in the compound. On bezoylation it gave a benzoyl derrivative (m. p. 145°C), which showed that the benzene ring contains one methyl group. Elemental analysis and high resolution mass spectrometry established the formula as $C_{11}H_8O_3$. The compound 'A' was identified as 2-methyl—5—hydroxy 1—4 – napthaquinone by chemical degradation [1] and spectral measurements. $\lambda\ max^{95\%\ EtOH}$ (log $\in$) ; 210 (4.53), 255 Sn (4.07), 267 (4.08), 424 (3.67) mm. $\lambda\ max^{95\% EtOH+OH^-}$ log ($\in$) ; 211 (5·04), 273 (4.00), 575 (3.84) mm. ; ν_{max}^{KBr} : 1160, 1640 cm^{-1}, NMR : δ (CCl_4) 2.02, d (J=1.5C/S), 2—CH_3 ; 6.58, $\underline{\rho}$ (J=1.5 C/S) C - 3H ; 6.92 – 7.08, $\underline{\rho}$ C – 6H, 7.33—7.46, t, C—7 and C—8H, 11.73, S, exchangable with D_2 O C – 5OH.

Its m. p. (77°C) is same as plumbagin reported in the literature. Finally the identity of the compound 'A' was established by T. I. R., N. M. R. and Mass spectral data.



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(*Dedicated to the memory of Professor Arthur Erdélyi*)

ON THE ABSOLUTE HARMONIC SUMMABILITY OF A LEGENDRE SERIES

By

SUSHIL SHARMA

Department of Mathematics, Madhav Vigyan Mahavidyalaya, Ujjain-**456001**, *M. P., India*

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ABSTRACT

The absolute harmonic summability of a Fourier series has been studied by Varshney [2]. In this paper we study the absolute harmonic summability of a Legendre series.

1. Introduction

Let f (x) be a Lebesgue integrable in $[-1, 1]$. The Legendre series corresponding to this function is

$$f(x) \sim \sum_{n=0}^{\infty} a_n P_n(x) \tag{1.1}$$

where

$$a_n = (n+\tfrac{1}{2}) \int_{-1}^{1} f(x) P_n(x)\, dx,$$

and $P_n(x)$ is defined by the following expansion :

$$(1-2xz+z^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} z^n P_n(x) \tag{1.3}$$

Also let $\{S_n\}$ be the sequence of partial sums of a given infinite series Σa_n and define the sequence $\{t_n\}$ by

$$t_n = \frac{(n+1)^{-1} S_0 + n^{-1} S_1 + \ldots + S_n}{p_n}$$

$$p_n = 1 + \tfrac{1}{2} + \dots + \frac{1}{n+1}.$$

The series $\Sigma\, a_n$ is said to be absolutely harmonic summable if the series

$$(1.4) \qquad \sum_{n=1}^{\infty} \left| t_n - t_{n-1} \right|$$

is convergent. It is known that this method of summability is absolutely regular and implies absolute Cesáro summability of every positive order (see [1]).

Varshney [2] has applied the method of absolute harmonic summaility to the series.

$$(1.5) \qquad \tfrac{1}{2} a_0 + \sum_{n=0}^{\infty} (a_n \cos nt + b_n \sin nt) \equiv \sum_{n=0}^{\infty} U_n,$$

where a_n and b_n are the Fourier coefficients of a function $f(x)$, which is periodic with period 2π and integrable (L) over $(-\pi, \pi)$.
He proved the following theorem :

heorem A (Varshney [2, p. 589]) : *If* f (x) *is of bounded variation in* $(-\pi, \pi)$ *and satisfies*

$$| f(x+h) - f(x) | \leqslant A \log^{-1-\varepsilon} (1/h), \; h \to 0 \; (\varepsilon > 0, \; 0 \leqslant x \leqslant 2\pi)$$

then the series 1.5) *is absolutely harmonic summable.*

Concerning the absolute Cesáro summability of the series (1.1) we have established the following theorem :

Theorem B (Sharma [3]) : *If* $\alpha > 1$.

$$\int_0^t | \psi(\theta, \omega) | \, d\omega = O\left(\frac{t}{(\log 1/t)\, \alpha} \right), \; t \to 0$$

and

$$\int_0^h | \phi(u) | \, du = O\left(\frac{h}{(\log 1/h)^{\alpha}} \right), \; h \to 0$$

then the series (1.1) *is summable* $|C, \delta|$ *for every* $\delta > 1$ *at a point* x *of the interval* $[-1+\varepsilon, 1-\varepsilon]$, ε *being a small fixed positive number.*

In this paper we prove a theorem on the absolute harmonic summability of the series (1.1) under the condition which is weaker than that of theorem B and corresponds to that of theorem A for Fourier series. In what follows we prove the following :

Theorem C : *If* f (x) *is of bounded variation in* $[-1, 1]$ *and satisfies*

$$|f(x+h)-f(x)| = \left(\frac{A}{\log^{1+a}\left(\frac{1}{h}\right)}\right), \; a>o, \; h\to o,$$

then the series (1.1) *is summable* $\left|N, \frac{1}{n+1}\right|$ *in the interval* $[-1+\varepsilon, 1-\varepsilon]$, $\varepsilon>o$.

In the proof of the theorem we require following lemma :

Lemma 1 (Sansone [4, p. 208]) :

$$(1.6) \qquad P_n(\cos\theta) = n^{-\frac{1}{2}} k(\theta)\left[\cos\left\{(n+\tfrac{1}{2})\,\theta-\frac{\pi}{4}\right\}\left(1-\frac{1}{4n}\right)\right.$$

$$\left.+\frac{1}{8n}\cot\theta\,\sin\left\{(n+\tfrac{1}{2})\,\theta-\frac{\pi}{4}\right\}\right]+O\left(n^{-5/2}\right)$$

$$k(\theta) = \sqrt{\frac{2}{\pi\sin\theta}} \text{ for } \epsilon' \leqslant \theta \leqslant \pi-\varepsilon', \; o<\varepsilon'<\frac{\pi}{2}.$$

2. **Proof of Theorem C** : We have :

$$t_n - t_{n-1} = \frac{1}{p_n\, p_{n-1}} \int_0^{\pi} \phi(\theta, \omega)\left[\sum_{k=o}^{n-1}\left(\frac{p_n}{k+1}-\frac{p_k}{n+1}\right)(n-k+\tfrac{1}{2})\, P_{n-k}(\cos\theta)\right.$$

$$\left.\times P_{n-k}(\cos\omega)\right] \sin\omega \; d\omega$$

where

$$\phi(\theta, \omega) = \{f(\cos\omega)-f(\cos\theta)\}$$

We denote the integral in the above expansion with the following:

$$t_n - t_{n-1} = \frac{1}{p_n p_{n-1}} \left[\int_0^{c/n} + \int_{c/n}^{\epsilon/2} + \int_{\varepsilon/2}^{\theta} + \int_{\theta}^{\pi-\varepsilon/2} + \int_{\pi-\epsilon/2}^{\pi-c/n} + \int_{\pi-c/n}^{\pi} \right]$$

$$= L_1 + L_2 + L_3 + L_4 + L_5 + L_6 \text{, say.}$$

Now

$$(2.1)\ L_1 = \frac{1}{p_n p_{n-1}} \int_0^{c/n} \phi(\theta,\omega) \sum_{k=0}^{n-1} \left(\frac{p_n}{k+1} - \frac{p_k}{n+1} \right) (n-k+\tfrac{1}{2})\, P_{n-k}(\cos\theta)\, P_{n-k}(\cos\omega) \sin\omega\, d\omega.$$

$= O(n^{-3/2})$; $\phi(\theta, \omega)$ is bounded.

Similarly

$$(2.2)\qquad L_6 = O(n^{-3/2})$$

Now using the first part of lemma 1, we have

$$(2.3)\quad L_3 = \frac{1}{p_n p_{n-1}} \int_{\epsilon/2}^{\theta} \phi(\theta,\omega) \sum_{k=0}^{n-1} \left(\frac{p_n}{k+1} - \frac{p_k}{n+1} \right) k(\theta)\, k(\omega) \frac{1}{4(n-k)}$$

$$\times \cos\left\{ (n-k+\tfrac{1}{2})\,\theta - \frac{\pi}{4} \right\} \cos\left\{ (n-k+\tfrac{1}{2})\,\omega - \frac{\pi}{4} \right\} \sin\omega\, d\omega$$

$$+ \frac{1}{p_n p_{n-1}} \int_{\epsilon/2}^{\theta} \phi(\theta,\omega) \sum_{k=0}^{n-1} \left(\frac{p_n}{k+1} - \frac{p_k}{n+1} \right) k(\theta)\, k(\omega) \frac{1}{8(n-k)}$$

$$\times \cos\left\{ (n-k+\tfrac{1}{2})\,\theta - \frac{\pi}{4} \right\} \sin\left\{ (n-k+\tfrac{1}{2})\,\omega - \frac{\pi}{4} \right\} \sin\omega\, d\omega$$

$$+ \frac{1}{p_n p_{n-1}}\, O\left[\sum_{k=0}^{n-1} \left(\frac{p_n}{k+1} - \frac{p_k}{n+1} \right) \frac{1}{(n-k)^2} \right]$$

$$= L_{3\cdot1} + L_{3\cdot2} + L_{3\cdot3} + L_{3\cdot4} \text{, say.}$$

$$(2.4)\quad L_{3\cdot1}=\frac{1}{2\,p_n\,p_{n-1}}\int_{\varepsilon/2}^{\theta}\phi\,(\theta,\,\omega)\sum_{k=o}^{n-1}\left(\frac{p_n}{k+1}-\frac{p_k}{n+1}\right)k\,(\theta)\,k\,(\omega)$$

$$\times\cos\left\{(n-k+\tfrac{1}{2})\,(\theta+\omega)-\frac{\pi}{2}\right\}\sin\omega\,d\omega$$

$$+\frac{1}{2\,p_n\,p_{n-1}}\int_{\varepsilon/2}^{\theta}\phi\,(\theta,\,\omega)\sum_{k=o}^{n-1}\left(\frac{p_n}{k+1}-\frac{p_k}{n+1}\right)k\,(\theta)\,k\,(\omega)$$

$$\cos\,\{(n-k+\tfrac{1}{2})\,(\theta-\omega)\}\times\sin\omega\,d\omega$$

$$=\quad L_{3\cdot1\cdot1}+L_{3\cdot1\cdot2}\text{ , say.}$$

Putting $\theta-\omega=t$, we obtain

$$(2.5)\quad L_{3\cdot1\cdot2}\leqslant\frac{1}{2\,p_{n-1}}\left|\int_{0}^{\theta-\varepsilon/2}\phi\,(t)\,k\,(\theta)\,k\,(\theta-t)\sum_{k=o}^{\infty}\frac{\cos\,(n-k+\frac{1}{2})t}{k+1}\right.$$

$$\left.\sin\,(\theta-t)\,dt\right|$$

$$+\frac{1}{2\,p_{n-1}}\left|\int_{0}^{\theta-\varepsilon/2}\phi\,(t)\left\{\sum_{k=n}^{\infty}\frac{\cos\,(n-k+\frac{1}{2})\,t}{k+1}+\sum_{k=o}^{n-1}\frac{p_k\cos\,(n-k+\frac{1}{2})\,t}{(n+1)\,p_n}\right\}\right.$$

$$\left.\times\;k\,(\theta)\;k\,(\theta-t)\sin\,(\theta-t)\;dt\right|$$

$$=\quad S_1+S_2+S_3+S_4\text{ , say.}$$

By Abel's transformation, we have on the line of Varshney [2].

$$\sum_{k=n}^{\infty}\frac{\cos\,(n-k+\frac{1}{2})\,t}{k+1}=\frac{1}{2\,(n+1)}\cos t/2+\frac{1}{2\,(n+1)}\sin t/2+O\,(n^{-2}\,t^{-2})$$

$$\text{for } t\geqslant\frac{1}{n}$$

$$\sum_{k=o}^{n-1}p_k\cos\,(n-k)\,t=O\left\{\left(1+\log\frac{1}{t}\right)t^{-1}\right\}-\tfrac{1}{2}\,p_{n-1}$$

$$(2.4)\quad L_{3\cdot 1} = \frac{1}{2\,p_n\,p_{n-1}} \int_{\varepsilon/2}^{\theta} \phi\,(\theta,\,\omega) \sum_{k=o}^{n-1} \left(\frac{p_n}{k+1} - \frac{p_k}{n+1}\right) k\,(\theta)\,k\,(\omega)$$

$$\times \cos\left\{ (n-k+\tfrac{1}{2})\,(\theta+\omega) - \frac{\pi}{2} \right\} \sin\omega\,d\omega$$

$$+ \frac{1}{2\,p_n\,p_{n-1}} \int_{\varepsilon/2}^{\theta} \phi\,(\theta,\,\omega) \sum_{k=o}^{n-1} \left(\frac{p_n}{k+1} - \frac{p_k}{n+1}\right) k\,(\theta)\,k\,(\omega)$$

$$\cos\{(n-k+\tfrac{1}{2})\,(\theta-\omega)\} \times \sin\omega\,d\omega$$

$$= L_{3\cdot 1\cdot 1} + L_{3\cdot 1\cdot 2}\ \text{, say.}$$

Putting $\theta-\omega=t$, we obtain

$$(2.5)\quad L_{3\cdot 1\cdot 2} \leqslant \frac{1}{2\,p_{n-1}} \left| \int_{0}^{\theta-\varepsilon/2} \phi\,(t)\,k\,(\theta)\,k\,(\theta-t) \sum_{k=o}^{\infty} \frac{\cos\,(n-k+\frac{1}{2})t}{k+1} \right.$$

$$\left. \sin\,(\theta-t)\,dt \right|$$

$$+ \frac{1}{2\,p_{n-1}} \left| \int_{0}^{\theta-\varepsilon/2} \phi\,(t) \left\{ \sum_{k=n}^{\infty} \frac{\cos\,(n-k+\frac{1}{2})\,t}{k+1} + \sum_{k=o}^{n-1} \frac{p_k \cos\,(n-k+\frac{1}{2})\,t}{(n+1)\,p_n} \right\} \right.$$

$$\left. \times\ k\,(\theta)\,k\,(\theta-t)\,\sin\,(\theta-t)\,dt \right|$$

$$= S_1 + S_2 + S_3 + S_4\ \text{, say.}$$

By Abel's transformation, we have on the line of Varshney [2].

$$\sum_{k=n}^{\infty} \frac{\cos\,(n-k+\frac{1}{2})\,t}{k+1} = \frac{1}{2\,(n+1)} \cos t/2 + \frac{1}{2\,(n+1)} \sin t/2 + O\,(n^{-2}\,t^{-2})$$

$$\text{for } t \geqslant \frac{1}{n}$$

$$\sum_{k=o}^{n-1} p_k \cos\,(n-k)\,t = O\left\{ \left(1+\log\frac{1}{t}\right) t^{-1} \right\} - \tfrac{1}{2}\,p_{n-1}$$

Similarly

$$\sum_{k=o}^{n-1} p_k \sin (n-k)\, t \;=\; O\left\{ (1+\log \frac{1}{t})\, t^{-1} \right\} - \tfrac{1}{2} p_{n-1}$$

From these resul s, we obtain

$$\sum S_2 \leqslant A \sum \frac{1}{p_{n-1}} \int_{c/n}^{\theta-\varepsilon/2} |\phi(t)|\, n^{-2} t^{-2}\, dt + A \sum \frac{1}{(n+1)\, p_n\, p_{n-1}} \int_{c/n}^{\theta-\varepsilon/2} |\phi(t)| \frac{\log 1/t}{t}\, dt$$

$$+ \sum \frac{1}{p_{n-1}} \left| \int_{c/n}^{\theta-\epsilon/2} \phi(t)\, k(\theta)\, k(\theta-t) \sin(\theta-t) \left[\frac{1}{n+1} \cos t/2 + \tfrac{1}{2} \frac{p_{n-1}}{(n+1)\, p_n} \sin t/2 - \frac{p_{n-1}}{2(n+1)\, p_n} \cos t/2 \right] dt \right|$$

$$= O\left[\sum \frac{1}{n (\log n)^{1+a}} \right]$$

$$= O(1).$$

$$\sum S_1 \leqslant \sum \frac{1}{2 p_{n-1}} \left| \int_0^{\theta-\epsilon/2} \phi(t) \left[\left\{ a(t) \cos t/2 + \beta(t) \sin t/2 \right\} k(\theta)\, k(\theta-t) \sin(\theta-t) \right] \times \sin t\, dt \right| + \sum \frac{1}{2 p_{n-1}} \left| \int_0^{\theta-\epsilon/2} \phi(t) \left[\left\{ \beta(t) \cos t/2 - a(t) \sin t/2 \right\} k(\theta)\, k(\theta-t) \sin(\theta-t) \right] \times \sin t\, dt \right|$$

where

$$a(t) = \sum_{k=o}^{\infty} \frac{\cos kt}{k+1} \quad ; \quad \beta(t) = \sum_{k=o}^{\infty} \frac{\sin kt}{k+1}$$

Now we write

$$\sum S_1 = \sum \frac{1}{2\,p_{n-1}} \left| \int_0^{\theta-\epsilon/2} \phi(t)\, p(t) \cos nt\, di \right|$$

$$+ \sum \frac{1}{2\,p_{n-1}} \left| \int_0^{\theta-\epsilon/2} \phi(t)\, q(t) \sin nt\, dt \right|$$

$$= \sum \frac{|a_n| + |b_n|}{2\,p_{n-1}}$$

where

$p(t) = k(\theta)\, k(\theta - t) \sin(\theta - t) \{ \alpha(t) \cos t/2 + \beta(t) \sin t/2 \}$ and
$q(t) = k(\theta)\, k(\theta - t) \sin(\theta - t) \{ \beta(t) \cos t/2 - \alpha(t) \sin t/2 \}$.

Since $\alpha(t)$ and $\beta(t)$ are continuous for $n \leqslant t \leqslant$ and for $0 < t \leqslant n$, their absolute values are each less than $A \log 1/t$. Hence $p(t)$ and $q(t)$ are also continuous and $|p(t)|$, $|q(t)|$ both are less than $A \log 1/t$.

Thus the constants a_n and b_n are Fourier coefficients of even and odd functions respectively and each of these function belongs to L^2 class. We define

$$\psi(t) = \phi(t) \text{ in } [0, \theta - \epsilon/2]$$

$$= 0 \text{ in } [\theta - \epsilon/2, \pi]$$

Therefore, from Parsvall's theorem, we have

$$\sum_{n=1}^{\infty} a_n^2 \sin^2 nh \leqslant A \int_0^{\pi} \{\psi(t+h)\, p(t+h) - \psi(t-h)\, p(t-h)\}^2\, dt$$

$$\leqslant 2A \{ I_1(h) + I_2(h) \} \text{ where}$$

$$I_1(h, \int_0^{\pi} \{p(t+h)\}^2 \{\psi(t+h) - \psi(t-h)\}^2\, dt$$

$$I_2(h) = \int_0^{\pi} \{\psi(t-h)\}^2 \{p(t+h) - p(t-h)\}^2\, dt$$

Now from the hypothesis we observe that

$$|\psi(t)| = |f \cos(\theta - t) - f(\cos\theta)| = \left(\frac{A}{(\log 1/t)^{1+\alpha}} \right)$$

Then for a positive h, we have

$$\psi\ (t+h)-\psi\ (t-h) \leqslant \left(\frac{A}{(\log 1/h)^{1+a}}\right).$$

Hence

$$I_1\ (h)\ \leqslant \left(\frac{A}{\log^{2a+2}\ 1/h}\right)\int_0^{\pi} \log^2 1/t \leqslant \left(\frac{A\ h}{\log^{\ 2a}(1/h)}\right).$$

On the line of McFadden [1], we have

$$(2.6)\ I_2\ (h) \leqslant A\int_{-h}^{h} \psi^2\ (t)\ p^2\ (t+2h)\ dt+\ A\int_{-h}^{h} \psi^2\ (t)\ p^2\ (t)\ +A$$

$$\int_{-h}^{\pi-h} \psi^2\ (t)\ \{p\ (t+2h)-p\ (t)\}^2\ dt$$

$$=\ I_{2\cdot1}\ +\ I_{2\cdot2}\ +\ I_{2\cdot3}\ ,$$

$$(2.7)\quad I_{2\cdot1}\ \leqslant A\int_{-h}^{h} \frac{1}{\log^{2a+2}\ 1/t}\ \log^2 \frac{1}{t+2\ h}\ dt \leqslant\left(\frac{A\ h}{\log^{2a} 1/h}\right).$$

Similarly

$$(2.8)\quad I_{2\cdot2}\ \leqslant\left(\frac{Ah}{\log^{2a} 1/h}\right).$$

Finally, we know that

$$\left|\frac{d}{dt}\left(\sum_{k=o}^{\infty}\ \frac{\cos kt}{k+1}\right)\ \right| = \left|-\sum_{k=o}^{\infty}\ (\sin\ kt)\ -\sum_{k=o}^{\infty}\ \frac{\sin\ kt}{k+1}\right| \leqslant \frac{A}{t}\ +A\ \log\ 1/t \leqslant \frac{A}{t}$$

By mean value theorem we have

$$(2.9)\quad I_{2\cdot3}\ \ =\int_{-h}^{\pi-h} \psi^2\ (t)\ \{p\ (t+2\ h)-p\ (t)\}^2\ dt \leqslant\left(\frac{Ah}{\log^{2a} 1/h}\right)$$

Now setting $h=\left(\frac{\pi}{2N}\right)$ and collecting (2.6), (2.7), (2 8) and (2.9) we find that

$$(2.10) \quad I_2(h) = \frac{A}{\mathcal{N} \log^{2\alpha} \mathcal{N}}$$

Since f is continuous and of bounded variation it is clear that ψ is also continuous and of bounded variation. Let $\omega(\delta)$ be the modulus of continuity of ψ and V the total variation of ψ over $(0, 2\pi)$.

We start from the inequality

$$\sum_{h=1}^{2\mathcal{N}} \left\{ p\left(t+\frac{k\pi}{\mathcal{N}}\right) \right\}^2 \left[\psi\left(t+\frac{k\pi}{\mathcal{N}}\right) - \psi\left(t+(k-1)\frac{\pi}{\mathcal{N}}\right) \right]^2$$

$$\leqslant A \log^2 1/t \frac{V}{\log^{1+\alpha} \mathcal{N}}$$

which we integrate over $(0, \pi)$. On account of the periodicity, replacing x by $x+\xi$, does not effect the value of the integral, and so all integrals formed from the left hand side are equal. Hence we have

$$(2.11) \qquad I_1\left(\frac{\pi}{2\mathcal{N}}\right) \leqslant \frac{A}{\mathcal{N} \log^{1+\alpha} \mathcal{N}}$$

Combining (2.10) and (2.11) we find that

$$\sum_{n=1}^{\infty} a_n^2 \sin^2\left(\frac{n\pi}{2\mathcal{N}}\right) \leqslant \frac{A}{\mathcal{N} \log^{2\alpha} \mathcal{N}} \quad (\alpha < 1)$$

Taking $\mathcal{N}=2^\nu$, we get

$$\sum_{n=2^{\nu-1}+1}^{2^\nu} a_n^2 \leqslant 2 \sum_{n=2^{\nu-1}+1}^{2^\nu} a_n^2 \sin^2 \frac{n\pi}{2^\nu+1} \leqslant A\,(2^{-\nu} \cdot \nu^{-2\alpha}$$

Therefore applying Schwartz's inequality we get

$$\sum_{n=2^{\nu-1}+1}^{2^\nu} |a_n| \log^{-1} n \leqslant \left\{ \sum_{n=2^{\nu-1}+1}^{2^\nu} a_n \right\}^{\frac{1}{2}} \left\{ \sum_{n=2^{\nu-1}+1}^{2^\nu} \log^{-2} n \right\}^2$$

$$\leqslant \frac{A}{\nu^{1+\alpha}}.$$

A similar relation holds in the case of b_n

$$\sum_{n=2^{\nu-1}+1}^{2^\nu} |b_n| \log^{-1} n \leqslant \frac{A}{\nu^{1+\alpha}}.$$

Hence

$$\sum_{n=1}^{\infty} \frac{a_n |+| b_n|}{p_{n-1}} \leqslant A \sum_{\nu=1}^{\infty} \sum_{n=2^{\nu-1}+1}^{2^\nu} \frac{|a_n|}{\log n} + A \sum_{\nu=1}^{\infty}$$

$$\sum_{n=2^{\nu-1}+1}^{2^\nu} \frac{|b_n|}{\log n} \leqslant A \sum_{\nu=1}^{\infty} \frac{1}{\nu^{1+\alpha}} = O(1).$$

Since

$$\sum_{k=n}^{\infty} \frac{\cos (n-k)\, t}{k+1} \leqslant \frac{4}{(n+1)\,|1-e^{-it}|}$$

We have

$$\sum S_3 \leqslant A \sum \frac{1}{p_{n-1}} \int_0^{c/n} |\phi(t)| \frac{1}{(n+1)\,t}\, dt = \sum \frac{1}{n (\log n)^{1+\alpha}}$$

$$= O(1)$$

Also

$$\sum S_4 = \sum \frac{A}{(n+1)\, p_n\, p_{n-1}} \left| \int_0^{c/n} \phi(t) \left(\sum_{k=o}^{n-1} p_k \cos(n-k)\, t \right) dt \right|$$

$$= O\left(\sum \frac{1}{n\,(\log n)^{1+\alpha}} \right)$$

$$= O(1).$$

Therefore

$$\sum L_{3\cdot1\cdot2} = O(1).$$

In the same way

$$\sum L_{3\cdot1\cdot1} = O(1).$$

Also it is evident that the absolute values of $L_{3\cdot2}$, $L_{3\cdot3}$, $L_{3\cdot4}$ are always less than that of $L_{3\cdot1}$. Therefore we get

$$\sum L_3 = O(1).$$

and finalising $\sum L_2 = O(1)$.

L_2 can also be treated in a similar manner.

Thus the theorem is completely proved.

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(*Dadicated to the memory of Professor Arthur Erdélyi*)

GROWTH OF THE MEAN VALUES OF ENTIRE FUNCTIONS REPRESENTED BY DIRICHLET SERIES

By

S. K. VAISH

Department of Mathematics, Sahu Jain College
Najibabad-246763, U. P., India

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1. INTRODUCTION

Let $f(s) = \sum_{n=1}^{\infty} a_n \exp. (s\lambda_n)$, where $s=\sigma+it$,

$$\lambda_{n+1} > \lambda_n, \lambda_1 \geqslant 0, \lim_{n\to\infty} \lambda_n = \infty \text{ and } \lim_{n\to\infty} \sup \frac{\log n}{\lambda_n} = 0$$

represent an entire function given by Dirichlet series. The Ritt-order ρ and lower order λ (in the sense of Ritt) of $f(s)$ are defined by [3] :

$$(1.1) \quad \lim_{\sigma\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log \log M(\sigma, f)}{\sigma} = \begin{matrix}\rho \\ \lambda\end{matrix}, \quad 0 \leqslant \lambda \leqslant \rho \leqslant \infty,$$

where $M(\sigma, f) = \sup \{ |f(\sigma+it)| : -\infty < t < \infty \}$.

Let us introduce the following mean values of $f(s)$:

$$(1.2) \quad \{ I_\delta(\sigma) \}^\delta = \{ I_\delta(\sigma, f) \}^\delta = \lim_{T\to\infty} \left\{ \frac{1}{2T} \int_{-T}^{T} |f(\sigma+it)|^\delta \, dt \right\}$$

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and

$$(1.3) \quad N_{\delta, k}(\sigma) = N_{\delta, k}(\sigma, f) = \exp.\left\{ e^{-k\sigma} \int_0^{\sigma} e^{kx} \log I_{\delta}(x, f)\, dx \right\}$$

where $o < \delta < \infty$ and $o < k < \infty$.

It is known that [2, p. 92] :

$$(1.4) \quad \lim_{\sigma\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log \log N_{\delta,k}(\sigma, f)}{\sigma} = \begin{matrix} \rho \\ \lambda \end{matrix}.$$

In this paper we have obtained the growth properties of the product of any finite number of mean values efined for entire fnnctions. Throughout this paper we shall assume that the function $f(s)$ is of finite non-zero oi der.

2. **Theorem** 1. Let $f_1(s), f_2(s)$ be two entire functions of orders ρ_1, ρ_2 and lower orders λ_1, λ_2, respectively. Then, if

$$(2.1) \log \log N_{\delta, k}(\sigma, f) \approx \log\left[\left\{ \log N_{\delta, k}(\sigma, f_1) \right\}\left\{ \log N_{\delta, k}(\sigma, f_2) \right\}\right]$$

the order ρ and lower order λ of the entire function $f(s)$ are such that

$$(2.2) \quad \lambda_1 + \lambda_2 \leqslant \lambda \leqslant \rho \leqslant \rho_1 + \rho_2,$$

and if

$$(2.3) \quad \log \log N_{\delta, k}(\sigma, f) \approx \left[\left\{\log \log N_{\delta, k}(\sigma, f_1)\right\}\left\{\log \log N_{\delta, k}(\sigma, f_2)\right\}\right]^{\frac{1}{2}},$$

then

$$(2.4) \quad (\lambda_1 \lambda_2)^{\frac{1}{2}} \leqslant \lambda \leqslant \rho \leqslant (\rho_1, \rho_2)^{\frac{1}{2}},$$

where $N_{\delta, k}(\sigma, f)$, $N_{\delta, k}(\sigma, f_1)$ and $N_{\delta, k}(\sigma, f_2)$ are the mean values of $f(s)$, $f_1(s)$ and $f_2(s)$, respectively.

Proof—Since the entire functions $f_1(s)$ and $f_2(s)$ are of orders ρ_1 and ρ_2, therefore from (1.4), we have

$$\lim_{\sigma \to \infty} \sup \frac{\log \log N_{\delta, k}(\sigma, f_1)}{\sigma} = \rho_1,$$

and

$$\lim_{\sigma \to \infty} \sup \frac{\log \log N_{\delta, k}(\sigma, f_2)}{\sigma} = \rho_2.$$

Hence, for any $\epsilon > 0$ and $\sigma > \sigma_0$,

$$(2.5) \quad \frac{\log \log N_{\delta, k}(\sigma, f_1)}{\sigma} < \rho_1 + \frac{\epsilon}{2}.$$

and

$$(2.6) \quad \frac{\log \log N_{\delta, k}(\sigma, f_2)}{\sigma} > \rho_2 + \frac{\epsilon}{2}.$$

Adding (2.5) and (2.6), we get

$$(2.7) \quad \frac{\log \left[\left\{ \log N_{\delta, k}(\sigma, f_1) \right\} \left\{ \log N_{\delta, k}(\sigma, f_2) \right\} \right]}{\sigma} < \rho_i + \rho_2 + \epsilon$$

Proceeding as above for the limit inferior, we find

$$(2.8) \quad \frac{\log \left[\left\{ \log N_{\delta, k}(\sigma, f_1) \right\} \left\{ \log N_{\delta, k}(\sigma, f_2) \right\} \right]}{\sigma} > \lambda_1 + \lambda_2 - \epsilon.$$

Now, if (2.1) holds, then from (2.7) and (2.8), we have, for any $\epsilon < o$ and sufficiently large σ,

$$\lambda_1 + \lambda_2 - \epsilon < \frac{\log \log N_{\delta, k}(\sigma, f)}{\sigma} < \rho_1 + \rho_2 + \epsilon.$$

Taking limits and using (1.4), it leads to (2.2).

Again, multiplying (2.5) and (2.6), we get

$$(2.9) \quad \frac{\left\{ \log \log N_{\delta, k}(\sigma, f_1) \right\} \left\{ \log \log N_{\delta, k}(\sigma, f_2) \right\}}{\sigma^2} < \left(\rho_1 + \frac{\epsilon}{2}\right)\left(\rho_2 + \frac{\epsilon}{2}\right),$$

for any $\epsilon > 0$ and sufficiently large σ.

Similarly, we have

$$(2.10)\quad \frac{\left\{\log\log \mathcal{N}_{\delta,\,k}(\sigma, f_1)\right\}\left\{\log\log \mathcal{N}_{\delta,\,k}(\sigma, f_2)\right\}}{\sigma^2} > \left(\lambda_1 - \frac{\epsilon}{2}\right)\left(\lambda_2 - \frac{\epsilon}{2}\right),$$

for any $\epsilon > 0$ and sufficiently large σ.

Further, if (2.3) holds, then from (2.9) and (2.10), on taking limits, (2.4) follows.

Corollary I. If $f_\xi(s)$, $(\xi = 1, 2, \ldots, n)$ are n entire functions of orders $\rho_1, \rho_2, \ldots \rho_n$ and lower orders $\lambda_1, \lambda_2, \ldots, \lambda_n$ and having the mean values $\mathcal{N}_{\delta,\,k}(\sigma, f_1), \mathcal{N}_{\delta,\,k}(\sigma, f_2), \ldots, \mathcal{N}_{\delta,\,k}(\sigma, f_n)$, respectively. Then, if

$$\log\log \mathcal{N}_{\delta,\,k}(\sigma, f) \approx \log\Big[\left\{\log \mathcal{N}_{\delta,\,k}(\sigma, f_1)\right\}\left\{\log \mathcal{N}_{\delta,\,k}(\sigma, f_2)\right\}\cdots$$
$$\cdots\left\{\log \mathcal{N}_{\delta,\,k}(\sigma, f_n)\right\}\Big],$$

the order ρ and lower order λ of $f(s)$ having the mean value $\mathcal{N}_{\delta,\,k}(\sigma, f)$ are such that

$$\lambda_1 + \lambda_2 + \ldots + \lambda_n \leqslant \lambda \leqslant \rho \leqslant \rho_1 + \rho_2 + \ldots \rho_n,$$

and if

$$\log\log \mathcal{N}_{\delta,\,k}(\sigma, f) \approx \Big[\left\{\log\log \mathcal{N}_{\delta,\,k}(\sigma, f_1)\right\}\left\{\log\log \mathcal{N}_{\delta,\,k}(\sigma, f_2)\right\}\cdots$$
$$\cdots\left\{\log\log \mathcal{N}_{\delta,\,k}(\sigma, f_n)\right\}\Big]^{1/n},$$

then

$$(\lambda_1 \lambda_2 \ldots \lambda_n)^{1/n} \leqslant \lambda \leqslant \rho \leqslant (\rho_1 \rho_2 \ldots \rho_n)^{1/n}.$$

Corollary 2. If $f_\xi(s)$, $(\xi = 1, 2, \ldots, n)$ are n entire functions of regular growth, of orders $\rho_1, \rho_2, \ldots, \rho_n$, respectively, then so is of $f(s)$, of order ρ, and

$$\rho = \rho_1 + \rho_2 + \ldots + \rho_n.$$

3. Let $\mu(\sigma, f) = \max_{n \geqslant 1} \{|a_n| \exp(\sigma \lambda_n)\}$ be the maximum term of $f(s)$ and $\nu(\sigma, f) = \max\{n : \mu(\sigma, f) = |a_n| \exp(\sigma \lambda_n)\}$ be its rank, then it is known that [1, p. 20] :

(3.1) $\log I_\delta(\sigma, f) \approx \log \mu(\sigma, f)$.

Theorem 2. Let $f(s)$ be an entire function of order ρ and lower order λ, $0 < \lambda < \rho < \infty$, then

$$(3.2)\quad \limsup_{\sigma \to \infty} \frac{\log N_{\delta, k}(\sigma, f)}{\sigma \lambda_{\nu(\sigma, f)}} \leqslant 1/k\,(1 - \lambda/\rho)$$

and

$$(3.3)\quad \limsup_{\sigma \to \infty} \frac{\log N_{\delta, k}(\sigma, f)}{\lambda_{\nu(\sigma, f)} \log \lambda_{\nu(\sigma, f)}} \leqslant 1/k\,(1/\lambda - 1/\rho).$$

Proof. From (1.3), we have

$$\log N_{\delta, k}(\sigma, f) = e^{-k\sigma} \int_0^\sigma e^{kx} \log I_\delta(x, f)\, dx \leqslant 1/k \log I_\delta(\sigma, f).$$

Hence,

$$(3.4)\quad \limsup_{\sigma \to \infty} \frac{\log N_{\delta, k}(\sigma, f)}{\sigma \lambda_{\nu(\sigma, f)}} \leqslant 1/k \limsup_{\sigma \to \infty} \frac{\log I_\delta(\sigma, f)}{\sigma \lambda_{\nu(\sigma, f)}}$$

$$= 1/k \limsup_{\sigma \to \infty} \frac{\log \mu(\sigma, f)}{\sigma \lambda_{\nu(\sigma, f)}}.$$

in view of (3.1).

We know [4, p. 84] that for $0 < \rho < \infty$,

$$(3.5)\quad \limsup_{\sigma \to \infty} \frac{\log \mu(\sigma, f)}{\sigma \lambda_{\nu(\sigma, f)}} \leqslant 1 - \lambda/\rho,$$

and

$$(3.6)\quad \limsup_{\sigma \to \infty} \frac{\log \mu(\sigma, f)}{\lambda_{\nu(\sigma, f)} \log \lambda_{\nu(\sigma, f)}} \leqslant 1/\lambda - 1/\rho.$$

Thus (3.2) follows (3.4) and (3.5).

Similarly, we can easily derive (3.3), if we use (3.6) instead of (3.5).

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ON THE CONVERGENCE AND RIESZ-SUMMABILITY OF A FOUIER-BESSEL SERIES OF SPECIAL KIND

By

S. R. Agrawal and C. M. Patel

Department of Mathematics, Faculty of Science
*The M. S. University of Baroda, Baroda-***390002,** *India*

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SUMMARY

In this paper the authors extend a convergence theorem of F. Kito [4] regarding the Fourier-Bessel series (1.3) below. The Riesz-summability of the series (1.3) corresponding to Lebesgue-integrable functions is also proved.

1. INTRODUCTION

Let

$$(1.1)\qquad Q_\nu(\alpha, \beta) = J_\nu(\alpha)\, Y'_\nu(\beta) - J'_\nu(\beta)\, Y_\nu(\alpha),$$

where $J_\nu(t)$ and $Y_\nu(t)$ denote Bessel functions of the first and second kinds respectively for $\nu \geqslant -1/2$ and $k_1, k_2, k_3, \ldots$ etc. the positive zeros of $S(z)$ in increasing order of magnitude, where

$$(1.2)\qquad S(z) = J'_\nu(bz)\, Y'_\nu(az) - J'_\nu(az)\, Y_\nu(a), \qquad 0 < a < b.$$

The series

$$(1.3) \qquad f(x) \sim \sum_{m=1}^{\infty} p_m \, Q_\nu (xk_m, ak_m) \qquad a \leqslant x \leqslant b$$

where

$$(1.3) \qquad p_m = \left[\int_a^b t\, f(t)\, Q_\nu (tk_m, ak_m)\, dt \right] / M(k_m)$$

and

$$(1.5) \qquad M(k_m) = \int_a^b t \; Q_\nu^2 (tk_m, ak_m)\, dt$$

$$= \frac{2}{\pi^2 k_m^2} \left[\left(1 - \frac{\nu^2}{b^2 k_m^2} \right) \left(\frac{J'_\nu (ak_m)}{J'_\nu (bk_m)} \right)^2 - \left(1 - \frac{\nu^2}{a^2 k_m^2} \right) \right].$$

corresponding to any function $f \in L[a, b]$, was first used by F. Kito [3] while studying the vibrations of a cylindrical shell immersed in water.[1]

The n-th partial sum of the series (1.3) is given by

$$(1.6) \qquad S_n(x) = \sum_{m=1}^{n} p_m Q_\nu (xk_m, ak_m) = \int_a^b t\, f(t)\, U_n(t, x)\, dt,$$

where

$$(1.7) \qquad U_n(t, x) = \sum_{m=1}^{n} \frac{Q_\nu (xk_m, ak_m)\; Q_\nu (tk_m, ak_m)}{M(k_m)}.$$

The series (1.3) is called Riesz-summable or summable (R) to a sum s, if for $k_n < L_n < k_{n+1}$,

$$\lim_{n \to \infty} \sum_{m=1}^{n} (1 - k_m/L_n)\, p_m \; Q_\nu (xk_m, ak_m) = s.$$

[1] In [4], Dr. Kito has used $G_\nu(z)$ instead of $Y_\nu(z)$, where

$$G_\nu(z) = -(\pi/2)\, Y_\nu(z).$$

The sum in the above limit, called the Riesz-sum corresponding to (1.3), is denoted by $R_n^{(\nu)}\ (x,f)$. By (1.4)

$$(1.8)\qquad R_n^{(\nu)}\ (x,f) = \int_a^b f(t)\, \emptyset_n\,(t,\, x/R)\, dt,$$

where

$$(1.9)\qquad \emptyset_n\,(t,\, x/R) = \sum_{m=1}^{n} \left(1-\frac{k_m}{L_n}\right) \frac{Q_\nu\,(xk_m,\, ak_m)\; Q_\nu\,(tk_m,\, ak_m)}{M\,(k_m)}.$$

The authors of the present paper have proved, ([1], [2]), certain properties of (1.3) regarding the order of its coefficients and convergence corresponding to functions of bounded veriation, square integrable and differentiable functions.

Kito [4] proved for a function $f \in L\,[a,b]$, which is also of bounded variation in $[a,b]$, that its Fourier-Bessel series (1.3), when ν is a positive integer, converges to the sum $\dfrac{f(x+o)+f(x-o)}{2}$.

In this paper we extend Kito's theorem to a general case in which $\nu \geqslant -1/2$, $\nu \neq 0$. We also establish Riesz-summability of (1.3).

Our theorems are as follows :

Theorem I. *If* $f \in L\,[a,b]$ *and has bounded variation in a neighbourhood* $x \in (a,b)$, *then the series (1.3) converges to the sum* $\frac{1}{2}\,\{f(x+0)+f(x-0)\}$.

Theorem 2. *If* $f \in L\,[a,b]$, $\nu \geqslant -1/2$, $\nu \neq 0$ *and if for* $x \in (a,b)$, *the limits* $f\,(x\pm 0)$ *exist, then the series* (1.3) *is summable* (R) *to the sum* $\frac{1}{2}\,\{f(x+0)+f(x-0)\}$. [1]

2. PREREQUISITES

The following lemmas, estimating $U_n\,(t,x)$, $\emptyset_n\,(t,\, x/R)$ and certain integrals concerning them, are needed to prove the above theorems (K_1, K_2, K_3,...etc. denote suitable positive constants throughout the paper) :

---

[1] This theorem is analoguous to Fejer's theorem

Lemma I. *Let* $F(w) = \pi w \dfrac{Q_\nu(tw, aw)\, Q_\nu(xw, bw)}{S(w)}$, $a < t < x < b$.

Then on the rectangle Γ, *whose vertices are* $\pm Li$, $L_n \pm Li$ *in the w-plane, where* $k_n < L_n < k_{n+1}$, *and* L *is to be made to tend to infinity,*

$$F(w) = O\left(\frac{e^{-|v|(x-t)}}{\sqrt{xt}}\right),$$

where $w = u + iv$ *and* n *is sufficiently large.*

Proof. We know that [6 ; pp. 228-229], $\varphi(t) = \sqrt{t}\, Q_\nu(tw, aw)$ is a solution of the differential equation

$$\frac{d^2\varphi}{dt^2} + \left(w^2 - \frac{\nu^2 - 1/4}{t^2}\right)\varphi = 0.$$

Hence [5 ; p. 10, equation (1.7.3)],

(2.1) $$Q_\nu(tw, aw) = \frac{2\cos(t-a)w}{\pi w\sqrt{at}} + O\left(\frac{e^{|v|(t-a)}}{|w|}\right),$$

where $a \leqslant t \leqslant b$ and $|w| \to \infty$.

Also [5 ; p.10, equation (1.7.6)],

(2.2) $$\varphi'(t) = -\frac{2\sin(t-a)w}{\pi\sqrt{a}} + O\left(e^{|v|(t-a)}\right),$$

where $a \leqslant t \leqslant b$ and $|w| \to \infty$. By (2.1) and (2.2),

(2.3) $$S(w) = -\frac{2\sin(b-a)w}{\pi w\sqrt{ab}} + O\left(\frac{e^{|v|(b-a)}}{|w|}\right),$$

when $|w| \to \infty$. Also [5 ; p. 13],

(2.4) $$|\sin(b-a)w| > K_1 e^{|v|(b-a)},$$

on the rectangle Γ and for arbitrary real $\alpha > 0$ and $w = u + iv$, it is true that

(2.5) $$|\cos \alpha w| < \tfrac{1}{2} e^{\alpha|v|},$$

The lemma, now, follows from (2 1), (2.3) to (2.5).

Lemma 2. *The following inequalities hold true for* a⩽x⩽b, a⩽t⩽b :

(2 6) $$| U_n (t, x)| \leqslant \frac{K_2}{| t-x |}, \; t \neq x ;$$

(2.7) $$\left| \int_a^t t^{\nu+1} U_n (t, x) \, dt \right| \leqslant \frac{K_3}{L_n} \left\{ \frac{1}{x-t} + \frac{1}{x-a} \right\}, \; x > t;$$

and

(2.8) $$\left| \int_t^b t^{\nu+1} U_n (t, x) \, dt \right| \leqslant \frac{K_4}{L_n} \left\{ \frac{1}{b-x} + \frac{1}{t-x} \right\}, \; t > x.$$

Proof. Using Watson, [7; p. 76, equation (6) and setting

$$\eta = \frac{J'_\nu (ak_m)}{J'_\nu (bk_m)} = \frac{Y'_\nu (ak_m)}{Y'_\nu (bk_m)}, \text{ we get}$$

$$S' (k_m) = b \left\{ J''_\nu (bk_m) \; Y'_\nu (ak_m) - J'_\nu (ak_m) \; Y''_\nu (bk_m) \right\} +$$

$$+ a \left\{ J'_\nu (bk_m) \; Y''_\nu (ak_m) - J''_\nu (ak_m) \; Y'_\nu (bk_m) \right\}$$

$$= -\frac{\pi k_m}{\eta} M (k_m).$$

Now F is a meromorphic function having poles at k_1, k_2, ...etc. The residue of $F (w)$ at k_m is given by

$$\frac{\pi k_m Q_\nu (tk_m, ak_m) \; Q_\nu (xk_m, bk_m)}{S' (k_m)} = -\frac{Q_\nu (tk_m, ak_m) \; Q_\nu (xk_m, ak_m)}{M (k_m)} .$$

Considering Γ to be the contour of integration, by (1.7),

(2.9) $$U_n (t, x) = \lim_{L \to \infty} \frac{1}{2 \pi i} \int_{L_n - Li}^{L_n + Li} F (w) \, dw,$$

Since by Lemma 1,

$$\lim_{L\to\infty}\int_{-Li}^{Li} F(w)\,dw = \lim_{L\to\infty}\int_{\pm Li}^{L_n\pm Li} F(w)\,dw = 0.$$

Hence by Lemma 1 and (2.9),

$$|U_n(t,x)| \leqslant \frac{2K_5}{\sqrt{xt}}\int_0^\infty e^{-v(x-t)}\,dt = \frac{K_2}{x-t},\quad x>t.$$

Similar is the case if $t > x$. This proves (2.6).

Now, by (2.9), for $x > t$,

$$\int_a^t t^{\nu+1} U_n(t,x)\,dt = \int_{L_n-\infty i}^{L_n+\infty i} \frac{w\,Q_\nu(xw,bw)}{S(w)}$$

$$\times\Big[\frac{t^{\nu+1}}{w}\Big\{J_{\nu+1}(tw)\,Y'_\nu(aw) - J'_\nu(aw)\,Y_{\nu+1}(tw)\Big\} -$$

$$-\frac{a^{\nu+1}}{w}\Big\{J_{\nu+1}(aw)\,Y'_\nu(aw) - J'_\nu(aw)\,Y_{\nu+1}(aw)\Big\}\Big]\,dw.$$

Therefore, by recurrence relations, [7, pp. 45, 66], and by (2.1) to (2.5), we obtain

$$\left|\int_a^t t^{\nu+1}\,U(t_n,x)\,dt\right| \leqslant \frac{t^{\nu+1}}{\sqrt{tx}}K_7\int_0^\infty \frac{e^{-(x-t)v}}{\sqrt{L_n^2+v^2}}\,dv +$$

$$+\frac{a^{\nu+1}}{\sqrt{at}}K_7\int_0^\infty \frac{e^{-(x-a)v}}{\sqrt{L_n^2+v^2}}\,dv$$

$$\leqslant \frac{K_3}{L_n}\left\{\frac{1}{x-t}+\frac{1}{x-a}\right\}.$$

The proof of (2.8) is similar.

Lemma 3. *For any real ν, ν not a negative integer or zero,*

$$\lim_{n\to\infty}\int_a^b t^{\nu+1}\,U_n(t,x)\,dt = x^\nu,\quad 0<a<x<b.$$

Proof. By (1.7) and recurrence relations,

$$\int_a^b t^{\nu+1} U_n(t,x)\,dt = \sum_{m=1}^{n} \frac{2\nu\left\{ b^{\nu-1} J'_\nu(ak_m) - a^{\nu-1} J'_\nu(bk_m)\right\} Q_\nu(xk_m, ak_m)}{k_m^3 M(k_m) J'_\nu(bk_m)}$$

The function

$$G(w) = \frac{2\nu\left\{ a^{\nu-1} Q_\nu(xw, bw) - b^{\nu-1} Q_\nu(xw, aw)\right\}}{w^2 S(w)},$$

has poles at zero and at k_m, $m=1,2,\ldots$.

Residue of $G(w)$ at $w=k_m$ is given by

$$\frac{2\nu\left\{ b^{\nu-1} J'_\nu(ak_m) - a^{\nu-1} J'_\nu(bk_m)\right\} Q_\nu(xk_m, ak_m)}{k_m^3 M(k_m) J'_\nu(bk_m)}$$

whereas at zero, the residue is $-2x^\nu$, since, for $\nu > 0$,

$$\lim_{w\to 0} \frac{J_\nu(aw)}{J_\nu(bw)} = \frac{a^\nu}{b^\nu} \text{ and } \lim_{w\to 0} \frac{Y_\nu(aw)}{Y_\nu(bw)} = \frac{b^\nu}{a^\nu},$$

when ν is not a negative integer or zero.

As in the proof of Lemma 2,

$$\int_a^b t^{\nu+1} U_n(t,x)\,dt = \frac{1}{2\pi i} \int_{L_n-\infty i}^{L_n+\infty i} G(w)\,dw) + x^\nu .$$

Also

$$\left| \frac{1}{2\pi i} \int_{L_n-\infty i}^{L_n+\infty i} G(w)\,dw \right| \leqslant \frac{\nu}{\pi} b^{\nu-1} K_8 \int_0^\infty \frac{e^{-v(x-a)} + e^{-v(b-x)}}{L_n^2 + v^2} dv$$

$$\leqslant \frac{K_9}{L_n^2} \left\{ \frac{1}{x-a} + \frac{1}{b-x} \right\}.$$

Since [1 ; Lemma 4],

(2.10) $$L_n \sim n, \text{ as } n \to \infty,$$

the lemma follows.

Lemma 4. *If* $|\lambda| < b-a$, then

$$\lim_{L\to\infty} \int_{L_n - Li}^{L_n + Li} \frac{\sin \lambda w \, dw}{w \sin (b-a) w} = O(1/n), \text{ as } n \to \infty.$$

Proof. Let $L_n = \dfrac{(2n+1)\pi}{2(b-a)}$. Then

$$\lim_{L\to\infty} \left| \int_{L_n - Li}^{L_n + Li} \frac{\sin \lambda w \, dw}{w \sin (b-a) w} \right| = \left| \int_{-\infty}^{\infty} \frac{\cos \{\lambda (L_n + iv) - L_n (b-a)\}}{(L_n + iv) \cos \{iv (b-a)\}} i \, dv \right|$$

$$\leq \frac{4}{L_n^2} \int_0^{\infty} \frac{v \sin h \lambda v}{\cos h \, v (b-a)} dv + \frac{4}{L_n} \int_0^{\infty} \frac{\cos h \lambda v \, dv}{\cosh v (b-a)}$$

$= O(1/n)$, as $n \to \infty$.

Lemma 5. *For any real* v, v *not a negative integer or zero and* $a<x<b$,

(2.11) $$\lim_{n\to\infty} \int_a^x t^{\nu+1} U_n(t, x)\, dt = \tfrac{1}{2} x^{\nu},$$

and

(2.12) $$\lim_{n\to\infty} \int_x^b t^{\nu+1} U_n(t, x)\, dt = \tfrac{1}{2} x^{\nu}.$$

Proof. By (2.9),

$$\int_a^x t^{\nu+1} U_n(t, x)\, dt = \frac{1}{2i} \int_{L_n - \infty i}^{L_n + \infty i} \frac{Q_\nu (xw, bw)}{S(w)} \Big[x^{\nu+1} .$$

$$\left\{ J_{\nu+1}(xw) Y_\nu'(aw) - J_\nu'(aw) Y_{\nu+1}(xw) \right\} -$$

$$- a^{\nu+1} \left\{ J_{\nu+1}(aw) Y_\nu'(aw) - J_\nu'(aw) Y_{\nu+1}(aw) \right\} \Big] dw$$

(2.13) $$= I_1 + I_2, \text{ say.}$$

As in Lemma 2, $|I_2| \to 0$, as $n \to \infty$, for $\sigma < \alpha$.

Similarly, using (2.1), (2.3), the recurrence relations and the asymptotic expressions [7; p. 199], we obtain,

$$I_1 = \frac{x^{\nu}}{2\pi i} \int_{L_n-\infty i}^{L_n+\infty i} \left\{ \frac{2\cos(b-x)w \sin(x-a)w}{w \sin(b-a)w} + O\left(\frac{1}{|w|^2}\right) \right\} dw$$

$$= \frac{x^{\nu}}{2\pi i} \int_{L_n-\infty i}^{L_n+\infty i} \left\{ \frac{1}{w} - \frac{\sin(b+a-2x)w}{w \sin(b-a)\, w} \right. dw + O(1/L_n)$$

(2.14) $= \frac{1}{2} x^{\nu} + O(1/n)$, as $n \to \infty$, by (2.10) and Lemma 4.

The lemma follows from (2.13), (2.14) and Lemma 3.

Lemma 6. *For* $a<t\leq b$, $a<x\leq b$, *we have,*

$$\int_a^t t^{\nu+1} U_n(t, x)\, dt = O(1/n), \text{ as } n\to\infty.$$

Proof. From the proofs of Lemmas 2 and 5,

(2.15) $\int_a^t t^{\nu+1} U_n(t, x)\, dt =$

$$\lim_{L\to\infty} \frac{1}{2i}\left[\int_{L_n-Li}^{L_n+Li} + \int_{-Li}^{L_n-Li} + \int_{Li}^{L_n+Li}\right] H(w)\, dw,$$

where

$$H(w) = \frac{2\, t^{\nu+1/2}}{\pi \sqrt{x}} \left\{ \frac{\sin(b+t-a-x)w}{w \sin(b-a)\, w} - \frac{\sin(b+a-t-x)\, w}{w \sin(b-a)\, w} \right\}.$$

By (2.4), the second and third limits in (2.15) are each zero. Hence the Lemma follows.

By inequalities of Lemma 2, the following Lemma is true [7; pp. 589-591],

Lemma 7. *If* $f \in L\,[a, b]$, $a\leq A<B\leq b$, *and if* $x \notin (A, B)$, *then*

$$\lim_{n\to\infty} \int_A^B t f(t)\, U_n(t, x)\, dt = 0.$$

The following lemma can be proved on the lines of Lemma 2 :

Lemma 8. *The following inequalities hold true for* $a<x<b$, $a\leq t\leq b$:

$$|\phi_n(t, x/R)| < \frac{K_{11}}{L_n(t-x)^2},\ \textit{if}\ x \neq t,$$

$$|\phi_n(t, x/R)| < K_{12}\, L_n,\quad \textit{for all } x \textit{ and } t.$$

Lemma 9. *For any real* ν, ν *not a negative integer or zero and* $a<x<b$,

$$\lim_{n\to\infty} \int_a^b t^{\nu+1}\, \phi_n(t, x/R)\, dt = x^\nu\ ;$$

$$\lim_{n\to\infty} \int_a^x t^{\nu+1}\, \phi_n(t, x/R)\, dt = \frac{x^\nu}{2}\ ;$$

$$\lim_{n\to\infty} \int_a^b t^{\nu+1}\, \phi_n(t, x/R)\, dt = \frac{x^\nu}{2}$$

This lemma follows from Lemmas 4 and 5 and the fact that convergence implies Riesz-summability.

3. Proof of Theorem 1

By (1.6) and Lemma 5,

$$\lim_{n\to\infty} \left\{ S_n(x) - \tfrac{1}{2}\{ f(x+0) + f(x-0)\} \right\} =$$

$$= \lim_{n\to\infty} \int_a^x t^{\nu+1} \left\{ t^{-\nu} f(t) - x^{-\nu} f(x-0) \right\} U_n(t, x)\, dt \ +$$

$$+ \lim_{n\to\infty} \int_x^b t^{\nu+1} \left\{ t^{-\nu} f(t) - x^{-\nu} f(x+0) \right\} U_n(t, x)\, dt$$

$$= \lim_{n\to\infty} (A+B), \text{ say.}$$

By hypothesis, $t^{-\nu} f(t) - x^{-\nu} f(x+0)$ is of bounded variation in a neighbourhood, say $(x-\eta, x+\eta)$, of x. Hence there are bounded increasing positive functions g_1 and g_2 of $t \in (x, v+\eta)$ such that

$t^{-\nu} f(t) - x^{-\nu} f(x+0) = g_1(t) - g_2(t)$, $t \in (x, x+\eta)$, and $g_1(x+0) = g_2(x+0) = 0$.

Therefore, for any $\varepsilon > 0$, there is $\delta > 0$, $\delta \leqslant \eta$, such that $0 \leqslant g_1(t) < \varepsilon$, $0 \leqslant g_2(t) < \epsilon$, *for* $x \leqslant t \leqslant x+\delta$.

Now,

$$B= \int_x^{x+\delta} t^{\nu+1} g_1(t)\, U_n(t,x)\, dt - \int_x^{x+\delta} t^{\nu+1} g_2(t)\, U_n(t,x)\, dt +$$

$$+ \int_{x+\delta}^{b} t^{\nu+1} \left\{ t^{-\nu} f(t) - x^{-\nu} f(x+0) \right\} U_n(t,x)\, dt$$

$= B_1 + B_2 + B_3$, say.

By the second mean value theorem and Lemma 6, B_1 and B_2 are both equal to O $(1/n)$, as $n \to \infty$, and by Lemma 7, $B_3 = $ o (1), as $n \to \infty$.

Hence, $B = $ o (1), as $n \to \infty$.

In a similar way, $A = o\,(1)$, as $n \to \infty$, which proves the theorem.

4. Proof of Theorem 2

As in the proof of Theorem 1, by (1.8) and Lemmas 5 and 9,

$$\lim_{n\to\infty} \left[R_n^{(\nu)}(x,f) - \tfrac{1}{2}\{ f(x+0) + f(x-0) \} \right] =$$

$$\lim_{n\to\infty} \int_a^x t^{\nu+1} \left\{ t^{-\nu} f(t) - x^{-\nu} f(x-0) \right\} \emptyset_n(t, x/R)\, dt +$$

$$\lim_{n\to\infty} \int_x^b t^{\nu+1} \left\{ t^{-\nu} f(t) - x^{-\nu} f(x+0) \right\} \emptyset_n(t, x/R)\, dt$$

$$= \lim_{n\to\infty} (I + I'), \text{ say.}$$

Given $\epsilon > 0$, let us choose $\delta > 0$ such that

$$\left| t^{-\nu} f(t) - x^{-\nu} f(x-0) \right| < \epsilon, \tag{4.1}$$

for $x - \delta < t < x$. If n is so large that $\delta > 1/L_n$, then

$$I = \int_a^{x-\delta} + \int_{x-\delta}^{x-1/L_n} + \int_{x-1/L_n}^{x} = I_1 + I_2 + I_3, \text{ say.}$$

By Lemma 8,

$$|I_1| \leqslant \frac{K_{11}}{L_n \delta^2} \int_a^{x-\delta} \left| t f(t) - t^{\nu+1} x^{-\nu} f(x+0) \right| dt = o\,(1), \text{ as } n \to \infty.$$

Also by (4.1),

$|I_2| \leqslant K_{11}\,\epsilon$, and $|I_3| \leqslant K_{13}\,\epsilon$.

Thus, $I = o(1)$, as $n \to \infty$. Similarly, I' is established and the theorem follows.

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(*Dedicated to the memory of Professor Arthur Erdélyi*)

A GENERAL CLASS OF UNIVARIATE DISTRIBUTIONS INVOLVING THE H-FUNCTOIN OF SEVERALS VARIABLES

By

R. K. AGRAWAL

Department of Mathematics, B. V. College of Arts and Science, Banasthali Vidyapith-**304022**, *Rajasthan, India*

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ABSTRACT

In this paper, an attempt has been made to present a unified theory of the classical statistical distributions associated with the generalized beta and gamma distributions of one variate. The probability density function is taken in terms of the H-function of several variables with general arguments. In particular, the characteristic function and the distribution function are investigated.

1. Introduction

In probability theory, a large number of statistical distributions have been studied from time to time by several authors. For example, Mathai and Saxena [5] introduced a general hypergeometric distribution, whose probability density functions involves a hypergeometric function ${}_2F_1$. Again, Srivastava and Singhal [7] studied another general class of distributions, whose probability density function involves the H-function of C. Fox ([4], p. 408]. It may be readily seen that the distributions, considered by Mathai and Saxena [5] and all other well-known classical statistical distributions, such as the generalized beta and gamma distributions, the exponential distribution, the generalized F-distribution, students t-distribution, the normal distribution, etc., can be derived as specialized or confluent cases of the class of distributions, considered by Srivastava and Singhal [7]. More recently, Exton [3] considered the family of distributions which have the probability density function in terms of the product of several generalized hypergeometric functions ${}_pF_q$.

In an attempt to present a further generalization of the probability distributions studied by Srivastava and Singhal [7], Exton [3], etc., here we introduce and study a general family of statistical probability distributions involving the H-function of several variables, which was defined and studied elsewhere by Srivastava and Panda (cf., e. g , [8], [9], [10] and [11]). Since the H-function of several variables includes almost all the special functions of one or more variables as its particulars cases, it can define a very general class of probability model. Thus all the classical statistical distributions, mentioned here and elsewhere, will be the special cases of our findings. The parameters of the H-function of several variables are to be resricted in such a way that the function is non-negative and finite in the region under consideration.

Following the notations explained fairly fully in the earlier papers by Srivastava and Panda [9] and [10], the H-function of n complex variables is defined in the manner given below :

$$H[z_1,\ldots,z_n] = H^{0,\lambda:(\mu',\nu');\ldots;(\mu^{(n)},\nu^{(n)})}_{A,C:[B',D'];\ldots;[B^{(n)},D^{(n)}]}\left[\begin{matrix}[(a):\theta',\ldots,\theta^{(n)}]:\\ [(c):\psi',\ldots,\psi^{(n)}];\end{matrix}\begin{matrix}[(b'):\emptyset'];\ldots;[(b^{(n)}):\emptyset'^{(n)}];\\ [(d'):\delta'];\ldots;[(d^{(n)}):\delta^{(n)}];\end{matrix}\; z_1,\ldots,z_n\right]$$

$$= \frac{1}{(2\pi\omega)^n}\int_{L_1}\cdots\int_{L_n}\Phi_1(\xi_1)\cdots\Phi_n(\xi_n)\,\psi(\xi_1,\ldots,\xi_n)\,z_1^{\xi_1}\cdots z_n^{\xi_n}\,d\xi_1\ldots d\xi_n,$$

$$\omega = \sqrt{-1} \tag{1.1}$$

here

$$\Phi_i(\xi_i) = \frac{\prod_{j=1}^{\mu^{(i)}}\Gamma[d_j^{(i)}-\delta_j^{(i)}\xi_i]\;\prod_{j=1}^{\nu^{(i)}}\Gamma[1-b_j^{(i)}+\emptyset_j^{(i)}\xi_i]}{\prod_{j=\mu^{(i)}+1}^{D^{(i)}}\Gamma[1-d_j^{(i)}+\delta_j^{(i)}\xi_i]\;\prod_{j=\nu^{(i)}+1}^{B^{(i)}}\Gamma[b_j^{(i)}-\emptyset_j^{(i)}\xi_i]}$$

$(i=1,\ldots,n)$;

$$\psi(\xi_1,\dots,\xi_n)=\frac{\prod_{j=1}^{\lambda}\Gamma[1-a_j+\sum_{i=1}^{n}\theta_j^{(i)}\xi_i]}{\prod_{j=\lambda+1}^{A}\Gamma[1-a_j+\sum_{i=1}^{n}\theta_j^{(i)}\xi_i]\prod_{j=1}^{C}\Gamma[1-c_j+\sum_{i=1}^{n}\psi_j^{(i)}\xi_i]} \qquad \dots(1\cdot3)$$

Also, let the associated positive numbers

$$\left.\begin{array}{l}\theta_j^{(i)}, j=1,\dots,A\,;\ \phi_j^{(i)}, j=1,\dots,B^{(i)}\,;\\ \psi_j^{(i)}, j=1,\dots,C\,;\ \delta_j^{(i)}, j=1,\dots,D^{(i)}\,;\ i=1,\dots,n\end{array}\right\} \qquad \dots(1.4)$$

be constrained by the inequalities :

$$\Lambda_i \equiv -\sum_{j=\lambda\ \ 1}^{A}\theta_j^{(i)}+\sum_{j=1}^{\nu^{(i)}}\phi_j^{(i)}-\sum_{j=\nu^{(i)}+1}^{B^{(i)}}\phi_j^{(i)}-\sum_{j=1}^{C}\psi_j^{(i)}$$

$$+\sum_{j=1}^{\mu^{(i)}}\delta_j^{(i)}-\sum_{j=\mu^{(i)}+1}^{D^{(i)}}\delta_j^{(i)}>0, \qquad \dots(1.5)$$

$$\Omega_i \equiv \sum_{j=1}^{A}\theta_j^{(i)}+\sum_{j=1}^{B^{(i)}}\phi_j^{(i)}-\sum_{j=1}^{C}\psi_j^{(i)}-\sum_{j=1}^{D^{(i)}}\delta_j^{(i)}<0, \qquad \dots(1.6)$$

Then it is known that the multiple Mellin-Barnes contour integral defining the H-function of several variables (1.1) would converge absolutely when

$$|\arg(z_i)|<\tfrac{1}{2}\Lambda_i\,\pi,\quad i=1,\dots,n, \qquad \dots(1.7)$$

Furthermore, from the known asymptotic expansion ([9], p. 131, Eq. (1.9)), we have

$$H[z_1,\dots,z_n]=\begin{cases}O(|z_1|^{\alpha}1\dots|z_n|^{\alpha}),\ \max\{|z_1|,\dots,|z_n|\}\to 0,\\ O(|z_1|^{\beta_1}\dots|z_n|^{\beta_n}),\ \lambda\equiv 0,\ \min\{|z_1|,\dots,|z_n|\}\to\infty\end{cases}$$

where, with $i=1,\ldots, n$,

$$\begin{cases} \alpha_i = \min \{ \operatorname{Re} (d_j^{(i)}) / \delta_j^{(i)} \}, & j = 1, \ldots, \mu^{(i)}, \\ \beta_i = \max \{ \operatorname{Re} (b_j^{(i)} - 1) / \emptyset_j^{(i)} \}, & j = 1, \ldots, \nu^{(i)}, \end{cases} \quad \ldots (1.9)$$

In terms of Fox's H-function, we find it worthwhile to record here the interesting relationship :

$$\lim_{z_2, \ldots, z_n \to 0} H^{0,\, \lambda : (\mu', \nu') ; (1, B'') ; \ldots ; (1, B^{(n)})}_{A,\, C : [B', D'] ; [B'', D''+1] ; \ldots [B^{(n)}, D^{(n)}+1]}$$

$$\left[\begin{matrix} [(a) : \theta', 1, \ldots, 1] : [(b') : \emptyset'] ; [(b'') : 1] ; \ldots ; [(b^{(n)}) : 1] ; \\ [(c) : \psi', 1, \ldots, 1] : [(d') : \delta'] ; (0,1), [(d'') : 1] ; \ldots ; (0, 1), [(d^{(n)}) : 1] ; \end{matrix} z_1, \ldots, z_n \right]$$

$$= \prod_{i=2}^{n} \left[\prod_{j=1}^{B^{(i)}} \Gamma (1 - b_j^{(i)}) \left\{ \prod_{j=1}^{D^{(i)}} \Gamma (1 - d_j^{(i)}) \right\}^{-1} \right]$$

$$H^{\mu', \nu' + \lambda}_{B'+A,\, D'+C} \left[z_1 \left| \begin{matrix} (a_j, \theta_j')_{1,\, \lambda}, (b_j', \emptyset_j')_{1,\, B'}, (a_j, \theta_j')_{\lambda+1,\, A} \\ (d_j', \delta_j')_{1, D'}, (c_j, \psi_j')_{1,\, C} \end{matrix} \right. \right] \quad \ldots (1.10)$$

Throught the present work, *we shall assume that the convergence (and existance) conditions given by (1.5), (1.6) and (1.7) are satisfied by each of the various H-functions involved.*

To simplify the space problem, we specify the parameters of the H-function of several variables in the following manner, throught the paper. Thus

$$H \left[\begin{matrix} [k : k_1, \ldots, k_n] ; \\ [u : u_1, \ldots, u_n] ; \end{matrix} z_1, \ldots, z_n \right]$$

would mean

$$H^{0,\, \lambda+1 : (\mu', \nu') ; \ldots ; (\mu^{(n)}, \nu^{(n)})}_{A+1,\, C+1 : [B'\, D'] ; \ldots ; [B^{(n)}, D^{(n)}]} \left[\begin{matrix} [k : k_1, \ldots, k_n], [(a) : \theta', \ldots, \theta^{(n)}] : \\ [(c) : \psi', \ldots, \psi^{(n)}], [u : u_1, \ldots, u_n] : \end{matrix} \right.$$

$$\left. \begin{matrix} [(b') : \emptyset'] ; \ldots ; [(b^{(n)}) ; \emptyset^{(n)}] ; \\ [(d') : \delta'] ; \ldots ; [(d^{(n)}) : \delta^{(n)}] ; \end{matrix} z_1, \ldots, z_n \right] \quad \ldots (1.11)$$

2. Probability Density Functions

This paper deals with certain classical statistical distributions associated with beta (or finite) and gamma (or infinite) distributions of one variate. The probability density function is taken in terms of the H-function of several variables, defined by (1.1). with generals arguments. First we find the probability density function.

Let a density function be defined by

$$f(x) = K\, x^{\sigma-1} (1-x)^{\rho-1}\, H\,[z_1 x^{\sigma_1} (1-x)^{\rho_1}, \ldots, z_n x^{\sigma_n} (1-x)]^{\rho_n},\ 0 \leqslant x \leqslant 1, \qquad \ldots(2.1)$$

for *finite distribution or generalized beta distribution*, and $f(x) = 0$, elsewhere. If $f(x)$ is a probability density function, then it should satisfy the relation

$$\int_{-\infty}^{\infty} f(x)\, dx = \int_0^1 f(x)\, dx \equiv 1. \qquad \ldots(2.2)$$

Putting the value of $f(x)$ given by (2.1) in (2.2) and evaluating the resulting integral with the help of Mellin-Barnes contour integral for the H-function of several variables, given by (1.1) and the well-known definition of beta function (see, e. g. [2], p 9, Eq. (1)), we find that

$$K^{-1} = H\left[\begin{matrix} [1-\sigma : \sigma_1, \ldots, \sigma_n]\ [1-\rho : \rho_1, \ldots, \rho_n]\ ; \\ [1-\sigma-\rho : \sigma_1+\rho_1, \ldots, \sigma_n+\rho_n]\ ; \end{matrix}\ z_1, \ldots, z_n\right] \qquad \ldots(2.3)$$

provided

$$\left.\begin{matrix} \mathrm{Re}\,(\sigma) + \sigma_i\, \alpha_i > 0, \text{ and} \\ \mathrm{Re}\,(\rho) + \rho_i\, \alpha_i < \forall\, i \in \{1, \ldots, n\} \end{matrix}\right\} \qquad \ldots(2\ 4)$$

where α_i is given by (1.9).

Again let

$$f(x) = Q\, e^{-sx}\, x^{\sigma-1}\, H\,[z_1 x^{\sigma_1}, \ldots, z_n x^{\sigma_n}] \qquad \ldots(2.5)$$

where $0<x<\infty$, Re $(s)>0$, Re $(\sigma) > 0$, Re $(\sigma)+\sigma_i\, \alpha_i > 0$, and

(α_i is given by (1.9))

$$Q^{-1} = s^{-\sigma}\, H\left[\begin{matrix} [1-\sigma : \sigma_1, \ldots, \sigma_n]\ ; \\ \text{———}\ ; \end{matrix}\ z_1 s^{-\sigma_1}, \ldots, z_n s^{-\sigma_n}\right] \qquad \ldots(2.6)$$

then $f(x)$ will be a probability density function for *infinite distribution or generalized gamma distribution*. By virtue of (2.2), we easily get

$$\int_0^{\infty} e^{-sx} \; x^{\sigma-1} H\left[z_1 x^{\sigma_1}, \ldots, z_n x^{\sigma_n}\right] dx = 1. \qquad \ldots(2.7)$$

Now, on evaluating the above integral with the help of (1.1) and the definition of gamma function (see, e. g. [2], p. 1, Eq (1)), we obtain the expression for Q given by (2.6).

Remark I. Complex values of the parameters hold little interest for statistics, but (2.1) and (2.5) can still define probability models ([1], p. 59 ; see also [6], p. 85, Eq. (4 2.2)).

Remark 2. Since Re (s) $>$ 0 in (2.5), the convergence of the integral (2.7) at its upper limit of integration can be guaranteed under the conditions stated already with the help of the following asymptototic expansion (see, e. g., 11], p. 357, Eq. (1.17)).

$$H[z_1, \ldots, z_n] = O\left(|z_1|^{\gamma_1} \ldots |z_n|^{\gamma_n}\right), \lambda \neq 0, \min\{|z_1|, \ldots, |z_n|\} \to \infty, \qquad \ldots(2.8)$$

for some $\gamma_1, \ldots, \gamma_n$; which would evidently complement the asymptotic expansions given by (1.8).

3. The Distribution function

The distribution function or the cumulative probability function $F(t)$ corresponding to a probability density function $f(x)$ is defined as

$$F(t) = \int_{-\infty}^{t} f(x)\, dx. \qquad \ldots(3.1)$$

We obtain here the distribution function for beta and gamma distributions separately.

If $f(x)$ is defined by (2.1) (with $\rho=1, \rho_1=\rho_2=\ldots=\rho_n=0$), the distribution function for *finite distribution* will be given by

$$F(t) = K\, t^{\sigma}\, H\left[\begin{matrix}[1-\sigma : \sigma_1, \ldots, \sigma_n] ; \\ [-\sigma : \sigma_1, \ldots, \sigma_n] ;\end{matrix}\; z_1 t^{\sigma_1}, \ldots, z_n t^{\sigma_n}\right] \qquad \ldots(3.2)$$

where

$$K^{-1} = H\left[\begin{matrix}[1-\sigma : \sigma_1, \ldots, \sigma_n]; \\ [-\sigma : \sigma_1, \ldots, \sigma_n];\end{matrix} \; z_1, \ldots, z_n\right], \qquad \ldots(3.3)$$

and $\mathrm{Re}(\sigma) + \sigma_i \alpha_i > 0$, α_i is given by (1.9), $i=1, \ldots, n$.

where, as already assumed in Section 1, the function $f(x) = 0$ for negative values of x.

Similarly, for *infinite distribution,* the distribution function $F(t)$ defined by (3.1) can also be obtained for (2.5), which is given below :

$$F(t) = Q \sum_{r=0}^{\infty} \frac{(-s)^r (t)^{\sigma+r}}{r!} H\left[\begin{matrix}[1-\sigma-r : \sigma_1, \ldots, \sigma_n]; \\ [-\sigma-r : \sigma_1, \ldots, \sigma_n];\end{matrix} \; z_1 t^{\sigma_1}, \ldots, z_n t^{\sigma_n}\right] \qquad \ldots(3.4)$$

where Q is given by (2.6).

We remark in passing that the distribution function, obtained by Srivastava and Singhal ([7], p. 6, Eq. (14)), can be deduced as a particular case of our function (3.2). This can be varified, if we put in (3.2), $\lambda=A=C=0$, $\mu^{(j)}=1$, $\nu^{(j)}=B^{(j)}$, $d_1^{(j)}=0$, $\delta_1^{(j)}=1$, $(j=2, 3, \ldots, n)$, take $z_2, z_3, \ldots, z_n \to 0$, and make a use of relationship given by (1.10). Also, the distribution function, recently given by Exton ([3], p 129, Eq. (7.1.1.5)) can alternatively be deduced as a particular case of (3.2). Indeed, if in (3.2) we put $\lambda=A=C=0$, $\mu^{(i)}=1$, $\nu^{(i)} = B^{(i)}$, $d_1^{(i)}=0$, $\delta_1^{(i)}=1$, replace $D^{(i)}$ by $D^{(i)}+1$ $(i=1, \ldots, n)$, make suitable changes in parameters and appeal to the known relationships ([6], p. 151) and ([8], p. 272, Eq (4.7)).

4. The Characteristic Function

The characteristic function denoted by $\phi(t)$ may be represented as $< e^{itx} >$, where the angle brackets denote "mathematical expectation" We may thus write the characteristic function as

$$\phi(t) = < e^{itx} > = \int_{-\infty}^{\infty} e^{itx} f(x)\, dx \qquad \ldots(4.1)$$

The characteristic function for *finite distribution*, when $f(x)$ is given by (2.1), is

$$\emptyset(t) = K \sum_{r=0}^{\infty} \frac{(i\,t)^r}{r!} H \left[\begin{matrix} [1-\sigma-r : \sigma_1, \ldots, \sigma_n], [1-\rho : \rho_1, \quad, \rho_n]; \\ [1-\sigma-\rho-r : \sigma_1+\rho_1, \ldots, \sigma_n+\rho_n] \quad ; \end{matrix} \; z_1, \ldots, z_n \right] \qquad \ldots(4.2)$$

where K is given by (2.3).

Also, the characteristic function for *infinite distribution*, when $f(x)$ is given by (2.5), is

$$\emptyset(t) = Q\,(s-i\,t)^{-\sigma} H \left[\begin{matrix} [1-\sigma : \sigma_1, \ldots, \sigma_n]; \\ \text{---} \quad ; \end{matrix} \; z_1 (s-i\,t)^{-\sigma_1}, \ldots, z_n (s-it)^{-\sigma_n} \right] \qquad \ldots(4.3)$$

where Q is given by (2.6).

On reduction of the H-function of several variables occurring in (4.3) in to Fox's H-function, with the help of relationship (1.10), we get the characteristic function considered by Srivastava and Singhal ([7], p. 5, Eq. (2)). Finally, by suitably specializing the parameters of the H-function of several variables and invoking the known result ([8], p. 272 Eq. (4.7)), the expression (4.3) will reduce to the characteristic function studied earlier by Exton ([3], p. 129, Eq. (7.1.1.5)).

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(Dedicated to the memory of Professor Arthur Erdélyi)

SOME REMARKS ON THE DERIVATIVE OF A POLYNOMIAL

By

M. A. MALIK

Department of Mathematics, Concordia University,
Loyola Campus, 7141 Sherbrooke Street West,
Montreal, Québec; Canada H4B 1R6

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Let $P(z)$ be a polynomial of degree n and $P'(z)$ denote its derivative. Concerning $|P'(z)|$ the following results are known :

Thaorem A ([3], p. 58). *If* $P(z)$ *is a polynomial of degree* n, *with* $|P(z)| \leqslant 1$ *on* $|z| \leqslant 1$ *and* $P(z)$ *has no zeros in the disk* $|z| < k$, $k \geqslant 1$ *then for* $|z| \leqslant 1$

$$(1) \qquad |P'(z)| \leqslant n/(1+k)$$

and the equality in (1) *holds for* $P(z)=[(z+k)/(1+k)]^n$.

Theorem B ([2], p. 544). *If* $P(z)$ *is a polynomial of degree* n, *with* $\max |P(z)| = 1$ *on* $|z| \leqslant 1$ *and* $P(z)$ *has all its zeros in the disk* $|z| \leqslant k$, $k \geq 1$ *then*

$$(2) \qquad \max_{|z|=1} |P'(z)| \geqslant n/(1+k^n)$$

and the equality in (2) *holds for* $P(z) = (z^n+k^n) / (1+k^n)$.

Theorem C ([4], p. 122). *If* $P(z)$ *is a plynomial of degree* n, *then for* $p \geqslant 1$

$$(3) \qquad \left[\int_0^{2\pi} |P'(e^{i\theta})|^p \, d\theta\right]^{1/p} \leqslant C_p\, n \left[\int_0^{2\pi} |\operatorname{Re} P(e^{i\theta})|^p \, d\theta\right]^{1/p}$$

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where $C^p_p = \sqrt{\pi}\,\dfrac{\Gamma(\frac{1}{2}p+1)}{\Gamma(\frac{1}{2}p+\frac{1}{2})}$ *and the equality in* (3) *holds for* $P(z)=az^n$ *where* a *is an arbitrary constant.*

In this note, we present another proof (short and simple) of the above theorems. Actually, we note that these theorems are a consequence of the observations due to be Bruijn [1] which are inspired by a result known as :

Laguerre Theorem. *If* $P(z)$ *is a polynomial of degree* n *and* $P(z)\neq 0$ *in* $z \in C$, *where* C *is circular domain, then for all* $z \in C$, $\xi \in C$,

(4) $\quad (\xi - z)P'(z) + nP(z) \neq 0.$

Remark : A circular domain is the image of the unit disk (open or closed) under a linear transformation.

In fact, with z_ν , $\nu = 1, 2, \ldots, n$ as the zeros of $P(z)$ in the right-hand side of (4) $\sum_{\nu=1}^{n} \dfrac{\xi - z_\nu}{z - z_\nu}$ cannot be zero as for fixed $\xi, z \in C$, the linear transformation $\dfrac{\xi-\lambda}{z-\lambda}$ maps z_ν , $\nu=1, 2, \ldots, n$ into a convex circular domain (the image of the complement of C) not containing zero.

With $P(z)$ replaced by $P(z)-\eta$, one can deduce from the Laguerre Theorem, the following

Lemma (*cf.* [1], p. 592). *Let* C *be a circular domain in the* z*-plane and* S *an arbitrary point set in the* w*-plane.* If $P(z)=w \in S$ *for* and $z \in C$, then we have for $z \in C$ *and* $\xi \in C$.

(5) $\quad \dfrac{\xi}{n}P'(z)+P(z)-z\dfrac{P'(z)}{n} \in S.$

Alternate Proof of Theorem A Since $P(z)$ has no zeros in $|z|>k$, $k\geqslant 1$, from Laguerre Theorem for $|z| \leqslant 1$, $0 < |\xi| < k$ and a suitable choice of argument ξ in (4), we have

(6) $\quad \left|\xi\,\dfrac{P'(z)}{n}\right| < \left|z\,\dfrac{P'(z)}{n}\,P(z)\right|$

or

(7) $$\left|\xi \frac{P'(z)}{n}\right| > \left| z\frac{P'(z)}{n} - P(z)\right|$$

But a sufficiently small value of $|\xi|$ contradicts (7), so (6) is true and letting $|\xi| \to k$, one has

(8) $$k\left|\frac{P'(z)}{n}\right| \leqslant \left| z\frac{P'(z)}{n} - P(z)\right|$$

for $|z| \leqslant 1$.

Now let C be the unit disk $|z| \leqslant 1$ and S be the disk $|w| \leqslant 1$. From the hypothesis, the image of the disk $|z| \leqslant 1$ under the maping $P(z)$ is contained in $|w| \leqslant 1$. As ξ varies in the unit disk, (5) implies that a disk of radius $\left|\frac{P'(z)}{n}\right|$ and centre $z\frac{P'(z)}{n} - P(z)$ (which also belong to S, take $\xi = 0$) must be contained in the disk $|w| \leqslant 1$. Since from (8) the centre is at a distance greater than $k\left|\frac{P'(z)}{n}\right|$ from the origin, we have $k\left|\frac{P'(z)}{n}\right| + \left|\frac{P'(z)}{n}\right| \leqslant 1$. This completes our proof of Theorem A.

Alternative Proof of Theorem B. Since $P(z) \neq 0$ for $|z| > k$, one has

(9) $$\xi \frac{P'(z)}{n} \neq z\frac{P'(z)}{n} - P(z)$$

for all $|\xi|$ and $|z| > k$. For these ξ and z, the inequality like (6) is violated for large $|\xi|$, thus

(10) $$k\left|\frac{P'(ke^{i\theta})}{n}\right| \geqslant \left|\frac{ke^{i\theta} P'(ke^{i\theta})}{n} - P(ke^{i\theta})\right|.$$

Since $P'(z)$ is a polynomial of degree $n-1$ add $k \geqslant 1$, making use of (10), we have

(11) $$k^n \max\left|\frac{P'(e^{i\theta})}{n}\right| \geqslant k \max\left|\frac{P'(ke^{i\theta})}{n}\right|$$

$$\geqslant \max\left|\frac{ke^{i\theta} P'(Re^{i\theta})}{n} - P(ke^{i\theta})\right.$$

$$\geqslant \max\left|\frac{e^{i\theta} P'(e^{i\theta})}{n} - P(e^{i\theta})\right|$$

$$\geqslant |P(e^{i\theta})| - \left|\frac{P'(e^{i\theta})}{n}\right|$$

which implies that

$$\max_{|z|=1} |P'(z)| \geqslant n / (1+k^n).$$

This completes our proof of Theorem B.

Alternative Proof of Theorem C. From a result of de Bruijn [1] we know that

$$(12) \qquad (e^{i(\eta+\theta)} - e^{i\theta}) \frac{P'(e^{i\theta})}{n} + P(e^{i\theta}) = \sum_{w} \lambda(n, w) P(we^{i\theta})$$

where w runs through the nth roots of $e^{i(n-1)\theta+\eta}$, $\lambda \geqslant 0$ and $\sum_{w} \lambda(n,w) = 1$.

Taking real parts of both sides in (12) and then integrating the pth power of the absolute value, one gets

$$(13) \qquad \int_0^{2\pi} \left| Re\, \{P(e^{i\theta}) - e^{i\theta} \frac{P'(e^{i\theta})}{n}\} + Re\, e^{i(\eta+\theta)} \frac{P'(e^{i\theta})}{n} \right|^p d\theta$$

$$\leqslant \int_0^{2\pi} \left| Re\, P(e^{i\theta}) \right|^p d\theta$$

where η is arbitrary. Put $A(\theta) = P(e^{i\theta}) - e^{i\theta} \frac{P'(e^{i\theta})}{n}$ and $B(\theta) = e^{i\theta} \frac{P'(e^{i\theta})}{n}$. Note that one can vary η in a sector of angle π so that both $Re\, A(\theta)$ and $Re\, e^{i\eta} B(\theta)$ have the same sign. In fact, if $Re\, A(\theta) \geq 0$, let $-\pi/2 - \arg B(\theta) \leqslant \eta \leqslant \pi/2 - \arg B(\theta)$ and in case $Re\, A(\theta) \leqslant 0$, choose $\pi/2 - \arg B(\theta) \leqslant \eta \leqslant 3\pi/2 - \arg B(\theta)$. This implies that for such a choice of η in (13), we have

$$(14) \qquad \int_0^{2\pi} \left| Re\, e^{i\eta} e^{i\theta} P'(e^{i\theta}) \right|^p d\theta \leqslant n^p \int_0^{2\pi} \left| Re\, P(e^{i\theta}) \right|^p d\theta.$$

From where

$$(15) \quad \int_0^{2\pi} \left| P'(e^{i\theta}) \right|^p \left| 1+e^{i(2\eta-\alpha(\theta))} \right|^p d\theta \leqslant \int_0^{2\pi} \left| Re\, P(e^{i\theta}) \right|^p d\theta;$$

$\alpha(\theta) = \arg e^{i\theta} \; P'(e^{i\theta})$. By integrating (15) with respect to η over the corresponding interval of length π, making a change of variable and then interchanging the order of integrations, one obtains

$$(16) \quad \int_0^{2\pi} \left| P'(e^{i\theta}) \right|^p d\theta \, . \int_0^{2\pi} \left| \frac{1+e^{i\eta}}{2} \right|^p d\eta \leqslant 2\pi\, n^p \int_0^{2\pi} \left| Re\, P(e^{i\theta}) \right|^p d\theta.$$

Consequently,

$$\int_0^{2\pi} \left| P'(e^{i\theta}) \right|^p d\theta \leqslant \sqrt{\pi}\, \frac{\Gamma(\frac{1}{2}p+1)}{\Gamma(\frac{1}{2}p+\frac{1}{2})}\, n^p \int_0^{2\pi} \left| Re\, P(e^{i\theta}) \right|^p d\theta.$$

This completes our proof of Theorem C.

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ON MULTIDIMENSIONAL INTEGRAL TRANSFORMS

By

K. C. GUPTA

Department of Mathematics, M. R. Engineering College, Jaipur-302004, India

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ABSTRACT

In this paper, two general theorems involving multidimensional integral transforms have been established. The first expresses an interesting relationship between images and originals of related functions in multiple integral transforms, while the second reveals the interconnections between images of related functions in the respective transforms. These theorems are sufficiently general in nature and shed light on the basic structure of integral transforms involved. They unify and extend a number of scattered theorems in the literature involving single and double integral transforms. Again by taking specific transforms in our theorems we can obtain a large number of results which may prove to be useful in solving certain boundary value problems.

1. INTRODUCTION AND DEFINITIONS

Let $f(x_1, \ldots, x_r)$ belong to a prescribed class of real or complex—valued functions or r real variables $x_1, \ldots, x_r$ defined over the region $R : 0 \leqslant x_i < \infty, i = 1, \ldots r$. Then (in line with the definition of linear integral transforms), the multidimensional integral transform $T\{f(x_1,\ldots,x_r) ; p_1, \ldots, p_r\}$ of the function $f(x_1, \ldots, x_r)$ is defined and represented as follows :

$$\phi(p_1,\ldots,p_r) = T\{f(x_1,\ldots,x_r);p_1,\ldots,p_r\}$$

$$=\int_0^\infty\ldots\int_0^\infty k(x_1,\ldots,x_r;p_1,\ldots,p_r)\,f(x_1,\ldots,x_r)\,dx_1\ldots dx_r, \qquad \ldots(1.1)$$

where the functions $f(x_1,\ldots,x_r)$ and $k(x_1,\ldots,x_r;p_1,\ldots,p_r)$ and the parameters $p_1,\ldots,p_r$ are always so chosen that the integral (1.1) is absolutely convergent, i. e.

(i) The product $(x_1)^{c_1}\ldots(x_r)^{c_r}f(x_1,\ldots,x_r)$ is integrable (in the sense of Lebesgue) over every finite region
$R(R_1,\ldots,R_r): \leqslant x_i \leqslant R_i,\ R_i > 0,\ i=1,\ldots,r$
where $k(x_1,\ldots,x_r;p_1,\ldots,p_r)$
$= O(x_1^{c_1}\ldots x_r^{c_r}),\ \max\{x_1,\ldots,x_r\}\to 0$
and

(ii) the limit of the finite form of the multiple integral in (1.1) with
$\int_0^\infty\ldots\int_0^\infty$ replaced by $\int_0^{R_1}\ldots\int_0^{R_r}$,
exists at the point $(p_1,\ldots,p_r)$ when $R_1,\ldots,R_r\to\infty$.

For a specific transform, $k(x_1,\ldots,x_r;p_1,\ldots p_r)$ is a definite function of $x_1,\ldots,x_r,p_1,\ldots,p_r$ and is known as the kernel of the transform. Also $\phi(p_1,\ldots,p_r)$ is called the image of the function $f(x_1,\ldots,x_r)$ in the transform defined by (1.1), and $f(x_1,\ \ ,x_r)$ the original. For a systematic study of two general classes of multidimensional integral transformations, in which the kernels involve the H-functions of several variables, the reader is referred to a series of recent papers by Srivastava and Panda [7] who also cite a number of special instances of their transforms in the literature (cf. [7], Part I, p. 119; see also pp. 122-124).

For an integral transform of the type given by (1.1), it is easy to verify that if

$$\phi_1(p_1,\ldots,p_r) = T\{f_1(x_1,\ldots,x_r);p_1,\ldots,p_r\} \qquad \ldots(1.2)$$

and

$$\phi_2(p_1,\ldots,p_r) = T\{f_2(x_1,\ \ ,x_r);p_1,\ldots p_r\} \qquad \ldots(1.3)$$

then, under appropriate conditions of convergence of the integrals involved, the following formula analogous to the Parseval-Goldstein type of formula for the Laplace transform in one and more variables holds :

$$\int_0^\infty \cdots \int_0^\infty f_2(x_1, \ldots, x_r)\, \emptyset_1(x_1, \ldots, x_r)\, dx_1 \ldots dx_r$$

$$= \int_0^\infty \cdots \int_0^\infty f_1(x_1, \ldots, x_r)\, \emptyset_2(x_1, \ldots, x_r)\, dx_1 \ldots dx_r. \qquad \ldots(1.4)$$

The above result will be referred to as the generalized Parseval-Goldstein formula and will be required in the sequel. For two interesting special forms of (1.4), see Srivastava and Panda [7, Part I, p. 129, Theorem 3].

2. We first establish a theorem which exhibits an interesting relationship between images and originals of related functions in the transforms T_1 and T_2 defined below

$T_1 \{f(x_1, \ldots, x_r) ; p_1, \ldots, p_r\}$

$$= \int_0^\infty \cdots \int_0^\infty f(x_1, \ldots, x_r)\, k_1(p_1 x_1, \ldots, p_r x_r)\, dx_1 \ldots dx_r, \qquad \ldots(2.1)$$

$T_2 \{f(x_1, \ldots, x_r) ; p_1, \ldots, p_r\}$

$$= \int_0^\infty \cdots \int_0^\infty f(x_1, \ldots, x_r)\, k_2(p_1 x_1, \ldots, p_r x_r)\, dx_1 \ldots dx_r, \qquad \ldots(2.2)$$

provided that the multiple integrals involved in (2.1) and (2.2) convergeo absolutely. Our result may be stated as

Theorem I. *If*

$$h_1(p_1, \ldots, p_r) = T_1 \{h_2(x_1, \ldots, x_r)\, g(x_1, \ldots, x_r) ; p_1, \ldots, p_r\}, \qquad \ldots(2.3)$$

and

$$h_2(p_1^{\sigma_1}, \ldots, p_r^{\sigma_r}) = T_2 \{f(x_1, \ldots, x_r) ; p_1, \ldots, p_r\}, \qquad \ldots(2.4)$$

then

$$h_1(p_1, \ldots, p_r) = \{\sigma_1 \ldots \sigma_r\} \int_0^\infty \cdots \int_0^\infty f(x_1, \ldots, x_r)$$

$$\cdot\, \emptyset(x_1, \ldots, x_r, p_1, \ldots, p_r)\, dx_1 \ldots dx_r, \qquad \ldots(2.5)$$

where

$$\emptyset(p_1,\dots,p_r,a_1,\dots,a_r) = T_2\,\{x_1^{\sigma_1-1}\dots x_r^{\sigma_r-1}\,g(x_1^{\sigma_1},\dots,x_r^{\sigma_r})$$

$$.k_1(a_1x_1^{\sigma_1},\dots,a_r x_r^{\sigma_r})\,;\,p_1,\dots,p_r\}, \qquad \dots(2.6)$$

the transforms T_1 *and* T_2 *are defined by means of equations* (2.1) *and* (2.2), $\sigma_1,\dots\sigma_r$ *are non-zero real numbers of the same sign, each* $a_1,\dots,a_r$ *is independent of* $p_1,\dots,p_r$ *and all the multiple integrals involved in* (2.3) *to* (2.6) *are assumed to be absolutely convergent.*

Proof. Applying the generalized Parseval-Goldstein formula given by (1.4) to the operational pairs (2.4) and (2.6), we get

$$\int_0^\infty\dots\int_0^\infty \{x_1^{\sigma_1-1}\dots x_r^{\sigma_r-1}\}\,g(x_1^{\sigma_1},\dots,x_r^{\sigma_r})\,h_2(x_1^{\sigma_1},\dots,x_r^{\sigma_r})$$

$$.k_1(a_1x_1^{\sigma_1},\dots,a_r x_r^{\sigma_r})\,dx_1\dots dx_r$$

$$=\int_0^\infty\dots\int_0^\infty \emptyset(x_1,\dots,x_r,a_1,\dots,a_r)\;f(x_1,\dots,x_r)\,dx_1\dots dx_r, \qquad \dots(2.7)$$

Now replacing $a_1,\dots,a_r$ by $p_1,\dots,p_r$ in (2.7), changing the variables of integration slightly on its left-hand side, and interpreting the result thus obtained in terms of (2.3), we easily arrive at Theorem 1.

The presence of the arbitrary function $g(x_1,\dots,x_r)$, and the general nature of kernel involved in the above theorem, enable it to yield general, deeper and useful results. Thus a large number of results, which express relationships between images and originals of related functions in any two integral transforms of the type given by (2.1) and (2.2), found recently by several authors and scattered in the literature, are all unified and extended by this theorem.

3. Special Cases of Theorem 1

If in the above theorem, we put $\sigma_1=\sigma_2=\dots=\sigma_r=1$ and take both the transforms T_1 and T_2 as Laplace transforms of r variables, $\emptyset(p_1,\dots,p_r,a_1,\dots,a_r)$ occurring in (2.6) can be given a slightly altered form In such a case we have

$$\emptyset(p_1,\dots,p_r,a_1,\dots,a_r) = L\,\{\,g(x_1,\dots,x_r)\,e^{-a_1x_1-\dots-a_rx_r}\,;\,p_1,\dots,p_r\,\}$$

$$=\theta(p_1+a_1,\dots,p_r+a_r), \qquad \dots(3.1)$$

where

$$\theta(p_1,\ldots,p_r) = L\{g(x_1,\ldots,x_r);\, p_1,\ldots,p_r\}$$

$$= \int_0^\infty \ldots \int_0^\infty e^{-p_1x_1-\ldots-p_rx_r}\, g(x_1,\ldots,x_r)\, dx_1\ldots dx_r\,, \qquad \ldots(3.2)$$

and the theorem takes the following interesting form :

Corollary I. *If*

$$h_1(p_1,\ldots,p_r) = L\{g(x_1,\ldots,x_r)\, h_2(x_1,\ldots,x_r);\, p_1,\ldots,p_r\}, \qquad \ldots(3.3)$$

and

$$h_2(p_1,\ldots,p_r) = L\{f(x_1,\ldots,x_r);\, p_1,\ldots,p_r\}, \qquad \ldots(3.4)$$

then

$$h_1(p_1,\ldots,p_r) = \int_0^\infty \ldots \int_0^\infty f(x_1,\ldots,x_r)\, \theta(x_1+p_1,\ldots,x_r+p_r)\, dx_1\ldots dx_r$$

where $\ldots(3.5)$

$$\theta(p_1,\ldots,p_r) = L\{g(x_1,\ldots,x_r);\, p_1,\ldots,p_r\}, \qquad \ldots(3.6)$$

$\mathrm{Re}(p_i) > 0$ $(i=1,\ldots,r)$, *and the various multiple integrals involved in (3.3) to (3.6) are absolutely convergent.*

The special case of Theorem 1, when $r=2$, is of interest in ilself Thus on suitable choices of the corresponding transforms T_1, T_2, and the function $g(x, y)$, we easily get the known results obtained earlier by Bose [2, p. 176], Goyal [3. p. 139], and Jain [6, p. 314]. The analogue of Theorem 1 for integral transforms involving one variable, i. e. when $r=1$, is also quite interesting. It generalizes a theorem of Agrawal [1, p. 538] and a very large number of other results scattered in the literature, as we pointed out in an earlier paper [4].

4. Now we establish another theorem which reveals the inter-connections between images and related functions in the transforms defined by (2.1) and (2.2). We first state

Theorem 2. *If*

$$h_1(p_1,\ldots,p_r) = T_1\{x_1^{(c_1/\sigma_1)-1}\ldots x_r^{(c_r/\sigma_r)-1} f(x_1,\ldots,x_r);\, p_1,\ldots,p_r\} \qquad \ldots(4.1)$$

and

$$h_2(p_1,\dots,p_r) = T_2\ \{ f(x_1^{-\sigma_1},\dots,x_r^{-\sigma_r})\ ;\ p_1,\dots,p_r \}, \qquad \dots(4.2)$$

then

$$h_1(p_1,\ \ ,p_r) = \{\sigma_1\dots\sigma_r\} \int_0^\infty \dots \int_0^\infty h_2(x_1,\dots,x_r)$$

$$\cdot\ \emptyset(x_1,\dots,x_r, p_1,\dots,p_r)\ dx_1\dots dx_r, \qquad \dots(4.3)$$

where

$$\{p_1^{-c_1-1}\dots p_r^{-c_r-1}\}\ k_1(a_1 p_1^{-\sigma_1},\dots,a_r p_r^{-\sigma_r})$$

$$= T_2\ \{\emptyset(x_1,\dots,x_r, a_1,\dots,a_r)\ ;\ p_1,\dots,p_r\}, \qquad \dots(4.4)$$

$\sigma_1,\dots,\sigma_r$ are non-zero real numbers of the same sign, each of $a_1,\dots,a_r$ is independent of $p_1,\dots,p_r$ and all the multiple integrals involved in equations (4.1) to (4.4) are assumed to converge absolutely.

Proof. Theorem 2 easily follows on applying (1.4) to the pairs given by (4.2) and (4.4) and proceeding on lines similar to those indicated in the proof of Theorem 1.

Theorem 2 also sufficiently general in nature. Its analogues for one-and two-dimensional integral transforms were obtained by the author in his earlier papers ([4], [5]). These analogues are of interest in themselves and unify and extend a vast number of results as pointed therein.

In conclusion, we remark that a number of theorems involving specific multidimensional integral transforms introduced from time to time by several authors can be obtained from Theorems 1 and 2 ; however, we do not record them here for want of space.

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(*Dedicated to the memory of Professor Arthur Erdélyi*)

COMPARISON OF THE ORDER PROPERTY OF A GENERALIZED LAPLACE-STIELTJES INTEGRAL

by

SUSHIL KUMAR SINHA

Qtrs. No. H/45, Hinoo, Ranchi-834002, Bihar, India

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ABSTRACT

We establish theorems concerning the order condition of $A_\alpha(t)$ as $t \to \infty$, by investigating the order relations existing between different expressions involving $A_\alpha(t)$, the α-th Cesáro sum of the determining function $A(t)$ from the summability property of the integral

$$\int_0^\infty (2st)^\lambda W_{k,m}(2st)\, dA(t)$$

when the integral is summable $|C, \alpha|$. Here $W_{k,m}$ is the Whittaker function [1], and $A(t)$ is a function of bounded variation in every finite interval $0 \leqslant t \leqslant T$, and $s = \sigma + i\,T$ is a complex variable. We shall also give a formula for the abscissa of absolute summability of the integral whenever it is positive.

I. INTRODUCTION

Goyal [3] gave a generalization of the Laplace-Stieltjes integral

$$f(s) = \int_0^\infty e^{-st}\, dA(t) \tag{1.1}$$

in the form :

$$f(s) = \int_0^\infty (2st)^\lambda\, W_{k,m}(2st)\, dA(t) \tag{1.2}$$

where $W_{k,m}$ is Whittaker function [1], and $A(t)$ is a function of bounded variation in any finite interval $0 \leqslant t \leqslant T$.

When $\lambda = -1/4$, $k = 1/4$, and $m = \pm 1/4$, the transform (1.2) reduce to (1.1).

Bosanquet [2] gave the order condition of $A_\alpha(t)$ as $t \to \infty$, by investigating the order relations existing between different expressions involving $A_\alpha(t)$, when the integral (1.1) is sumable $| C, \alpha |$.

The purpose of the present paper is to obtain similar theorems for (1.2), and also to deduce a formula for the abscissa of absolute summability of integral (1 2) whenever it is positive.

2 NOTATIONS

We introduce the following notations :

(i) $\emptyset_{k,m}(2st) = (2st)^\lambda W_{k,m}(2st)$

(ii) $[\alpha]$ denotes the integral part of α. If $\alpha > 1$, we denote by h the greatest integer less than α ; if $0 < \alpha \leqslant 1$, then $h=0$, and if $\alpha=0$, then $h=-1$.

(iii) σ and T stand for the real and imaginary parts of the complex number s, $s = \sigma + iT$.

(iv) k and m are taken to be real.

(v) In (1.2), we shall take $k + \lambda \geqslant 0$, unless otherwise stated.

(vi) $A_\alpha(x)$ denotes the α-th Cesáro sum of $A(x)$.

3. Main Results

Teorem 3.1 If $\alpha \geqslant 0$, $\sigma_0 > 0$ and

$$x^{-\alpha} \emptyset_{k,m}(2s_0 x) A_\alpha(x) = o(1) \,|\, C, 0 \,|, (x \to \infty),$$

then $\displaystyle\int_1^x | d[u^{-\alpha} A_\alpha(u)] | = o[2\sigma_0 x)^{-\lambda-k} e^{\sigma_0 x}]$.

Proof : We set $g(x) = x^{-\alpha} \emptyset_{k,m}(2s_0 x) A_\alpha(x)$

$$h(x) = x^{-\alpha} A_\alpha(x)$$

$$\text{and } \overset{*}{g}(x) = \int_x^{\infty} | \, dg\,(u) \, |.$$

Then $| g(x) | \leqslant \overset{*}{g}(x) = o\,(1)$, and hence by the use of a Theorem of Pollard ([4], 261) and by integration by part, we have

$$\int_1^x | \, dh\,(u) \, | = \int_1^x | \, d\,[u^{-a} A_a\,(u)\,] \, |$$

$$\leqslant \int_1^x | \, d\,[\,g\,(u)\,/\,\phi_{k,m}\,(2s_0 u)\,] \, |$$

$$\leqslant [\,1\,/\,|\,\phi_{k,m}\,(2s_0 1)\,|\,]\,\overset{*}{g}\,(1)$$

$$+2\int_1^x \overset{*}{g}\,(u)\,|\,(d/du)\,[1/\phi_{k,m}\,(2s_0 u)\,]\,|\,du.$$

$$= o\,[\,(2\,\sigma_0\,x)^{-\lambda-k}\; e^{\sigma_0 x}\,], \qquad (x \to \infty)$$

(By [5] ; Lemma 3.3)

Thus the theorem is completely established.

Theorem 3.2 If $a \geqslant 0$ and $\sigma_0 > 0$ and if

$$(3.1) \quad \int_1^x | \, dA_a\,(u) \, | = o\,[\,x^a\,(2\,\sigma_0\,x)^{-\lambda-k}\,e^{\sigma_0 x}\,],\ (x \to \infty),$$

then

$$(3.2) \quad \int_1^x | \, d\,[u^{-a}\,A_a\,(u)] \, | = o\,[\,(2\,\sigma_0\,x)^{-\lambda-k}\,e^{\sigma_0 x}\,],$$

and conversely.

Proof : We first assume that (3.1) holds, and $\overset{*}{A}(x) = \int_1^x | \, dA_a\,(u) \, |$;

then $| A_a\,(x) | \leqslant | A_a\,(1) | + \overset{*}{A}(x) = o\,[\,x^a\,(2\,\sigma_0 x)^{-\lambda-k}\,e^{\sigma_0 x}\,]$

and hence by the use of a Theorem of Pollard ([4], 261) and by integration by part, we have

$$\int_1^x | \, d\,[\,u^{-a}\,A_a\,(u)\,] \, |$$

$$\leqslant x^{-\alpha} \overset{*}{A}(x) + \int_1^x \alpha u^{-\alpha-1} [\overset{*}{A}(u) + | A_\alpha (u) |] du$$

$$= o [(2 \sigma_0 x)^{-\lambda-k} e^{\sigma_0 x}] + o [x^{-1} (2 \sigma_0 x)^{-\lambda-k} e^{\sigma_0 x}]$$

$$= o [(2 \sigma_0 x^{-\lambda-k} e^{\sigma_0 x}]$$

which gives (3.2).

Conversely, we assume that (3.2) holds, and

$$C_\alpha (x) = [A_\alpha (x) / x^\alpha] ;$$

then $\int_1^x | dC_\alpha (u) | = \int_1^x | d [u^{-\alpha} A_\alpha (u)] |$

$$= o [(2 \sigma_0 x)^{-\lambda-k} e^{\sigma_0 x}].$$

We set $\overset{*}{C}(x) = \int_1^x | dC_\alpha (x) | = o [(2 \sigma_0 x)^{-\lambda-k} e^{\sigma_0 x}]$

then $| C_\alpha (x) | = \left| \int_0^x dC_\alpha (x) \right| \leqslant C_\alpha (1) + \overset{*}{C}(x)$

$$= o [(2 \sigma_0 x)^{-\lambda-k} e^{\sigma_0 x}], (x \to \infty).$$

Now $\int_0^x | dA_\alpha (x) | \leqslant \int_0^1 | dA_\alpha (u) | + \int_1^x | d [u^\alpha C_\alpha (u)] |$

$$\leqslant \int_0^l | dA_\alpha (u) | + x^\alpha \overset{*}{C}(x) + \alpha \int_1^x u^{\alpha-1} [\overset{*}{C}(u) + | C_\alpha (u) |] du$$

$$= o [x^\alpha (2 \sigma_0 x)^{-\lambda-k} e^{\sigma_0 x}], (x \to \infty),$$

which gives (3.1).

Thus the theorem is completely established.

Theorem 3.3 If $\alpha \geq 0$, $\sigma_0 > 0$ and the integral (1.2) is summable $| C, \alpha |$ for $s = s_0$, then

$$\int_0^x | dA_\alpha (u) | = o [x^\alpha (2 \sigma_0 x)^{-\lambda-k} e^{\sigma_0 x}], (x \to \infty)$$

Proof : Applying [7] ; Theorem 1, to our assumption, we have

$$x^{-\alpha}\, \emptyset_{k,m}(2s_0x)\, A_\alpha(x) = o(1) \mid C, 0 \mid, \; (x \to \infty).$$

This implies, by virtue of Theorem 3.1, that

$$(3.3) \quad \int_1^x \mid d\,[u^{-\alpha} A_\alpha(u)] \mid = \int_1^x \mid d\,[u^{-\alpha} \{u^\alpha / \emptyset_{k,m}(2s_0u)\}] \mid$$

$$= o\,[(2\,\sigma_0 x)^{-\lambda-k}\, e^{\sigma_0 x}] \quad (x \to \infty).$$

Hence, by applying the converse part of Theorem 3.2 to (3.3), we get

$$\int_0^x dA_\alpha(u) \mid = o\,[x^\alpha\, (2\,\sigma_0\, x^{-\lambda-k}\, e^{\sigma_0 x}],\; (x \to \infty).$$

Hence the theorem.

4. Abscissa of Absolute Summability

The following theorem gives the existence and expression for the abscissa of absolute summability of the integral (1.2) whenever it is positive.

Theorem 4.1 If $\alpha \geqslant 0, \bar{\sigma}_\alpha$ is the abscissa of absolute summability $\mid C, \alpha \mid$ of (1.2) and if $\bar{\sigma}_\alpha \geqslant 0$, then

$$\bar{\sigma}_\alpha = \overline{\lim_{x \to \infty}} \left[(1/x) \log \int_0^x \mid dA_\alpha(x) \mid \right].$$

Proof : Let

$$(4.1) \quad \overline{\lim_{x \to \infty}} \left[(1/x) \log \int_0^x \mid dA_\alpha(u) \mid \right] = \sigma_a$$

then we have $(1/x) \log \int_0^x \mid dA_\alpha(u) \mid < \sigma_a + \epsilon$, where ϵ is an arbitrary small positive number.

Hence, from [6] ; Theorem 2, the integral (1.2) is summable $\mid C, \alpha \mid$ for $\sigma > \sigma_a$.

Hence

$$(4.2) \quad \bar{\sigma}_\alpha \leqslant \sigma_a .$$

Now, if possible, let the integral (1.2) be summable $| C, \alpha |$ for $\sigma < \sigma_a - \epsilon$, $\epsilon > 0$, then by theorem 3.3, we have

$$(1/x) \log \int_0^x | dA_\alpha (x) | < \sigma_a - \epsilon,$$

which contradicts (4.1).

Hence, we have $\bar{\sigma}_\alpha = \sigma_a$.

This establishes the theorem.

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A TYPE OF GENERALISED OSCILLATORY REGRESSION

By

Harold Exton

27, Hollinhurst Avenue, Penwortham Preston Lancashire, PR1 OAE U.K.

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SUMMARY

Bessel-Clifford function of quardratic argument are employed in a statistical time-series regression model to provide a type of least-squares fit which generalises the use of standard Fourier analysis. An example from econometrics is given and a further generalisation is indicated.

1. INTRODUCTION

If $Y_1(t)$ and $Y_2(t)$ constitute a fundamental system of solutions of a linear ordinary differential equation of the second order, then we may model a statistical time-series by means of the expression

$$y(t) = A_1 Y_1(\lambda t) + A_2 Y_2(\lambda t) + \in(t), \qquad \text{...(1.1)}$$

where A_1, A_2 and λ are constants and where $\in(t)$ represents the correspon-ding residuals wh ch are to be minimised by considering the sum of their squares as usual. The model (1.1) is envisaged as a generalisation of the simple Fourier case, where Y_1 and Y_2 are then the circular functions Cos and Sin, respectively. {See Bloomfield [1], p. 11, for example. } It is assumed that the mean and any secular component of the time-series have been removed, and that the functions Y_1 and Y_2 are oscillatory over the range of the independent variable under consideration.

In this paper, we let the functions $Y_1(t)$ and $Y_2(t)$ take the form

$$\mathrm{Ec}(b, t) = {}_0F_1\left(-; \frac{1+b}{2}; -\frac{t^2}{4}\right) \qquad \text{...(1.2)}$$

and $$Es\,(b, t) = t^{1-b}\,{}_0F_1\,(-\,;\,\frac{3-b}{2}\,;-\frac{t^2}{4})\,, \qquad \ldots(1.3)$$

where the parameter b is less than unity. We note that Ec and Es reduce, respectively, to Cos and Sin when we let b become zero.

The two functions defined by (1.2) and (1.3) are linearly independent solutions of the Bessel-Clifford equation provided that b is not equal to a negative odd integer because $Ec\,(b, t)$ is not defined under those circumstances. {See Hayek [4].}

The function ${}_0F_1\,(-\,;\,b\,;\,x)$ is a case of the generalised hypergeometric function of one variable defined by the relation

$${}_AF_B\left(\begin{matrix}a_1, \ldots, a_A\,; \\ b_1, \ldots, b_B\,;\end{matrix}\; x\right) = \sum_{r=0}^{\infty} \frac{(a_1)_r \cdots (a_A)_r}{(b_1)_r \cdots (b_B)_r}\,\frac{x^r}{r\,!}, \qquad \ldots(1.4)$$

$$(a)_r = a\,(a+1)\,(a+2)\ldots(a+r-1),\ (a)_0 = 1. \qquad \ldots(1.5)$$

{ See Slater [6], p. 40. }

It will be seen that the functions (1.2) and (1.3) are related to the Bessel function of the first kind, but are preferred in the present context because the corresponding fundamental system of solutions of the Bessel equation $J_\nu\,(t)$ and $J_{-\nu}(t)$, where ν is not an integer or zero, is such that one member vanishes at the origin and the other becomes infinite there, depending upon the sign of ν.

The use of these generalisations of the circular functions seems to be most useful in dealing with relatively small segments of a longer time-series, and the least-squares fit can often dramatically improved by suitably adjusting the parameter b.

2. A Simple Least-Squares Model—When applying the method of least-squares to the model.

$$y\,(t) = Ac\;Ec\,(b, \lambda\,t) + As\;Es\,(b, \lambda\,t) + \in(t), \qquad \ldots(2.1)$$

we minimise the expression

$$T = \sum_{r=0}^{n} \{\,y\,(t_r) - Ac\;Ec\,(b, \lambda\,t_r) - As\;Es\,(b, \lambda\,t_r)\,\}^2 \qquad \ldots(2.2)$$

where $n+1$ is the number of elements of the time-series, which are taken to be equally spaced. After putting the partial derivatives $\frac{\partial T}{\partial A_C}$ and $\frac{\partial T}{\partial A_S}$ equal to zero we may solve the resulting normal equations which yield

$$A_C = \frac{\sum_{r=0}^{n} y(t_r)\, E_C(b,\lambda t_r) \sum_{r=0}^{n} \{E_S(b,\lambda t_r)\}^2 - \sum_{r=0}^{n} y(t_r) E_S(b,\lambda t_r) \sum_{r=0}^{n} E_S(b,\lambda t_r) E_C(b,\lambda t_r)}{\sum_{r=0}^{n} \{E_C(b,\lambda t_r)\}^2 \sum_{r=0}^{n} \{E_S(b,\lambda t_r)\}^2 - \left\{\sum_{r=0}^{n} E_S(b,\lambda t_r)\, E_C(b,\lambda t_r)\right\}^2} \quad \ldots(2.3)$$

and

$$A_S = \frac{\sum_{r=0}^{n} y(t_r)\, E_S(b,\lambda t_r) \sum_{r=0}^{n} \{E_C(b,\lambda t_r)\}^2 - \sum_{r=0}^{n} y(t_r) E_C(b,\lambda t_r) \sum_{r=0}^{n} E_S(b,\lambda t_r) E_C(b,\lambda t_r)}{\sum_{r=0}^{n} \{E_C(b,\lambda t_r)\} \sum_{r=0}^{n} \{E_S(b,\lambda t_r)\}^2 - \left\{\sum_{r=0}^{n} E_S(b,\lambda t_r)\, E_C(b,\lambda t_r)\right\}} \quad \ldots(2.4)$$

The values of λ and b which minimise the function T must then be obtained by numerical computation. When $b = 0$, T will generally be approximately minimised when $\sqrt{A_C^2 + A_S^2}$ is maximised, a fact which is exploited in classical periodogram analysis. For general values of b, however, this does not hold, and the exact expression for T must be used.

3. Practical Computation—The series representations (1.2) and (1.3) converge quite rapidly when the argument is not too large. In fact, these functions may be computed to an accuracy of two decimal places for $t \leqslant 14$, and a greater degree of accuracy would seldom if ever be required in practice. For larger values of t an asymptotic expansion could be employed, but since circular functions feature prominently in this expansion, no additional advantage will accrue for $t > 14$, approximately. This is the main reason why the generalised analysis discussed here is best applicable to short segments of time-series where each segment does not include more than about two cycles of the principal oscillatory component.

The author has found, in practice, that each segment should be in length somewhat less than one cycle and more than one-half of a cycle of the principal component. From a numerical point of view, the behaviour of the fnnction T is more sensitive to changes in λ than to changes in b, so that we may find the approximate value of λ using circular functions directly which is a much quicker procedure that using series expansions. This gives the optimum minimisation of T for $b=0$. The computation is then repeated for progressively varying values of b within the range of λ already determined.

As an example, we use the time-series of relative prices of the export of steel bars from the U. S. A. for the period 1957–1965 given in the following table :—

1957	1958	1959	1960	1961	1962	1963	1964	1965
1.06	1.02	1.15	1.12	1.10	1.13	1.23	1.24	0.99

{ See Tarr [7].}

After removal of the mean and multiplying by 100 for convenience, we obtain the time-series.

$$-6.7, -0.7, 2.3, -0.7, -2.7, 0.3, 10.3, 11.3, -13.7.$$

The parameter λ is computed for $b=0$ by the straight forward use of circular functions and b is then varied until the sum of the squares, T, is minimised. The results are set out below :

T	λ	b
185.95	11.9	0
151.04	12.0	−0.5
96.03	12.0	−2 5
79.16	11.9	−3.5
61.94	10.8	−4.5
53.79	10.7	−5.5
50.82	10.7	−6.5
51.21	10.7	−7.5
50.65	10.70	−6.8

The values of the constants Ac and As corresponding to the last entry for optimised T are -0.18 and -9.61×10^{-5}, respectively.

One interpretation of the period of the principal local oscillation of a time-series whose segments are analysed by the above method may be taken to be twice the interval between the first two positive zeros of the function

$$F(t) = Ac\, Ec\,(b, \lambda\, t) + As\, Es\,(b, \lambda\, t), \qquad \text{...(3.1)}$$

multiplied by a factor equal to the length of time occupied by the segment of the time-series under discussion. An indication of the amplitude of this component may be furnished by taking the mean of the moduli of the turning values of F (t) in the interval $0 \leqslant t \leqslant 1$. The local period and amplitude of the above series were found to be 5.98 years and 7 units respectively. All these calculations were carried out by means of a micro-computer.

4. A Further Generalisation. Conclusion. A basic analogue of the system $Ec\,(b, t)$ and $Es\,(b, t)$ is given by

$$Ec\,(q\,;\, b, t) = \sum_{r=0}^{\infty} \frac{q^{\frac{1}{2}r(r-1)}(-t^2/4)^r}{[\frac{1}{2}+\frac{1}{2}b]_r\,[r]\,!} \qquad \text{.. (4.1)}$$

and

$$Es\,(q\,;\, b, t) = \sum_{r=0}^{\infty} \frac{q^{\frac{1}{2}r(r-1)}(-t^2\, q^{\frac{1}{2}-\frac{1}{2}b}/4)^r}{[3/2-\frac{1}{2}b]_r\,[r]\,!}, \qquad \text{...(4.2)}$$

where q is theoretically any uumber, real or complex, called the base. Also

$$[a]_r = [a]\,[a+1]\,[a+2]\,\ldots\,[a+r-1], \qquad [a]_0 = 1,$$

$$[r]\,! = [1]\,[2]\,[3]\,\ldots \qquad \text{and } [a] = (1-q^a)/(1-q).$$

{ See Jackson [5], for example.}

In the present application, the base is taken to be real and restricted to the range $0 < q \leqslant 1$. When $q \to 1$, $Ec\,(q\,;\, b, t)$ and $Es\,(q\,;\, b, t)$ tend to $Ec\,(b, t)$ and $Es\,(b, t)$, respectively. The functions (4.1) and (4.2) are for general values of b linearly independent solutions of a basic analogue of the Bessel-Clifford equation. {See Exton [2] and [3].}

The basic functions $Ec\,(q\,;\, b, t)$ and $Es\,(q\,;\, b, t)$ may be used as a basis for the model (1.1), and it is envisaged that segments of time series will arise in which the corresponding least-squares fit will be further improved by varying q as an additional parameter after first finding the optimum values of λ and b.

The method outlined in this paper constitutes a generalisation of complex demodulation of statistical time-series ; see Bloomfield [1], Chapter 6. An estimate of the spectrum of each segment of the time series under discussion may be built-up by repeated application of the above analysis.

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(*Dedicated to the memory of Professor Arthur Erdélyi*)

THE DISTRIBUTION OF A LINEAR COMBINATION AND THE RATIO OF PRODUCTS OF RANDOM VARIABLES ASSOCIATED WITH THE MULTIVARIABLE H-FUNCTION

By

S. P. Goyal and R. K. Agrawal

*Department of Mathematics, B. V. College of Arts and Science, Banasthali Vidyapith-***304022***, Rajasthan, India*

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ABSTRACT

The present paper deals with the probability density function of a linear combination and the ratio of products of random variables having as their probability density function in terms of the *H*-function of several complex variables, defined by H. M. Srivastava and R. Panda. The results obtained in this paper are quite general and useful in nature. The results established recently by A. M. Mathai and R. K. Saxena, R. K. Saxena and S. P. Dash, and several others, follow as particular cases of our findings.

1. INTRODUCTION

In a large variety of statistical problems such as total time of service required in a medical check-up, problem of inter-live-birth interval, etc., the distribution of a linear combination of random variables plays an important role. Again, the linear combination and the ratio of independent random variables (when density function belongs to the same family) are useful in the theory of sampling distribution. There is a vast litrature on the distribution of linear combination and the ratio of products of random variables, when individual variables are assumed to have particular type of probability density models. For example, Mathai and Saxena [2], Srivastava and Singhal [7], and Saxena and Dash [4] have discussed these problems, where each component variable is assumed to have a density associated with Gauss's ${}_2F_1$—function or Fox's H-function. In this paper, we have studied the aforementioned problems for the probability density function in terms of the multivariable H-function defined by Srivastava and Panda [5, p. 271, Eq. (4.1) *et seq.*]. The probability density function considered here contains (as particular cases) a large variety of such functions introduced in the literature from time to time. Thus our findings will unify and extend the results on linear combination and the ratio of products of random variables studied by several research workers. Indeed, as long as one can find the practical situations, when the introduction of a more general function is justifiable, the generalization can be put to practical use. The technique employed here to derive the results is that of the Laplace transform and its inverse.

The multivariable H-function occurring in this paper is a special case of the general multivariate H-function introduced and studied earlier by Srivastava and Panda ([5] and [6]). The parameters of this function will be displayed in the following contracted notation, which slightly differs from that of Srivastava and Panda [6, p. 130, Eq. (1.3)]:

$$H[x_1,\ldots,x_r] = H^{0,\,n:\,1,\,n_1;\ldots;\,1,\,n_r}_{p,\,q:\,p_1,\,q_1;\ldots;\,p_r,\,q_r}\left[\begin{matrix} x_1 \\ \vdots \\ x_r \end{matrix}\,\middle|\,\begin{matrix} (a_j;\alpha_j',\ldots,\alpha_j^{(r)})_{1,p}: (c_j',\epsilon_j')_{1,p_1};\ldots;(c_j^{(r)},\epsilon_j^{(r)})_{1,p_r} \\ (b_j;\beta_j',\ldots,\beta_j^{(r)})_{1,q}: (0,1),(d_j',\delta_j')_{2,q_1};\ldots;(0,1),(d_j^{(r)},\delta_j^{(r)})_{2,q_r} \end{matrix}\right]$$

$$= (2\pi\omega)^{-r}\int_{L_1}\cdots\int_{L_r}\phi(s_1,\ldots,s_r)\prod_{i=1}^{r}\{\theta_i(s_i)\,\Gamma(-s_i)\,(x_i)^{s_i}\,ds_i\}, \qquad \ldots(1.1)$$

where $\omega = \sqrt{-1}$, and

$$\phi(s_1,\ldots,s_r) = \prod_{j=1}^{n}\Gamma\left(1-a_j+\sum_{i=1}^{r}\alpha_j^{(i)}s_i\right)$$

$$\cdot\left[\prod_{j=n+1}^{p}\Gamma\left(a_j-\sum_{i=1}^{r}\alpha_j^{(i)}s_i\right).\prod_{j=1}^{q}\Gamma\left(1-b_j+\sum_{i=1}^{r}\beta_j^{(i)}s_i\right)\right]^{-1}, \quad \ldots(1.2)$$

$$\theta_i(s_i)=\prod_{j=1}^{n_i}\Gamma\left(1-c_j^{(i)}+\epsilon_j^{(i)}s_i\right)\left[\prod_{j=n_i+1}^{p_i}\Gamma\left(c_j^{(i)}-\epsilon_j^{(i)}s_i\right)\right.$$

$$\left.\prod_{j=2}^{q_i}\Gamma\left(1-d_j^{(i)}+\delta_j^{(i)}s_i\right)\right]^{-1} \quad \ldots(1.3)$$

$$(i=1,\ldots,r)$$

i in the superscript (i) stands for the number of primes, e. g, $b^{(1)}=b'$, $b^{(2)}=b''$, and so on ; $(a_j\,;\alpha_j',\ldots,\alpha_j^{(r)})_{1,p}$ would abbreviate $(a_1;\alpha_1',\ldots,\alpha_1^{(r)})$, $\ldots,(a_p\,;\alpha_p',\ldots,\alpha_p^{(r)})$ and $(c_j\,,\ \epsilon_j)_{n+1,p}$ the $(p-n)$ parmeter squence $(c_{n+1},\epsilon_{n+1}),\ldots,(c_p\,,\epsilon_p)$ for integers n and p such that $0\leqslant n\leqslant p$, and so on.

The conditions of convergence for the mnltiple contour integral (1.1), and other details for the H-function of several variables can be found in the papers by Srivastava and Panda ([5] and [6]).

2. Useful Results

The following results will be required in the course of our analysis :

Result 1

$$H[x_1,\ldots,x_r]=\sum_{v_1=0}^{\infty}\cdots\sum_{v_r=0}^{\infty}\emptyset(v_1,\ldots,v_r)\prod_{i=1}^{r}\left\{\frac{\theta_i(v_i)\ (-x_i)^{v_i}}{v_i\,!}\right\} \quad \ldots(2.1)$$

where $\emptyset(v_1,\ldots,v_r)$ is defined by (1.2) and $\theta_i(v_i)$ is defined by (1.3).

The above result follows easily from a series expansion given by Saxena [3, p. 225, Eq. 4.1)].

Result 2

$$\int_0^\infty \cdots \int_0^\infty \prod_{i=1}^{r} \left\{ (x_i)^{\sigma_i - 1} \exp(-k_i x_i) \right\} H\left[y_1 x_1^{\lambda_1}, \ldots, y_r x_r^{\lambda_r} \right] dx_1 \ldots dx_r$$

$$= \prod_{i=1}^{r} \left\{ (k_i)^{-\sigma_i} \right\} H^{0, n : 1, n_1+1; \ldots; 1, n_r+1}_{p, q : p_1+1, q_1; \ldots; p_r+1, q_r} \left[\begin{matrix} y_1 k_1^{-\lambda_1} \\ \vdots \\ y_r k_r^{-\lambda_r} \end{matrix} \right|$$

$$(a_j ; \alpha_j', \ldots, \alpha_j^{(r)})_{1,p} : (1-\sigma_1, \lambda_1), (c_j', \epsilon_j')_{1,p} ; \ldots ; (1-\sigma_r, \lambda_r),$$

$$(b_j ; \beta_j', \ldots, \beta_j^{(r)})_{1,q} : (0, 1), (d_j', \delta_j')_{2,q_1} ; \ldots ; (0, 1),$$

$$\left. \begin{matrix} (c_j^{(r)}, \epsilon_j^{(r)})_{1,p_r} \\ (d_j^{(r)}, \delta_j^{(r)})_{2,q_r} \end{matrix} \right] = J(k_1, \ldots, k_r ; \sigma_1, \ldots, \sigma_r) \qquad \ldots(2.2)$$

where

$$\min_{1 \leq i \leq r} \{Re(k_i), Re(\sigma_i)\} > 0, \lambda_i > 0, U_i > 0, |\arg y_i| < (1/2) U_i \pi, \qquad \ldots(2.3)$$

and

$$U_i = -\sum_{j=n+1}^{p} \alpha_j^{(i)} - \sum_{j=1}^{q} \beta_j^{(i)} + \sum_{j=1}^{n_i} \epsilon_j^{(i)} - \sum_{j=n_i+1}^{p_i} \epsilon_j^{(i)} - \sum_{j=2}^{q_i} \delta_j^{(i)} + 1 \qquad \ldots(2.4)$$

The evaluation of the above result is quite straightforward, and therefore we omit the details.

Result 3. Let the probability density function

$$f_{x_1, \ldots, x_r}(k_1, \ldots, k_r ; \sigma_1, \ldots, \sigma_r) = \{ J(k_1, \ldots, k_r; \sigma_1, \ldots, \sigma_r)\}^{-1}$$

$$\prod_{i=1}^{r} \left\{ (x_i)^{\sigma_i - 1} \exp(-k_i x_i) \right\} H\left[y_1 x_1^{\lambda_1}, \ldots, y_r x_r^{\lambda_r} \right], \qquad \ldots(2.5)$$

provided that the inequalities given with (2.3) are satisfied, and

$f_{x_1, \ldots x_r}(k_1, \ldots, k_r ; \sigma_1, \ldots, \sigma_r) = 0$, elsewhere.

Since there exists at least one set of parameters for which the function $f_{x_1,\ldots,x_r}(k_1,\ldots,k_r;\sigma_1,\ldots,\sigma_r)$ in (2.5) is non-negative, it is assumed that the parameters are such that

$f_{x_1,\ldots,x_r}(k_1,\ldots,k_r;\sigma_1,\ldots,\sigma_r) \geqslant 0$ for $0 < x_i < \infty$ $(i=1,\ldots,r)$.

It can be remarked here that, on putting $n = p = q = 0$, the function given by (2.5) breaks up into the probability density function involving several Fox's H-functions, which happens to be a particular case of the probability density function considered by Saxena and Dash [4]. Incidentally, the value of C (τ, σ, ξ, d) involved in the function taken by Saxena and Dash ([4], p. 59, Result 3) contains some misprints.

3. Distribution of a Linear Combination of Several Random Variables

The theorem given below gives us the distribution of a linear combination of several independent variables associated with the probability density function defined by (2.5).

Theorem I. *Let* X_i $(i=1,\ldots,r)$ *be the* r *independent random variables, where* X_i *has the probability density function defined by (2.5). Then the probability function* $h(u)$ *of*

$U=\sum_{i=1}^{r} z_i X_i$ *is as follows :*

$$h(u)=\sum_{v_1=0}^{\infty}\cdots\sum_{v_r=0}^{\infty}\phi(v_1,\ldots,v_r)\prod_{i=1}^{r}\left\{\frac{\theta_i(v_i)\,\Gamma(R_i)\,(-y_i)^{v_i}\,(z_i)^{-R_i}}{v_i\,!}\right\}$$

$$\cdot\frac{u^{\sum_{i=1}^{r}(R_i)-1}\,\phi_2\left(R_1,\ldots,R_r;\sum_{i=1}^{r}R_i;-\frac{k_1}{z_1}u,\ldots,-\frac{k_r}{z_r}u\right)}{\Gamma\left(\sum_{i=1}^{r}R_i\right)\,J(k_1,\ldots,k_r;\sigma_1,\ldots,\sigma_r)},$$

where

$$R_i = \sigma_i + \lambda_i v_i \qquad \ldots(3.2)$$

$$\emptyset_2 (c_1, \ldots, c_r ; d ; z_1', \ldots, z_r') = \sum_{m_1=0}^{\infty} \ldots \sum_{m_r=0}^{\infty} \frac{(c_1)_{m_1} \cdots (c_r)_{m_r} (z_1')^{m_1} \cdots (z_r')^{m_r}}{(d)_{m_1+\ldots+m_r} \; m_1! \ldots m_r!}, \quad \ldots (3.3)$$

and $\emptyset (v_1, \ldots, v_r)$ *and* $\theta_i (v_i)$ *are defined by* (1.2) *and* (1.3), *respectively. The density function (3.1) is valid under the following conditions :*

$\lambda_i > 0, \quad \min_{1 \leq i \leq r} \{\mathrm{Re}\,(k_i), \mathrm{Re}\,(\sigma_i)\} > 0, \; |\arg y_i| < (1/2)\, U_i\, \pi, \; (i=1, \ldots, r),$

(U_i is given by (2.4) and the series on the right-hand side of (3.1) *converges absolutely.*

Proof : Let $\bar{\emptyset}(s, u)$ denote the Laplace transform of U. Then

$$\bar{\emptyset}(s, U) = E\,[\exp(-sU)] = E\,[\exp(-s \sum_{i=1}^{r} z_i X_i)]$$

$$= \frac{\int_0^{\infty} \cdots \int_0^{\infty} \prod_{i=1}^{r} \left[\exp\{-(k_i + s z_i) x_i\} (x_i)^{\sigma_i - 1} \right] H \left[y_1 x_1^{\lambda_1}, \ldots, y_r x_r^{\lambda_r} \right] dx_1 \ldots dx_r}{J (k_1, \ldots, k_r ; \sigma_1, \ldots, \sigma_r)}$$

$$= \frac{J (k_1 + s z_1, \ldots, k_r + s z_r ; \sigma_1, \ldots, \sigma_r)}{J (k_1, \ldots, k_r ; \sigma_1, \ldots, \sigma_r)}, \quad \ldots (3.4)$$

where E stands for "Mathematical expectation".

Expanding the multivariable H-function involved in numerator of (3.4) with the help of (2.1), we get, after a little simplification, that

$$\bar{\emptyset}(s, U) = \{ J (k_1, \ldots, k_r ; \sigma_1, \ldots, \sigma_r) \}^{-1} \sum_{v_1=0}^{\infty} \ldots \sum_{v_r=0}^{\infty} \emptyset (v_1, \ldots, v_r)$$

$$\prod_{i=1}^{r} \left\{ \frac{\theta_i (v_i)\, \Gamma (R_i)\, (-y_i)^{v_i} (k_i + s z_i)^{-R_i}}{(v_i)!} \right\} \quad \ldots (3.5)$$

Using a known result [1, p. 238, Eq. (9)] for finding out the inverse Laplace transform of the terms $\prod_{i=1}^{r} (k_i + s z_i)^{-R_i}$, involving s in (3.5), we arrive at the right-hand side of $h(u)$ given by (3.1),

4. Distribution of the Ratio of Products of several Random variables

Let $W = \dfrac{X_1 \ldots X_h}{X_{h+1} \ldots X_r}$ $(1 \leqslant h \leqslant r)$, where X_i $(i=1,\ldots, r)$ be the r independent variables and the probability density function for X_i is given by (2.5). The Theorem given below gives the probability density function $g(w)$ of W.

Theorem 2. *Let*

$$V = \log_e W = \sum_{i=1}^{h} \log_e X_i - \sum_{i=h+1}^{r} \log_e X_i .$$

Then the density function g (w) *is given by*

$$g(w) = \{ J(k_1, \ldots, k_h ; \sigma_1, \ldots \sigma_h) J(k_{h+1}, \ldots, k_r ; \sigma_{h+1}, \ldots, \sigma_r)\}^{-1}$$

$$\sum_{v_1=0}^{\infty} \ldots \sum_{v_r=0}^{\infty} \phi(v_1, \ldots, v_h) \phi(v_{h+1}, \ldots, v_r) \prod_{i=1}^{r} \left\{ \frac{\theta_i (v_i (-y_i)^{v_i} (k_i)^{-R_i}}{v_i !} \right\}$$

$$G^{h, r-h}_{r-h, h}\left(\frac{k_1 \ldots k_h}{k_{h+1} \ldots k_r} W \,\middle|\, \begin{matrix} (1-R_j)_{h+1, r} \\ (R_j)_{1, h} \end{matrix} \right). \qquad \ldots (4.1)$$

where $1 \leqslant h \leqslant r$ *and* R_i *is defined by (3.2). Incidently, the conditions of existenee for* g (w) *are same as those for* h (u) *given by Theorem 1.*

Proof : Suppose $\bar{\phi}(s, V)$ is the Laplace transform of V. Then

$$\bar{\phi}(s, V) = E[\exp(-sV)] = E\left[\prod_{i=1}^{h} (X_i)^{-s} \prod_{i=h+1}^{r} (X_i)^{s} \right]$$

$$= \left[\int_0^{\infty} \cdots \int_0^{\infty} f_{x_1, \ldots, x_h}(k_1, \ldots, k_h ; \sigma_1, \ldots, \sigma_h) \left\{ \prod_{i=1}^{h} (x_i)^{-s} \right\} dx_1 \ldots dx_h \right].$$

$$\cdot \left[\int_0^{\infty} \cdots \int_0^{\infty} f_{x_{h+1}, \ldots, x_r}(k_{h+1}, \ldots, k_r ; \sigma_1, \ldots \sigma_r) \left\{ \prod_{i=h+1}^{r} (x_i)^{s} \right\} dx_{h+1} \ldots dx_r \right]$$

$$= \frac{J(k_1,\ldots,k_h\,;\,\sigma_1^{-s},\ldots,\sigma_h^{-s})\;J(k_{h+1},\ldots,k_r\,;\,\sigma_{h+1}+s,\ldots,\sigma_r+s)}{J(k_1,\ldots,k_h\,;\,\sigma_1,\ldots,\sigma_h)\;\;J(k_{h+1},\ldots,k_r\,;\,\sigma_{h+1},\ldots,\sigma_r)} \qquad \ldots(4.2)$$

Expand the multivariate H-function involved in the numerator of (4.2) with the help of (2.1) and collect the terms involving s, which are

$$\prod_{i=1}^{h} (k_i)^s\,\Gamma(R)-s_i \cdot \prod_{i=h+1}^{r} (k_i^{-s})\,\Gamma(R_i+s)\,. \qquad \ldots(4.3)$$

Now taking the inverse Laplace transform of the expression (4.3) and substituting the value thus obtained in the expanded form of (4.2), we arrive at (4.1).

5. Special Cases

At the outset we should remark that the multivariable H-function defined by (1.1) includes a large variety of elementary special functions involving one or more variables as its particular cases. Thus for probability density function given by (2.5) is quite general in nature and from it all the known statistical distributions, such as generalized beta & gamma distributions, generalized F-distribution, student's t-distribution, normal distribution, exponential distribution, etc, can be derived as specialized or limiting cases of our distribution (2.5). Indeed for all these distributions, the probability density function for U and W can be obtained from (3.1) and (4.1) by suitably specializing the various parameters involved. We, however, prefer to omit the details.

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(*Dedicated to the memory of Professor Arthur Erdélyi*)

REDUCTION FORMULAE FOR SOME TRIPLE HYPERGEOMETRIC FUNCTIONS

By

PER W. KARLSSON

Department of Physics, The Technical University,
DK-2800 Lyngby Denmark

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The functions considered in this paper belong to the class of triple Gaussian hypergeometric functions, i. e., functions that reduce to ${}_2F_1$ when ever exactly one variable is non-zero. Reduction formulae expressing some of these functions in terms of double Gaussian hypergeometric functions (the fourteen functions introduced by Appell and by Horn; *cf.*, *e.g.*, [1, § 5.7.1]), or even simpler functions, will be given. The parameters of the triple series will satisfy *two* conditions and the variables will be $(x, y, -y)$.

To state the results conveniently we shall use a modification of the earlier notations (*cf.* [4] and [5]). For a triple Gaussian hypergeometric series with the general term $C(m, n, p)\, x^m y^n z^p$, imagine the pattern

$$\begin{matrix} m, m\ ;\ n, n\ ;\ p, p \\ -m;\ \ -n;\ \ -p \end{matrix}$$

and the symbol

$$F\left[\begin{matrix} ,\ ;\ ,\ ;\ , \\ ;\ \ ; \end{matrix}\ \middle|\ x\ ;y\ ;z\right]$$

with nine blank spaces for the parameters. Each parameter is now written in the space (s) corresponding to its Pochhammer symbol subscript. An example wiil clarify the principle :

$$(1)\qquad F\left[\begin{matrix} a, a\,;\, b, d\,;\, c, d \\ e\,;\quad a\,;\quad a\end{matrix}\middle|\, x\,;y\,;\,z\right] =$$

$$= \sum_{m,n,p} \frac{(a)_{2m-n-p}\,(b)_n\,(c)_p\,(d)_{n+p}\,x^m\,y^n\,z^p}{(e)_m\, m\,!\,n\,!\,p\,!}.$$

If two parameters are equal, one will be enclosed in square brackets. All summations are from 0 to ∞.

The reduction formulae will be obtained from the result*

$$(2)\quad \sum_{m,n,p} \frac{A\,(m, n+p)\,(b)_n\,(b)_p\,x^m\,y^n\,(-y)^p}{m\,!\,n\,!\,p\,!} =$$

$$= \sum_{m,k} \frac{A\,(m,\,2k)\,(b)_K\,x^m\,y^{2k}}{m\,!\,k\,!},$$

which we applied to some similar problems in a previous paper [3] ; absolute convergence is, of course, assumed. Proofs will merely be outlined; the elementary rules for the Pochhammer symbol are, of course, applied.

The formula (2), which actually implies one parameter condition, is applicable to a few triple Gaussian series (the one considered in (1) is among these). In each case a higher-order double hypergeometric series S is obtained. S may reduce to a double Gaussian hypergeometric series after introduction of a suitable parameter condition; in this way the following formulae (as well as variants having the denominator parameters $2b$, $b+\frac{1}{2}$ replaced by $2b-1$, $b-\frac{1}{2}$) are found :

$$(3)\quad F\left[\begin{matrix} c, d\,;\, a, b\,;\, a, [b] \\ a\,;\quad 2b\,;\quad 2b\end{matrix}\middle|\, x\,;y\,;\,-y\right] = H_7\,(a, c, d, b+\tfrac{1}{2}\,;\,\tfrac{1}{4}y^2, x),$$

$$(4)\quad F\left[\begin{matrix} a, a\,;\, a, b\,;\, a, [b] \\ d\,;\quad 2b;\quad 2b\end{matrix}\middle|\, x\,;y\,;\,-y\right] = F_4\,(\tfrac{1}{2}a, \tfrac{1}{2}+\tfrac{1}{2}a, d, b+\tfrac{1}{2};\,4x, y^2),$$

$$(5)\quad F\left[\begin{matrix} a, c\,;\, a, b\,;\, a, [b] \\ d\,;\quad 2b\,;\quad 2b\end{matrix}\middle|\, x\,;y\,;\,-y\right] = H_4\,(a, c, b+\tfrac{1}{2}, d\,;\,\tfrac{1}{4}y^2, x).$$

---

* Notice that (2) is an obvious triple-series analogue of an identity due to Srivastava [6, p. 297 (17)] who indeed gave several other classes of reduction formulas involving double series with essentially arbitrary terms.

In other cases, the double series is written in the form

$$(6)\qquad S=\sum_k B(k)\, y^{2k} F(k,x),$$

where F is a ${}_2F_1$ with variable depending upon x and parameters involving k. It may reduce to power functions after introduction of one parameter condition on account of [1, § 2.8 (5), (6) . This procedure leads to the following reduction formulae :

$$(7)\quad F\left[\begin{matrix} a,\, a\,;\, a,\, b\,;\, a,\, [b] \\ \tfrac{1}{2};\quad d\,;\quad d \end{matrix}\;\middle|\; x\,;\, y\,;\, -y\right] =$$

$$= \tfrac{1}{2}(1+2\sqrt{x})^{-a}\, {}_3F_2(\tfrac{1}{2}a,\, \tfrac{1}{2}+\tfrac{1}{2}a,\, b\,;\, \tfrac{1}{2}d,\, \tfrac{1}{2}+\tfrac{1}{2}d\,;\, y^2\,(1+2\sqrt{x})^{-2})$$
$$+\tfrac{1}{2}(1-2\sqrt{x})^{-a}\, {}_3F_2(\tfrac{1}{2}a,\, \tfrac{1}{2}+\tfrac{1}{2}a,\, b\,;\, \tfrac{1}{2}d,\, \tfrac{1}{2}+\tfrac{1}{2}d\,;\, y^2\,(1-2\sqrt{x})^{-2}),$$

$$(8)\quad F\left[\begin{matrix} a,\, a\,;\, d,\, b\,;\, d,\, [b] \\ \tfrac{1}{2};\quad a\,;\quad a \end{matrix}\;\middle|\; x\,;\, y\,;\, -y\right] =$$

$$= \tfrac{1}{2}(1+2\sqrt{x})^{-a}\, {}_3F_2(\tfrac{1}{2}d,\, \tfrac{1}{2}+\tfrac{1}{2}d,\, b\,;\, \tfrac{1}{2}-\tfrac{1}{2}a,\, 1-\tfrac{1}{2}a\,;\, y^2(1+2\sqrt{x})^2)$$
$$+\tfrac{1}{2}(1-2\sqrt{x})^{-a}\, {}_3F_2(\tfrac{1}{2}d,\, \tfrac{1}{2}+\tfrac{1}{2}d,\, b\,;\, \tfrac{1}{2}-\tfrac{1}{2}a,\, 1-\tfrac{1}{2}a\,;\, y^2(1-2\sqrt{x})^2),$$

$$(9)\quad F\left[\begin{matrix} a,\, a\,;\, 1-a,\, b\,;\, 1-a,\, [b] \\ 1-a\,;\quad a\,;\quad a \end{matrix}\;\middle|\; x\,;\, y\,;\, -y\right] =$$

$$=(\tfrac{1}{2}+\tfrac{1}{2}\sqrt{(1+4x)}\,)^{1-a}\,(1-y^2\,(\tfrac{1}{2}+\tfrac{1}{2}\sqrt{(1+4x)}\,)^2)^{-b}/\sqrt{(1+4x)},$$

$$(10)\quad F\left[\begin{matrix} a,\, a;\, -a,\, b\,;\, -a,\, [b] \\ -a\,;\quad a\,;\quad a \end{matrix}\;\middle|\; x\,;\, y\,;\, -y\right] =$$

$$= (\tfrac{1}{2}+\tfrac{1}{2}\sqrt{(1+4x)}\,)^{-a}\, {}_2F_1\,(-\tfrac{1}{2}a,\, b\,;\, 1-\tfrac{1}{2}a;\, y^2(\tfrac{1}{2}+\tfrac{1}{2}\sqrt{(1+4x)})^2),$$

$$(11)\quad F\left[\begin{matrix} a,\, a\,;\, a,\, b\,;\, a,\, [b] \\ 1+a;\, 1+a;\, 1+a \end{matrix}\;\middle|\; x;\, y;\, -y\right] =$$

$$=(\tfrac{1}{2}+\tfrac{1}{2}\sqrt{(1-4x)}\,)^{-a}\, {}_2F_1\,(\tfrac{1}{2}a,\, b\,;\, 1+\tfrac{1}{2}a;\, 4y^2\,(1+\sqrt{(1-4x)}\,)^{-2}).$$

Finally, to prove the formula

$$(12)\quad F\left[\begin{matrix} a,\, c\,;\, d,\, 1-\tfrac{1}{2}a;\, d,\, [1-\tfrac{1}{2}a] \\ d\,;\quad a\,;\quad a \end{matrix}\;\middle|\; x;\, y\,;\, -y\right] =$$
$$=(1+x)^{-c}\, H_7(d,\, 1-a-d,\, c,\, \tfrac{1}{2}-\tfrac{1}{2}a;\, \tfrac{1}{4}\,y^2,\, -x/1+x),$$

and the variant obtained by interchanging the parameters $1-\frac{1}{2}a$ and $\frac{1}{2}-\frac{1}{2}a$, we perform a linear transformation of $F(k,\varkappa)$ in (6) before introducing the second parameter condition.

Some of the functions considered have occurred in the literature. Thus, we have $D^{1,3}_{(3)}$ in (3), F_G in (5), ${}^{(1)}H^{(3)}_{3}$ in (11), and G_B in (12). (Definitions of these functions can be found in [2, Ch. 3].)

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A NOTE ON THE NUMBER OF DERANGEMENTS

By

J. -L. Lavoie

Départment de Mathématiques, Université Laval,
Québec City, P. Q., Canada G1K 7P4

and

Richard Tremblay

Departments of Mathematics, Royal Military College of Canada
Kingston, Ontario, Canada K7L 2W3

(Received : May 4, 1980)

Numerical evidences led Prof. G. Kreweras* to conjecture that

$$(1)\quad \sum_{k=0}^{n} (-1)^{k+1} \frac{d\,(n+k+2,\, k+1)}{n+k+2} = \frac{B_{n+1}}{n+1}\,,\ n=0,\, 1,\ldots,$$

where $d\,(n, k)$ is the number of derangements of N, $|N| = n$, with k orbits, or permutations with k cycles of length $\geqslant 2$. These numbers have the generating function [1, p. 256]

$$(2)\qquad e^{-ut}\,(1-t)^{-u} = \sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} d\,(n,\, k)\, u^k\, \frac{t^n}{n!}\,.$$

The B_k are the Bernoulli numbers. Let $B_k^{(n)}\,(x)$ be the Bernoulli polynomials of order n, [2, p. 124ff]. Special cases of these polynomials appear in a variety of contexts. We have

$$(3)\qquad B_k \equiv B_k^{(1)}(0) = (-1)^k\, B_k^{(1)}(1),\ k = 0,\, 1,\ldots,$$

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* Institut de Statistique des Universités de Paris, Paris, France. Private communication.

and, in a moment, we shall use the expansion

$$\left[\frac{t}{ln(1+t)}\right]^n = \sum_{k=0}^{\infty} B_K^{(k-n+1)}(1)\ \frac{t^k}{k!}.$$

It is easy to see that $d(0,0)=1$, and

$d(n,0) = 0$ for $n = 1, 2, \ldots$; hence the right-hand side of (2) can be written

$$1+\sum_{n=0}^{\infty}\sum_{k=0}^{n} d(n+k+2, k+1)\ u^{k+1}\ \frac{t^{n+k+2}}{(n+k+2)!}$$

and replacing u by u/t, we have

$$(4)\quad \sum_{k=0}^{n} \frac{d(n+k+2, k+1)}{(n+k+2)!} u^{k+1} = \frac{1}{(n+1)!}\ D_t^{n+1}\left[e^{-u}(1-t)^{-u,t}\right]\Big|_{t=0},$$

$$n = 0, 1. 2, \ldots,$$

using Taylor's theorem.

In order to establish (1), consider the operational formula obtained by replacing u by $D_x \equiv d/dx$ in the last relation. Operating each member of the resulting equation on x^{-n-1}, we have

$$\sum_{k=0}^{n} \frac{d(n+k+2, k+1)}{(n+k+2)!}\cdot (-1)^{k+1}\ \frac{(n+k+1)!}{n!}\ x^{-n-k-2}$$

$$= \frac{1}{(n+1)!}\ D_t^{n+1}\left[e^{-\left(1+\frac{ln(1-t)}{t}\right)D_x} x^{-n-1}\right]\Bigg|_{t=0}$$

$$= \frac{1}{(n+1)!}\ D_t^{n+1}\left[\left(x-1-\frac{ln(1-t)}{t}\right)^{-n-1}\right]\Big|_{t=0}.$$

Thus, for $x=1$,

$$\sum_{k=0}^{n} (-1)^{k+1}\frac{d(n+k+2, k+1)}{n+k+2}$$

$$= \frac{1}{n+1} D_t^{n+1} \left[\frac{-t}{ln(1-t)}\right]^{n+1}\Bigg|_{t=0}$$

$$= \frac{1}{n+1} \sum_{k=n+1}^{\infty} (-1)^k B_k^{(k-n)}(1) \frac{t^{k-n-1}}{(k-n-1)!}\Bigg|_{t=0}$$

$$= \frac{(-1)^{n+1}}{n+1} B_{n+1}^{(1)}(1) = \frac{B_{n+1}}{n+1}.$$

This completes our derivation of (1).

Next, differentiating (2) with respect to t, we have

$$\sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} d(n+2, k+1) \frac{u^k t^n}{(n+1)!}$$

$$= e^{-ut} (1-t)^{-u-1}$$

$$= (1-t)^{-1} \sum_{r=0}^{\infty} \sum_{k=0}^{[r/2]} d(r, k) \frac{u^k t^r}{r!}$$

$$= \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \sum_{k=0}^{[r/2]} d(r, k) \frac{u^k t^{n+r}}{r!}$$

$$= \sum_{n=0}^{\infty} \sum_{r=0}^{n} \sum_{k=0}^{[r/2]} d(r, k) \frac{u^k t^n}{r!}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} \sum_{r=2k}^{n} d(r, k) \frac{u^K t^n}{r!}.$$

Hence

$$(5) \qquad d(n+2, k+1) = (n+1)! \sum_{r=2k}^{n} \frac{d(r, k)}{r!}, \; n = 0, 1, \ldots .$$

Iterating this recurrence, we find the explicit representation :

$$(6) \quad d(n+2, k+1)=(n+1)!\sum_{t_1=0}^{n-2k}\ \sum_{t_2=0}^{t_1}\cdots\sum_{t_k=0}^{t_{k-1}} \frac{1}{(t_1+2k)(t_2+2k-2)\ldots(t_k+2)},$$

for $n = 2k, 2k+1, \ldots$.

A similar representation can be obtained for the Bernoulli nnmber by merely substituting the last expression in (1).

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ON THE RATIO OF TWO GAMMA FUNCTIONS

By

YUDELL L. LUKE

Department of Mathematics, University of Missouri
Kansas City, Missouri 64110, U. S. A.

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We are concerned with evaluation of $\Gamma(z+a)\,/\,\Gamma(z+b)$ where a and b are fixed and z is large. In view of the property $\Gamma(y+1)=y\Gamma(y)$, we can always arrange matters so that the essential part of the calculation allows the pertinent variable to be sufficiently large in some sense. This is readily seen by writing

$$\frac{\Gamma(x+a)}{\Gamma(x+b)}=\frac{(x+b)_m}{(x+a)_m}\,\frac{\Gamma(z+a)}{\Gamma(z+b)},\; z = x + m, \tag{1}$$

where m is a positive integer to insure that z is sufficiently large. Further in the case of real variable and parameter we can take $a=0$ and $0<b<1$. See later remarks. Considerable information in this topic will be found in the volumbes by Luke [1, 2].

In [1951] Tricomi and Erdélyi [3] gave the asymptotic expansion

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} = z^{a-b}\sum_{k=0}^{N-1}\frac{(-)^k\,B_k^{(c)}(a)\,(b-a)_k\,z^{-k}}{k\,!}$$

$$+z^{a-b}\,O(\,z^{-N}\,), c = a-b+1,$$

$$|z|\rightarrow\infty,\; |\,arg\,(z+a)\,| \leqslant \pi-\epsilon,\; \epsilon>0, \tag{2}$$

where $B_k^{(c)}\,(a)$ is the generalized Bernoulli polynomial. This expansion is noted in the Luke references. In rather recent times, several workers have dealt essentially with the special case $a=0$, $b=\frac{1}{2}$ without noting the Tricomi-Erdélyi expansion.

Consider

$$h(n)=\frac{(2\pi)^{\frac{1}{2}}\,(2n+3)^{\frac{1}{2}}\,\Gamma(n+2)}{2^{n+2}\,\Gamma(n+{}^{5}/_{2})}=\frac{\pi^{\frac{1}{2}}}{2^{n+1}}\,H(n). \tag{3}$$

Phillips and Sahney [4] give the first four terms in the asymptotic expansion of $H(n)$. Their result is but a special case of (2) with $z=n+{}^{3}/_{2}$, $a=\frac{1}{2}$, $b=1$ and with z^{-k} in the descending series portion of (2) expanded in descending powers of n. They also showed that the first two (three) terms of this series gives a lower (upper bound) for $H(n)$.

Bowman and Shenton [5] gave the first 16 coefficients in the asymptotic expansion for

$$g(m)=\frac{(m/2)^{\frac{1}{2}}\,\Gamma(m/2)}{\Gamma(m/2+\frac{1}{2})} \tag{4}$$

and the corresponding coefficients in the continued J-fraction. They note that these representations go back to Stieltjes, but they do not seem to recognize that their asymptotic expansion is a special case of (2) with $a=0$, $b=\frac{1}{2}$.

All of this is a prelude to the point that all the expansions noted above are inferior to a result of Fields [6] who proved that

$$\frac{\Gamma(z+a)}{\Gamma(z+b)}=\omega^{a-b}\sum_{k=0}^{N}\frac{B_{2k}^{(2\rho)}(\rho)\,(1-2\rho)_{2k}\,\omega^{-2k}}{(2k)!}$$

$$+\omega^{a-b}\,O(\omega^{-2N}),\ \omega=z+a-\rho,$$

$$2\rho=1+a-b,\ |z|\to\infty,\ |\arg(z+a)|\leqslant\pi-\varepsilon,\ \varepsilon>0. \tag{5}$$

Further, we will show that certain Padé fractions based on the series portion in (5) give rise to upper and lower bounds which are also quite powerful and certainly supperior to the inequalit previously noted. (In this connection, see the concluding paragraph of this paper.) It appears that (5) was also unknown to the above authors. It is also recorded in [1, 2] and in [1], some expressions for early Bernoulli polynomials $B_{2k}^{(2\rho)}(\rho)$ are posted. Further polynomials can be found using a recursion formula which is also given in [1, 2]. In a follow up paper Fields [7] gave an improved order estimate for the remainder. Note that (5) is an even series in ω^{-1} and it is this feature which makes (5) more powerful than (2). In the following, we give further remarks based on heuristic evidence to support this contention. If one is committed to the use of (2), and the variable and parameters are real, one can always adjust the variable so that in effect $a=0$ and $0<b<1$ whence $0<c<1$. In this event, the series in (2) is not alternating. Indeed, it can be shown that $B_k^{(c)}(0)$ vanishes for $k=2$, 4 and 6 for a value of c, $0<c<1$. On the other hand, in the use of (5), with real variable and parameters, simple adjustments can be made so that in essence, $0<\rho<\frac{1}{2}$. In this situation $b_{2k}^{(2\rho)}$ (∂) is positive (negative) when k is even (odd). Thus the series in (5) is alternating and so we get lower and upper bounds for N even odd respectively.

To illustrate the relative merits of (2) and (5), we use the model $a=\frac{1}{2}$, $b=1$ whence $\rho=\frac{1}{4}$.

From (2),

$$U(z)=\frac{\Gamma(z+\frac{1}{2})}{\Gamma(z+1)}=z^{-\frac{1}{2}}V_N(z)+O(z^{-N-\frac{1}{2}}),$$

$$V_N(z)=\sum_{k=0}^{N-1} a_k z^{-k}, \tag{6}$$

where for example $a_0=1$, $a_1=-1/8$, $a_2=1/128$, and $a_3=5/1024$. From (5)

$$U(z)=\frac{\Gamma(z+\frac{1}{2})}{\Gamma(z+1)}=2y^{-\frac{1}{2}}W_N(z)+O(z^{-2N-\frac{1}{2}}),$$

$$W_N(z)=\sum_{k=0}^{N-1}(-)^k b_k y^{-2k}, \quad y=4z+1, \tag{7}$$

where for example $b_0=1$, $b_1=1/4$, $b_2=21/32$, $b_3=671/128$ and $b_4=180323/2048$. The coefficients b_5, b_6, and b_7 are given in [1, 2]. In the following tables, we show computations for $U(z)$ using the above approximations for $z=3$ and $z=4$.

$z = 3, U(z) = .55389\ 18284$

N	$z^{-1/2}\,V_N(z)$	Error	$2y^{-1/2}\,W_N(z)$	Error
1	.57735 02692	-.235 (-1)	.55470 01962	-.808 (-3)
2	.55329 40080	.598 (-3)	.55387 96338	.122 (-4)
3	.55379 51801	.976 (-4)	.55389 23792	-·551 (-6)
4	.55389 95909	-.777 (-5)	.55389 17768	.516 (-7)

$z = 4, U(z) = \cdot 48465\ 53499$

N	$z^{-1/2}\,V_N(z)$	Error	$2y^{-1/2}\,W_N(z)$	Error
1	.5	-.154 (-1)	·48507 12501	-.416 (-3)
2	.48437 5	.280 (-3)	.48465 16383	.371 (-5)
3	.48461 91406	.362 (-4)	.48465 54496	-.997 (-7)
4	.48465 72876	-.194 (-5)	.48465 53443	.56 (-8)

The superiority of (7) is manifest. Note that the errors in the approximants from (7) alternate in sign as N sncreases and so we get the bounds as previously noted.

We now consider $W_N(z)$. Let $U_{n,n}$ be the main diagonal Padé approximant of order n. Let $U_{n,n+1}$ be the first subdiagonal Padé approximant. In the former case, the numerator and denominator are polynomials in $1/y^2$ of degree n, while in the latter case, the numerator and denominator are polynomials in $1/y^2$ of degree n and $n+1$ respectively. We find it convenient to write the equivalent forms

$$U_{0,0}(z) = 1,\ U_{0,1}(z) = \frac{4y^2}{4y^2+1},\ U_{1,1}(z) = \frac{8y^2+19}{8y^2+21},$$

$$U_{1,2}(z) = \frac{608y^4+5048y^2}{608y^4+5200y^2+901},$$

$$U_{2,2}(z) = \frac{57664y^4+1202712y^2+1719379}{57664y^4+1217128y^2+1985819}. \tag{8}$$

For $z>0$, we have the inequalities

$$U_{n,n+1}(z) < (y^{1/2}/2)\,U(z) < U_{n,n}(z). \tag{9}$$

The following tables illustrate use of the Padé approximants for $z = 1\,(1)\,4$. Calculations for $2y^{-1/2}\,U_{2,3}\,(z)$ are not provided.

$z = 1,\; U(z) = .88622\;69255$

n	$2y^{-1/2}\,U_{n,n}\,(z)$	Error	$2y^{-1/2}\,U_{n,n+1}\,(z)$	Error
0	.89442 71910	–.820 (–2)	.88557 14762	.656 (–3)
1	.88633 28273	–.106 (–3)	.88619 72165	.297 (–4)
2	.88623 68357	–.791 (–5)	———	———

$z = 2,\; U(z) = .66467\;01941$

n	$2y^{-1/2}\,U_{n,n}(z)$	Error	$2y^{-1/2}\,U_{n,n+1}\,(z)$	Error
0	.66666 66667	–.200 (–2)	.66461 53847	.547 (–3)
1	.66467 36423	–.345 (–5)	.66466 97750	.417 (–6)
2	.66467 02639	–.698 (–7)	———	———

$z = 3,\; U(z) = .55389\;18284$

n	$2y^{-1/2}\,U_{n,n}\,(z)$	Error	$2y^{-1/2}\,U_{n,n+1}\,(z)$	Error
0	.55470 01962	–.808 (–3)	.55388 08459	.110 (–4)
1	.55389 21843	–.156 (–6)	.55389 18052	.232 (–7)
2	.55389 18306	–.22 (–8)	———	———

$z = 4,\; U(z) = .48465\;53499$

n	$2y^{-1/2}\,U_{n,n}\,(z)$	Error	$2y^{-1/2}\,U_{n,n+1}(z)$	Error
0	.48507 12501	–.416 (–3)	.48465 20010	.335 (–5)
1	.48465 54153	–.654 (–7)	.48465 53472	.27 (–8)
2	.48465 53500	–.1 (–9)	———	———

Neglecting the remainder in (7), we can rearrange this expression to read

$$\pi \sim \frac{4}{y}\left\{\frac{2^{2z}\,\Gamma^2\,(z+1)}{\Gamma\,(2z+1)}\right\}^2 \{W_N(z)\}^2 \tag{10}$$

If z is a positive integer, we can evaluate the right hand side and so have an approximation for π for each N. Equation (6) can be treated in a similar fashion. Numerics showing the superiority of (7) will be found in [1, 2].

In a previous paper (8), I noted the works of numerous authors concerning two sided inequalities for $\Gamma(z+1)/\Gamma(z+\frac{1}{2})$. Further inequalities for the latter have been given by Slavic [9] and Shafer [10]. For the same number of terms, the inequalities noted in (8, 9) are much sharper than any others known to me,

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(Dedicated to the memory of Professor Arthur Erdélyi)

ON A NEW INTEGRAL TRANSFORM. I

By

S. L. Kalla and Alfredo Villalobos

Division de Postgrado, Facultad de Ingenieria,
Universidad del Zulia, Maracaibo, Venezuela

(Received : June 12, 1980)

ABSTRACT

In the present paper, we introduce a new integral transform, involving Bessel functions as kernel. Inversion formula is established and some properties are given. The transform can be used to solve certain class of mixed boundary value problems.

1. Definition and Inversion Formula We consider the Bessel's differential equation [3]

(1) $$x^2 y'' + xy' + (\lambda^2 x^2 - \nu^2) y = 0$$

with the conditions

(2) $$y(a) = 0, \quad y(b) + h y'(b) = 0.$$

The general solution of (1) can be written as

(3) $$y(x) = C_1 J_\nu(\lambda x) + C_2 y_\nu(\lambda x)$$

where C_1, C_2 are arbitrary constants, and $J_\nu(x)$ and $y_\nu(x)$ are the Bessel's functions of first and second kind respectively.

We want to obtain solutions of (1) which satisfies the conditions (2). Hence, we have

(4) $$C_1 J_\nu(\lambda a) + C_2 y_\nu(\lambda a) = 0$$

(5) $$C_1 \left[J_\nu(\lambda b) + h\,\lambda\, J'_\nu(\lambda b) \right] + C_2 \left[y_\nu(\lambda b) + h\,\lambda\, y'_\nu(\lambda b) \right] = 0.$$

From (5) and (6), we can deduce

(6) $$\frac{C_1}{C_2} = -\frac{y_\nu(\lambda a)}{J_\nu(\lambda a)} = -\frac{\left[y_\nu(\lambda b) + h\,\lambda\, y'_\nu(\lambda b) \right]}{\left[y_\nu(\lambda b) + h\,\lambda\, J'_\nu(\lambda b) \right]}.$$

Thus, the function given by (3) is the solution of the equation (1), with conditions (2), if λ is a root of the transcendental equation

(7) $$J_\nu(\lambda a) \left[y_\nu(\lambda b) + h\lambda\, y'_\nu(\lambda b) \right] - y_\nu(\lambda a) \left[J_\nu(\lambda b) + h\,\lambda\, J'_\nu(\lambda b) \right] = 0.$$

Introducing the following notations,

$$A_\nu(\lambda x) = J_\nu(\lambda x) + h\,\lambda\, J'_\nu(\lambda x)\,;$$

$$B_\nu(\lambda x) = y_\nu(\lambda x) + h\,\lambda\, y'_\nu(\lambda x)$$

(7) can be written as

(8) $$J_\nu(\lambda a)\, B_\nu(\lambda b) - y_\nu(\lambda a)\, A_\nu(\lambda b) = 0.$$

Let λ_i ($i = 1, 2, 3, \ldots$) be the positive roots of the equation (8). Then, from (4) and (5), we have

$$y_i(x) = \frac{C_1}{y_\nu(\lambda_i a)} \left[J_\nu(\lambda_i x)\, y_\nu(\lambda_i a) - y_\nu(\lambda_i x)\, J_\nu(\lambda_i a) \right]$$

(9) $$y_i(x) = \frac{C_1}{B_\nu(\lambda_i b)} \left[J_\nu(\lambda_i x)\, B_\nu(\lambda_i b) - y_\nu(\lambda_i x)\, A_\nu(\lambda_i b) \right].$$

Then the following functions are the solutions of the equation (1), with conditions (2) :

(10) $$C_\nu(\lambda_i x) = \left[y_\nu(\lambda_i a) + B_\nu(\lambda_j b) \right] J_\nu(\lambda_i x) - \left[J_\nu(\lambda_i a) + A_\nu(\lambda_i b) \right] y_\nu(\lambda_i x).$$

Now according the theory of Sturm-Liouville [1], the functions of the system (10) are orthogonal on the interval $[a, b]$ with weight function x, that is

(11) $$\int_a^b x\, C_\nu(\lambda_i x)\, C_\nu(\lambda_j x)\, dx = 0, \quad i \neq j$$

(12) $$\int_a^b x\, C^2_\nu(\lambda_i x)\, dx = \| C_\nu(\lambda_i x) \|^2.$$

If a function $f(x)$ and its first derivative are piecewise continuous on the interval $[a, b]$, then the relation

$$T[f(x), a, b, \nu; \lambda_i] = \bar{f}_\nu(\lambda_i) = \int_a^b x f(x) C_\nu(\lambda_i x)\, dx$$

defines an integral transform, where λ_i are the positive roots of the equation (8). To obtain the inversion formula, let

(13)
$$f(x) = \sum_{i=1}^{\infty} a_i C_\nu(\lambda_i x).$$

Multiplying both sides by $x C_\nu(\lambda_j x)$, (j ; *fixed*) integrating with respect to x between a and b, we get

(14)
$$a_i = \frac{\int_a^b x f(x) C_\nu(\lambda_i x)\, dx}{\| C_\nu(\lambda_i x) \|^2} = \frac{\bar{f}_\nu(\lambda_i)}{\| C_\nu(\lambda_i x) \|^2}.$$
$$i = 1, 2, 3, \ldots$$

Hence

(15)
$$f(x) = \sum_{i=1}^{\infty} \frac{\bar{f}_\nu(\lambda_i)}{\| C_\nu(\lambda_i x) \|^2} C_\nu(\lambda_i x)$$

where the summation is taken over all the positive roots of the equation (8).

Using some well known properties of the Bessel functions [2, p. 634, 968, 969] we can easily derive

(16)
$$\| C_\nu(\lambda_i x) \|^2 = \tfrac{1}{2} M^2(\lambda_i, a, b) \{ b^2 P(\lambda_i; b; \nu) - a^2 P(\lambda_i; a; \nu)\}$$
$$- M(\lambda_i; a, b) N(\lambda_i, a, b) \{ b^2 Q(\lambda_i, b, \nu) - a^2 Q(\lambda_i, a, \nu) \} +$$
$$+ \tfrac{1}{2} N^2(\lambda_i, a, b) \{ b^2 R(\lambda_i, b, \nu) - a^2 R(\lambda_i, a, \nu)\}$$

where

$$M(\lambda_i, a, b) = y_\nu(\lambda_i a) + B_\nu(\lambda_i b)$$

$$N(\lambda_i, a, b) = J_\nu(\lambda_i a) + A_\nu(\lambda_i b)$$

$$P(\lambda_i, \mu; \nu) = J^2_\nu(\lambda_i \mu) - J_{\nu-1}(\lambda_i \mu) J_{\nu+1}(\lambda_i \mu)$$

$$Q(\lambda_i, \mu; \nu) = J'_\nu(\lambda_i \mu)\, y_{\nu-1}(\lambda_i \mu) - \frac{1}{\lambda_i \mu} J_{\nu-1}(\lambda_i \mu)\, y_\nu(\lambda_i \mu)$$

$$-J'_{\nu-1}(\lambda_i \mu)\, y_\nu(\lambda_i \mu)$$

(17) $$R(\lambda_i, \mu; \nu) = y_\nu^2(\lambda_i \mu) - y_{\nu-1}(\lambda_i \mu)\, y_{\nu+1}(\lambda_i \mu)$$

$(\mu = a, b)$.

2. Some Properties of the Transform. The following propcrties can be easily varified from the definition of the transform

(18) $$T[\alpha f(x) + \beta g(x), a, b, \nu; \lambda_i] = \alpha T[f(x), a, b, \nu; \lambda_i] + \beta T[g(x), a, b, \nu; \lambda_i]$$

(19) $$T\left[f(ax), a, b, \nu; \lambda_i\right] = \frac{1}{a^2} T\left[f(x), a\alpha, b\alpha, \nu; \frac{\lambda_i}{a}\right].$$

Transform of $g(x) = \dfrac{d^2 f}{dx^2} + \dfrac{1}{x}\dfrac{df}{dx} - \dfrac{\nu^2}{x^2} f$

Let

(20) $$I = \int_a^b x\left(f''(x) + \frac{1}{x} f'(x)\right) C_\nu(\lambda_i x)\, dx$$

$$= \int_a^b x f''(x)\, C_\nu(\lambda_i x)\, dx + \int_a^b f'(x)\, C_\nu(\lambda_i x)\, dx$$

$$= \left\{ x\, C_\nu(\lambda_i x) f'(x) \right\}_a^b - \lambda_i \int_a^b x f'(x)\, C'_\nu(\lambda_i x)\, dx$$

$$= \left\{ x\left[C_\nu(\lambda_i x) f'(x) - \lambda_i C'_\nu(\lambda_i x) f(x) \right] \right\}_a^b$$

$$+ \int_a^b x^{-1} \left[\lambda_i^2 x^2 C''_\nu(\lambda_i x) + (\lambda_i x) C'_\nu(\lambda_i x) \right] f(x)\, dx.$$

As the functions $C_\nu(\lambda_i x)$ satisfies the Bessel's differential equation we have

(21) $$\bar{g}_\nu(\lambda_i) = \int_a^b x\left(f'' + \frac{1}{x} f' - \frac{\nu^2}{x^2} f\right) C_\nu(\lambda_i x)\, dx$$

$$= \left\{ x \left[C_\nu (\lambda_i x) f'(x) - \lambda_i C'_\nu (\lambda_i x) f(x) \right] \right\}_a^b - \lambda_i^2 \bar{f}_\nu(\lambda_i),$$

which, on simplification gives

$$\bar{g}_\nu(\lambda_i) = \frac{b}{h} C_\nu(\lambda_i b) \left[f + h \frac{df}{dx} \right]_{x=b} + \lambda_i a C'_\nu(\lambda_i a) f(a) - \lambda^2 \bar{f}_\nu(\lambda_i)$$

(22)

Transform of x^ν. From the definition we have

$$T[x^\nu, a, b, \nu; \lambda_i] = \int_a^b x^{\nu+1} C_\nu(\lambda_i x)\, dx.$$

Using the result [2, p. 634]

$$\int x^{\rho+1} Z_\rho(x)\, dx = x^{\rho+1} Z_{\rho+1}(x)$$

where $Z_\rho(x)$ is one of the Bessel functions, and after a little simplification, we obtain

$$(23) \qquad T[x^\nu, a, b, \nu; \lambda_i] = \frac{b^{\nu+1}}{\lambda_i^2} \left(\frac{\nu}{b} + \frac{1}{h} \right) C_\nu(\lambda_i b) + \frac{a^{\nu+1}}{\lambda_i} C'_\nu(\lambda_i a)$$

Transform of a constant. We can easily derive

$$(24) \qquad T[a, a, b, 0; \lambda_i] = \frac{a}{\lambda_i^2} \left[\frac{b}{h} C_0(\lambda_i b) + a \lambda_i C'_0(\lambda_i a) \right.$$

3. Applications. The transform introduced in the previous sections can be used to solve a certain class of mixed boundary value problems. For example, the transform can be applied in problems of conduction of heat in hollow cylinders, when one lateral face is kept at a prescribed temperature, while the other radiates heat in the surrounding medium. A systematic use of this transform along with some other suitable transform can be used to solve problems of hollow cylinders (finite, semi-infinite or infinite).

In the second part of this paper the authors propose to solve some problems of conduction of heat in concentric hollow cylinders.

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(*Dedicated to the memory of Professor Arthur Erdélyi*)

A TRANSFORMATION FORMULA FOR A GENERAL HYPERGEOMETRIC FUNCTION OF THREE VARIABLES

By

S. P. Chhabra and K. C. Rusia

Department of Mathematics, Government Engineering College, Ujjain - 456009, M. P., India

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1. INTRODUCTION

The aim of this note is to establish a transformation formula involving the general hypergeometric function $F^{(3)}\,[x, y, z]$ of three variables introduced earlier by Srivastava [7]. Due to the very general nature of the transformation formula obtained here, a number of results (known as well as new) follows as particular cases.

The hypergeometric function $F^{(3)}\,[x, y, z]$ of three variables has been defined by Srivastava in the form (see [7], p. 428) :

$$(1) \qquad F^{(3)}\left[\begin{matrix}(a)::(b);(b');(b''):(c);(c');(c'');\\ (f)::(g);(g');(g''):(h);(h');(h'');\end{matrix}\; x, y, z\right]$$

$$= \sum_{m,n,p=0}^{\infty} \frac{((a))_{m+n+p}\,((b))_{m+n}\,((b'))_{n+p}\,((b''))_{p+m}\,((c))_m\,((c'))_n}{((f))_{m+n+p}\,((g))_{m+n}\,((g'))_{n+p}\,((g''))_{p+m}\,((h))_m\,((h'))_n} \cdot \frac{((c''))_p\, x^m\, y^n\, z^p}{((h''))_p\, m!\, n!\, p!},$$

which is valid under certain restrictions on the parameters, etc. Here (a) is taken to denote the sequence of A parameters $a_1, a_2, .., a_A$, that is,

unless otherwise stated, there are A of the a parameters, B of the b parameters, and so on. Thus $((a))_m$ is to be interpreted as $\prod_{j=1}^{A} (a_j)_m$; with similar interpretations for $((b))_m$, etc. Further $(a)_m = \Gamma(a+m) / \Gamma(a)$.
An empty product in (1) is to be treated as unity.

2. Main Result. We establish the following transformation formula :

(2) $$(1-x)^{-\lambda} (1-y)^{-\mu} (1-z)^{-\nu} F^{(3)} \left[\begin{matrix} (a) :: (b); (b'); (b'') : (c); (c'); (c''); \\ (f) :: (g); (g'); (g'') : (h); (h'); (h''); \end{matrix} \left(\frac{-\alpha\, x}{(1-x)(1-y)(1-z)} \right)^{\delta}, \left(\frac{-\beta\, y}{(1-x)(1-y)(1-z)} \right)^{\delta}, \left(\frac{-\gamma\, z}{(1-x)(1-y)(1-z)} \right)^{\delta} \right]$$

$$= \sum_{R,S,T,=0}^{\infty} \frac{(\lambda)_R (\mu)_S (\nu)_T \, x^R \, y^S \, z^T}{R!\ S!\ T!}$$

$$. F^{(3)} \left[\begin{matrix} (a) :: \Delta(\delta, \nu+T), (b); \Delta(\delta, \lambda+R), (b'); \Delta(\delta, \mu+S), (b'') : \\ \Delta(\delta, \lambda), \Delta(\delta, \mu), \Delta(\delta, \nu), (f) :: (g); (g'); (g'') : \end{matrix} \begin{matrix} \Delta(\delta, -R), (c); \Delta(\delta, -S), (c'); \Delta(\delta, -T), (c''); \\ (h); (h'); (h''); \end{matrix} \alpha^{\delta}, \beta^{\delta}, \gamma^{\delta} \right]$$

where δ is a positive integer, and $\Delta(m, a)$ stands for the m parameters a/m, $(a+1)/m, \ldots, (a+m-1)/m$.

The formula (2) is valid under the conditions
$A+B+B''+C \leqslant F+G+G''+H$, $A+B+B'+C' \leqslant F+G+G'+H'$, $A+B'+B''+C'' \leqslant F+G'+G''+H''$, and $|x| < 1$, $|y| < 1$, $|z| < 1$;
but if $A+B+B''+C = F+G+G''+H+1$, $A+B+B'+C' = F+G+G'+H'+1$, $A+B'+B''+C'' = F+G'+G''+H''+1$, then $|x|$, $|y|$, $|z|$, $|\alpha|$, $|\beta|$, and $|\gamma|$ are to be restricted appropriately, so that the series involved are either terminating of convergent.

Proof : Substituting for $F^{(3)}$ in the left-hand side of (2) from (1) and making use of the elementry expansion :

$$(3)\quad \sum_{r=0}^{\infty} \frac{(n)_r\, x^r}{r\,!} = (1-x)^{-n}\,,\ |x| < 1,$$

we obtain

$$\sum_{m,n,p=0}^{\infty} \frac{((a))_{m+n+p}((b))_{m+n}\,((b'))_{n+p}\,((b''))_{p+m}\,((c))_m\,((c'))_n\,((c''))_p}{((f))_{m+n+p}\,((g))_{m+n}\,((g'))_{n+p}\,((g''))_{p+m}\,((h))_m\,((h'))_n\,((h''))_p}$$

$$\cdot \frac{(-\alpha)^{m\delta}\,(-\beta)^{n\delta}\,(-\gamma)^{p\delta}}{m!\ n!\ p!} \sum_{r,s,t=0}^{\infty} \frac{(\lambda+m\delta+n\delta+p\delta)_r}{r\,!\ s\,!\ t\,!}$$

$$\cdot\,(\mu+m\delta+u\delta+p\delta)_s\,(\nu+m\delta+n\delta+p\delta)_t\ x^{r+m\delta}\,y^{s+n\delta}\,z^{t+p\delta}.$$

Inverting the order of summation, which is justified due to absolute convergence of the series involved, substituting $r+m\delta = R$, $s+n\delta = S$, $t+p\delta=T$, making use of the relationships [6, pp. 22, 32] :

$$(4)\qquad (a)_{mk} = m^{mk} \prod_{i=1}^{m} \left(\frac{a+i-1}{m}\right)_k;\ (-n)_k = \frac{(-1)^k\, n\,!}{(n-k)\,!},\ 0\leqslant k\leqslant n\ ;$$

and simplifying, we get the right-hand side of (2).

3. Special Cases—We give below some of the particular cases which are believed to be new.

(i) Putting $\delta=1$ in (2) and replacing x by x/λ, y by y/μ, z by z/ν, α bp $\lambda\alpha$, β by $\mu\beta$ and γ by $\nu\gamma$, we obtain

$$(1-x/\lambda)^{-\lambda}\,(1-y/\mu)^{-\mu}\,(1-z/\nu)^{-\nu}\,F^{(3)}\left[\begin{matrix}(a)::(b);\,(b');\,(b''):(c);\,(c');\,c'');\\(f)::(g);\,(g');\,(g''):(h);\,(h');\,(h'');\end{matrix}\right.$$

$$\left.\frac{-\alpha x}{(1-x/\lambda)\,(1-y/\mu)\,(1-z/\nu)},\ \frac{-\beta y}{(1-x/\lambda)\,(1-y/\mu)\,(1-z/\nu)},\ \frac{-\gamma z}{(1-x/\lambda)(1-y/\mu)(1-z/\nu)}\right]$$

$$= \sum_{R,S,T=0}^{\infty} \frac{(\lambda)_R\,(\mu)_S\,(\nu)_T\,(x/\lambda)^R\,(y/\mu)^S\,(z/\nu)^T}{R!\ S!\ T!} \sum_{r,s,t=0}^{\infty} \frac{((a))_{r+s+t}}{((\lambda))_{r+s+t}}$$

$$\frac{(\nu+T)_{r+s}\,((b))_{r+s}\,(\lambda+R)_{s+t}\,((b'))_{s+t}\,((\mu+S)_{t+r}\,((b''))_{t+r}}{(\mu)_{r+s+t}\,(\nu)_{r+s+t}\,((f))_{r+s+t}\,((g))_{r+s}\,((g'))_{s+t}\,((g''))_{t+r}}$$

$$\frac{(-R)_r\,((c))_r\,(-S)_s\,((c'))_s\,(-T)_t\,((c''))_t\,((\lambda\alpha))^r\,((\mu\beta))^s\,(\nu\gamma)^t}{((h))_r\,((h'))_s\,((h''))_t\;r!\;s!\;t!}.$$

Proceeding to limits as $\lambda\to\infty$, $\mu\to\infty$, $\nu\to\infty$, we obtain

$$(5)\quad e^{x+y+z}\,F^{(3)}\left[\begin{matrix}(a)::(b);(b');(b''):(c);(c');(c'');\\(f)::(g);(g');(g''):(h);(h');(h'');\end{matrix}\;-\alpha x,\,-\beta y,\,-\gamma z\right]$$

$$=\sum_{R,S,T=0}^{\infty}\frac{x^R\,y^S\,z^T}{R!\,S!\,T!}\,F^{(3)}\left[\begin{matrix}(a)::(b);(b')(b''):-R,(c);-S,(c');-T,(c'');\\(f)::(g);(g');(g''):(h);(h');(h'');\end{matrix}\;\alpha,\beta,\gamma\right]$$

(ii) If we take $\delta=2$, replace α^2 by $\lambda^2\alpha$, β^2 by $\mu^2\beta$, γ^2 by $\nu^2\gamma$, x by x/λ, y by y/μ, z by z/ν and proceed as in (i) above, we get the transformation formula

$$(6)\quad e^{x+y+z}\,F^{(3)}\left[\begin{matrix}(a)::(b);(b');(b''):(c);(c');(c'');\\(f)::(g);(g');(g''):(h);(h');(h'');\end{matrix}\;\alpha x^2,\,\beta y^2,\,\gamma z^2\right]$$

$$=\sum_{R,S,T=0}^{\infty}\frac{x^R\,y^S\,z^T}{R!\,S!\,T!}\,F^{(3)}\left[\begin{matrix}(a)::(b);(b');(b''):\Delta(2,-R),(c);\Delta(2,-S),(c');\\(f)::(g);(g');(g''):(h);(h');(h''):\end{matrix}\right.$$

$$\left.\begin{matrix}\Delta(2,-T),(c'');\\------;\end{matrix}\;4\alpha,\,4\beta,\,4\gamma\right].$$

(iii) On taking $A=B=B'=B''=F=G=G'=G''=0$ in (2), $F^{(3)}$ breaks into the product of three hypergeometric functions and it takes the form

$$(7)\quad (1-x)^{-\lambda}\,(1-y)^{-\mu}\,(1-z)^{-\nu}\;{}_CF_H\left[\begin{matrix}(c);\\(h);\end{matrix}\left(\frac{-\alpha x}{(1-x)\,(1-y)\,(1-z)}\right)^{\delta}\right]$$

$$\cdot\;{}_{C'}F_{H'}\left[\begin{matrix}(c');\\(h');\end{matrix}\left(\frac{-\beta y}{(1-x)\,(1-y)\,(1-z)}\right)^{\delta}\right]\;{}_{C''}F_{H''}\left[\begin{matrix}(c'');\\(h'');\end{matrix}\left(\frac{-\gamma z}{(1-x)(1-y)(1-z)}\right)^{\delta}\right]$$

$$=\sum_{R,S,T=0}^{\infty}\frac{(\lambda)_R\,(\mu)_S\,(\nu)_T\,x^R\,y^S\,z^T}{R!\,S!\,T!}\,F^{(3)}\left[\begin{matrix}-::\Delta(\delta,\nu+T);\Delta(\delta,\lambda+R;\\\Delta(\delta,\lambda),\,\Delta(\delta,\mu),\,\Delta(\delta,\nu)::\end{matrix}\right.$$

$$\triangle\ (\delta,\mu+S) : \triangle(\delta,-R),\ (c);\triangle\ (\delta,-S),\ (c');\ \triangle(\delta,-T),\ (c'');\ \alpha^{\delta},\beta^{\delta},\gamma^{\delta} \Bigg].$$
$$\text{———} : (h)\ ;\ (h')\ ;\ (h'')\ ;$$

On specializing the parameters in $F^{(3)}$, it yields the fourteen hypergeometric functions $F_1, F_2, \ldots, F_{14}$ of three variables, defined explicitly by Lauricella [4, p. 114], and the three additional functions H_A, H_B and H_C, defined by Srivastava (*cf.*, *e.g.*, [8], pp. 99-100); it also reduces to Appell functions and other functions of similar nature. Thus the transformation discussed here becomes a key result. In particular, the results of Mahajan [5], Brafman [1], Chaundy [3], and Jain and Dave [4], can be obtained by the proper choices of the parameters and variables involved.

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A NOTE ON A GENERATING RELATION FOR THE MULTIVARIABLE H-FUNCTION

By

Mirdula Garg

Department of Mathematics, M. R. Engineering College.
Jaipur - **3020024**, *Rajasthan, India*

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ABSTRACT

The aim of the present note is to illustrate the application of the differential operator $T_k = x\,(k+xD)$, $D \equiv d/dx$, in obtaining generating relations of a general type. Thus we have derived here an interesting known generating relation for the multivariate H-function introduced by H. M. Srivastava and Rekha Panda [4]. The corresponding generating relation for the H-function of two variables, which is also quite general in nature, follows as its particular case.

1. Introduction

Srivastava and Panda [5] defined the H-function of several complex variables $z_1, z_2, \ldots, z_r$ by means of the following multiple contour integral [4, p. 121, Eq. (1.10)]

$$H^{0,\,\lambda\,:\,(\mu',\,\nu')\,;\ldots;\,(\mu^{(r)},\,\nu^{(r)})}_{A,\,C,\,:\,[B',D']\,;\ldots;\,[B^{(r)},\,D^{(r)}]}\begin{pmatrix}[(a):\theta',\ldots,\theta^{(r)}]\,:\,[(b'):\phi']\,;\ldots;\,[(b^{(r)}):\phi^{(r)}]\,; & \\ & z_1,\ldots,z_r \\ [(c):\psi',\ldots,\psi^{(r)}]\,:\,[(d'):\delta']\,;\ldots;\,[(d^{(r)}):\delta^{(r)}]\,; & \end{pmatrix}$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r} \Phi_1(\zeta_1)\cdots\Phi_r(\zeta_r)\,\psi(\zeta_1,\ldots,\zeta_r)\, z_1^{\zeta_1}\cdots z_r^{\zeta_r}\, d\zeta_1\ldots d\zeta_r,\ \omega = \sqrt{-1}\,, \qquad \ldots(1.1)$$

where

$$\Phi_i(\zeta_i)=\frac{\prod_{j=1}^{\mu^{(i)}}\Gamma[d_j^{(i)}-\delta_j^{(i)}\zeta_i]\prod_{j=1}^{\nu^{(i)}}\Gamma[1-b_j^{(i)}+\phi_j^{(i)}\zeta_i]}{\prod_{j=\mu^{(i)}+1}^{D^{(i)}}\Gamma[1-d_j^{(i)}+\delta_j^{(i)}\zeta_i]\prod_{j=\nu^{(i)}+1}^{B^{(i)}}\Gamma[b_j^{(i)}-\phi_j^{(i)}\zeta_i]}$$

$$i=1,\ldots,r; \qquad \ldots(1.2)$$

$$\Psi(\zeta_1,\ldots,\zeta_r)=\frac{\prod_{j=1}^{\lambda}\Gamma[1-a_j+\sum_{i=1}^{r}\theta_j^{(i)}\zeta_i]}{\prod_{j=\lambda+1}^{A}\Gamma[a_j-\sum_{i=1}^{r}\theta_j^{(i)}\zeta_i]\prod_{j=1}^{C}\Gamma[1-c_j+\sum_{i=1}^{r}\psi_j^{(i)}\zeta_i]} \qquad \ldots(1.3)$$

where the various symbols, notations and conditions on parameters can be referred to in the papers by Srivastava and Panda ([4], [5]). Also, following Srivastava and Panda [*loc. cit.*], we shall use a contracted notation and write the first member of (1.1) in the abbreviated form

$$H^{0,\lambda:(\mu',\nu');\ldots;(\mu^{(r)},\nu^{(r)})}_{A,C:[B',D'];\ldots;[B^{(r)},D^{(r)}]}\begin{pmatrix} z_1 \\ \vdots \\ z_r \end{pmatrix} \text{ or } H\begin{pmatrix} z_1 \\ \vdots \\ z_r \end{pmatrix}$$

whenever no ambiguity or confusion arises.

The multiple integral in (1.1) converges absolutely if

$$|\arg(z_i)| < \tfrac{1}{2}\pi\,\Delta_i,\; i=1,\ldots,r \qquad \ldots(1.4)$$

where

$$\Delta_i \equiv -\sum_{j=\lambda+1}^{A}\theta_j^{(i)}+\sum_{j=1}^{\nu^{(i)}}\phi_j^{(i)}-\sum_{j=\nu^{(i)}+1}^{B^{(i)}}\phi_j^{(i)}$$

$$-\sum_{j=1}^{C}\psi_j^{(i)}+\sum_{j=1}^{\mu^{(i)}}\delta_j^{(i)}-\sum_{j=\mu^{(i)}+1}^{D^{(i)}}\delta_j^{(i)}>0,\; i=1,\ldots,r.$$

2. Generating relations for the *H*-function of several complex variables.

We shall make use of the following operational formula, which is essentially the same as that given by Mittal [1], in obtaining the known generating relation (2.4) given below.

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} T^n_{\alpha+1+\beta n}\, f(x) = \frac{(1+v)^{\alpha+1}}{1-\beta v}\, f[x(1+v)] \qquad \ldots(2.1)$$

where $v = xz\,(1+v)^{\beta+1}$, β being a constant, $f(x)$ admits a formal power series in x, $T_k = x\,(k+x\,D)$, $D \equiv d/dx$, and $T_k{}^n$ means that the operator T_k is repeated n times. Taking

$$f(x) = H\begin{pmatrix} y_1/x^{\sigma_1} \\ \vdots \\ y_r/x^{\sigma_r} \end{pmatrix}$$

in (2.1) and assuming that T_k operates on x alone, the left-hand side of (2.1) is given by

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} T^n_{\alpha+1+\beta n} H\begin{pmatrix} y_1/x^{\sigma_1} \\ \vdots \\ y_r/x^{\sigma_r} \end{pmatrix}$$

$$= \sum_{n=0}^{\infty} \frac{z^n}{n!} \frac{1}{(2\pi\omega)^r} \int_{L_1} \cdots \int_{L_r} \Phi_1(\zeta_1) \cdots \Phi_r(\zeta_r)$$

$$\cdot\, \Psi(\zeta_1, \ldots, \zeta_r)\, y_1^{\zeta_1} \ldots y_r^{\zeta_r}\, T^n_{\alpha+1+\beta n}\left(x^{\sum_{i=1}^{r} \sigma_i \zeta_i}\right) d\zeta_1 \ldots d\zeta_r \qquad \ldots(2.2)$$

The interchange of the order of integration and differentiation is easily seen to be permissible under the conditions stated. Now evaluate

$$T^n_k\, x^\gamma = (k+\gamma)_n\, x^{\gamma+n} \qquad \ldots(2.3)$$

and substitute the values in (2.1) from (2.2) and (2.3). Further, putting $z = t/x$, $y_1 = z_1\, x^{\sigma_1}, \ldots, y_r = z_r\, x^{\sigma_r}$, and interpreting the result by means of (1.1) and (1.3), we obtain the following known generating relation for the multivariate H-function.

$$\sum_{n=0}^{\infty} (-1)^n \frac{t^n}{n!} H^{0,\, \lambda+1 : (\mu', \nu') ; \ldots ; (\mu^{(r)}, \nu^{(r)})}_{A+1,\, C+1 : [B', D'];\ldots; [B^{(r)}, D^{(r)}]}$$

$$\left(\begin{matrix} [1+\alpha+\beta n : \sigma_1 ,\ldots, \sigma_r] : [(a) : \theta' ,\ldots, \theta^{(r)}] : [(b') : \emptyset'] ;\ldots; \\ [1+\alpha+\beta n+n : \sigma_1 ,\ldots, \sigma_r] : [(c) : \psi' ,\ldots, \psi^{(r)}] : [(d') : \delta'] ;\ldots; \end{matrix} \begin{matrix} [(b^{(r)}) : \emptyset^{(r)}] ; \\ [(d^{(r)}) : \delta^{(r)}] ; \end{matrix} z_1 , \ldots , z_r \right)$$

$$= \frac{(1+v)^{\alpha+1}}{1-\beta v} H^{0,\, \lambda : (\mu', \nu') ;\ldots; (\mu^{(r)}, \nu^{(r)})}_{A,\, C : [B', D'];\ldots; [B^{(r)}, D^{(r)}]} \begin{pmatrix} z_1/(1+v)^{\sigma_1} \\ \vdots \\ z_r/(1+v)^{\sigma_r} \end{pmatrix} \qquad \ldots(2.4)$$

where $v=t\,(1+v)^{\beta+1}$, $\sigma_i > 0$, $i=1 ,\ldots, r$ and the conditions (1.4) and (1.5) are also satisfied.

Particular cases

(1) For $r=2$, we have the following generating relation for the H function of two variables

$$\sum_{n=0}^{\infty} (-1)^n \frac{t^n}{n!} H^{0,\, n_1+1 : m_2, n_2 ; m_3, n_3}_{p_1+1,\, q_1+1, : p_2, q_2; p_3, q_3}$$

$$\left(\begin{matrix} (1+\alpha+\beta n; \sigma_1, \sigma_2) ; (a_j ; \alpha_j, A_j)_{1,p_1} : (c_j : \gamma_j)_{1,p_2} ; (e_j : E_j)_{1,p_3} \\ (1+\alpha+\beta n+n; \sigma_1, \sigma_2); (b_j ; \beta_j, B_j)_{1,q_1} : (d_j : \delta_j)_{1,q_2} ; (f_j : F_j)_{1,q_3} \end{matrix} \middle| \begin{matrix} x \\ y \end{matrix} \right)$$

$$= \frac{(1+v)^{\alpha+1}}{1-\beta v} H^{0,\, n_1 : m_2, n_2 ; m_3, n_3}_{p_1, q_1 : p_2, q_2 ; p_3, q_3}$$

$$\left(\begin{matrix} (a_j : \alpha_j, A_j)_{1,p_1} : (c_j : \gamma_j)_{1,p_2} ; (e_j : E_j)_{1,p_3} \\ (b_j : \beta_j, B_j)_{1,q_1} : (d_j : \delta_j)_{1,q_2} ; (f_j : F_j)_{1,q_3} \end{matrix} \middle| \begin{matrix} x/(1+v)^{\sigma_1} \\ y/(1+v)^{\sigma_2} \end{matrix} \right) \qquad \ldots(2.5)$$

where the contracted notation for the (Mittal-Gupta [2]) H-function of two variables, used in (2.5), is due to Srivastava and Panda [5, p. 266, Eq. (1.5) *et seq.*].

(2) Taking all α's, β's, γ's, δ's, A's, B's, E's, F's, σ_1, and σ_2 equal to unity in (2.5), we obtain the corresponding generating relation for G-function of two variables which is recently being derived by Shukla [3].

The result (2.4) was obtained earlier by Srivastava and Panda [4, p. 130, Eq. (4.3)] adopting a different method. The present note shows the importance and utility of the differential operator T_k in obtaining the generating relation. A number of other generating relations involving different special functions can also be obtained by this method, but we omit the details.

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[illegible] for his valuable suggestions and also to Dr. K. C. Gupta for his help and guidance during the preparation of the paper.

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(*Dedicated to the memory of Professor Arthur Erdélyi*)

USE OF ROUTING ALGORITHM TO OBTAIN APPROXIMATE SOLUTIONS OF VARIATIONAL PROBLEMS

By

R. Bellman

Department of Electrical Engineering, Mathematics, and Medicine

and

M. Roosta

Department of Electrical Engineering University of Southern California Los Angeles, California 90007, U. S. A.

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I. Introduction

The purpose of this paper is to show how the algorirhm of the routing problems may be used to obtain approximate solutions of variational problems.

In Sections 2 we give a difficult variational problem and explain why the Euler-Lagrange equation cannot be used directly. In Section 3, we show how the routing algorithm can be applied. In Section 4, we show how this could be modified to include the free boundary problem. Finally, in Section 5, we make some remarks about how a smooth approximate can be obtained.

2. An Example

Let us consider the functional

$$\mathcal{J}(u) = \int_0^{10} (u'^2 + t^2 u^4)\, dt$$

Let us assume that u is subject to the constraint $u(0) = u(10) = \frac{1}{2}$.

Elementary use of functional analysis establishes the existance of the solution of a minimum. This minimization function satisfies the Euler-Lagrange equation $u''-2t^2\, u^3=0,\ u(0) = u(10) = \frac{1}{2}$.

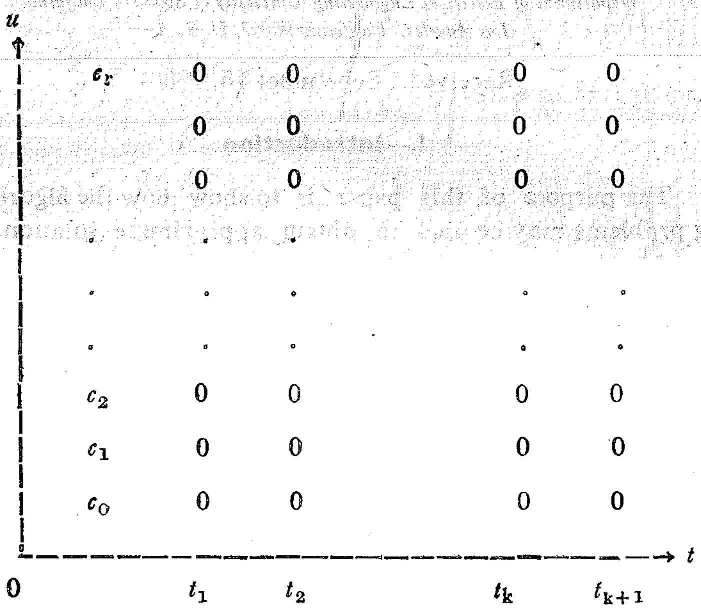
This is a non-linear differential equation subject to a two-point boundary condition. It must be solved by a method of successive approximation. Because of the length of the interval, this method may not converge.

3. Use of the Routing Algorithm

Let us add the constraint $0 \leqslant u \leqslant 1$. We know then that the desired solution lies in the rectangle $0 \leqslant u \leqslant 1$, $0 \leqslant t \leqslant 10$.

In general, either from mathematical or physical considerations, we can determine the region in which the solution must lie. Let us take this rectangle and take cross-sections at t interval of 1/10. Along each cross-section, let us take a gride of 100 points.

The number of points we take depends upon the accuracy we want and the time we are willing to devote to the problem. See the figure below.



The nodes are now the points on the cross-sections. The "time" to go from the i-th node from ohe cross-section to the j-th node on the next cross-section is given by

$$t_{ij} = \min_u J\,(t_k,\ t_{k+1},\ c_i,\ c_j,\ u) = \min \int_{t_k}^{t_{k+1}} (\, u^2 + t^2\ u^4)\ dt,$$

where $\quad u\,(t_k) = c_i,\ \ u\,(t_{k+1}) = c_j$.

From each point there are 100 possible points to go to. However in reality there are not. We can bound the slope of the solution a priori or we can ask for a smooth approximation.

These are the generalized times. The t_{ij}'s may be calculated ahead of time or as we need them. To calculate them we use a method of successive approximation. A good initial approximation will just be the straight line connecting the two modes.

Once we have calculated the $t_{ij}'s$, we can use the routing algorithm to determine the approximation.

4. Free boundary Problem

Let us now assume that the boundary conditions are $u\,(0)=\frac{1}{2}$, $u'\,(10)=0$. At the final step we impose the last condition to determine the t_{ij} . A good initial approximation is now a line parallel to the t-axis.

5. Conclusion

The approximation obtained in this way may be smoothed in various ways. Alternatively, it can be used as an initial approximation for another method.

[illegible]

From each point there are two possible [illegible] in reality [illegible] the approximate [illegible] solution [illegible] approximation.

[illegible]

Once we have introduced [illegible]

[illegible]

§ 5. Conclusion

The approximation obtained in this way may be applied [illegible] another method.

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(Dedicated to the memory of Professor Arthur Erdélyi)

ON A UNIFICATION OF THE GENERALIZED HUMBERT AND LAGUERRE POLYNOMIALS

By

J. P. Singhal and Savita Kumari

Department of Mathematics, M. S. University
Baroda - 390002, India

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ABSTRACT

The object of this paper is to initiate a systematic study of a new class of polynomials which unifies (as well as provides extensions of) several known sysstems of polynomials which are embraced by the generalized Humbert polynomials of H. W. Gould [3] and the recent generalization of Laguerre, Jacobi, Hermite, and other polynomials that was introduced into analysis by R. Panda [5] (see also [1], [4] and [7]).

1. INTRODUCTION

Recently, having been motivated by the earlier works of Rainville [7, p. 137, Theorem 48], Chandel ([1], [2]) and Jain [4], Rekha Panda [5] introduced an elegant generalization of several known polynomial systems belonging to (or providing extensions of) the families of the classical Jacobi Hermite and Laguerre polynomials poynomials by means of the generating relation

$$(1.1) \qquad (1-t)^{-c}\; G\left[\frac{xt^s}{(1-t)^r}\right] = \sum_{n=0}^{\infty} g^c_n\,(x, r, s)\; t^n ,$$

where

$$(1.2) \qquad G[z] = \sum_{n=0}^{\infty} \gamma_n z^n, \quad \gamma_0 \neq 0,$$

c is an arbitrary complex number, r is any integer, positive or negative, and $s=1, 2, 3$ . .

A comparison of (1.1) with the generating function [3, p. 697]

$$(1.3) \qquad (C-mxt+yt^m)^p = \sum_{n=0}^{\infty} P_n(m, x, y, p, C)\, t^n$$

for the generalized Humbert polynomials of Gould [3] suggests that it would be interesting and worthwhile to study a new class of polynomials $\{f^c_n(x, y, r, m) \mid n=0, 1, 2, \ldots\}$ defined by the generating relation

$$(1.4) \quad (1+yt^m)^{-c}\, G\left[\frac{xt}{(1+yt^m)^r}\right] = \sum_{n=0}^{\infty} f_n^c(x, y, r, m)\, t^n,$$

where, as in (1.1) and (1.2),

$$(1.4) \qquad G[z] = \sum_{n=0}^{\infty} \gamma_n z^n, \quad \gamma_0 \neq 0,$$

$m \geqslant 1$ is an integer and other parameters are unrestricted in general.

From (1.4) and (1.5) it is easy to deduce that $f_n^c(x, y, r, m)$ is a polynomial of degree n in x with its explicit representation as

$$(1\ 6) \qquad f_n^c(x, y, r, m) = \sum_{k=0}^{[n/m]} \binom{-c-nr+mrk}{k} y^k \gamma_{n-mk}\, x^{n-mk}.$$

On choosing γ_n appropriately and restricting r to integral values, the right member of (1.6) may also be put in hypergeometric form.

It is obvious that when $y = -1$, and $m=1$, (1·4) would correspond to the special case $s=1$ of (1.1), whereas on taking $\gamma_n = (-m)^n \binom{p}{n}$, $c = -p$

and $r=1$, (1.4) would transform into (1.3) with $C=1$. Thus the class of polynomials $\{f_n^c(x, y, r, m) \mid n=0, 1, 2, \ldots\}$ defined by (1.4) and (1.5) provides an interesting unification as well as generalization of the various polynomials included $g_n^c(x, r, 1)$ and the generalized Humbert polynomial $P_n(m, x, y, p, 1)$ which it self is a generalization of several known polynomials including those of Legendre, Gegenbauer, Humbert, Tchebycheff, Princherle, and many others. For the different conditions on the parameters of $f_n^c(x, y, r, m)$ under which it reduces to the polynomials mentioned above and many others, e. g., the polynomials of Sister Celine, Jacobi, Rice reference may be made to [3], [4] and [5].

Being motivated by the observations mentioned in the above paragraphs, we initiate with this paper a systematic study of ihe polynomials $f_n^c(x, y, r, m)$. In Section 2 of this paper we derive a generating function for $f_n^{c+\sigma n}(x, y, r, m)$ and discuss its various particular cases and their applications in the derivations of some expansion formulae. Section 3 incorporates a number of recurrence relations for $f_n^c(x, y, r, m)$. In Section 4 we give some additional results for the particular case $\gamma_n=1/n\,!$ of $f_n^c(x, y, r, m)$ which has been denoted by $\Gamma_n^c(x, y, r, m)$.

In what follows, for the sake of brevity, we shall abbreviate $f_n^c(x, y, r, m)$ by $f_n^c(x)$ and $\Gamma_n^c(x, y, r, m)$ by $\Gamma_n^c(x)$ unless there is any ambiguity regarding other parameters.

2. Generating Function for $f_n^{c+\sigma n}(x)$

The generating function that we propose to derive in this section is

$$(2.1)\qquad \sum_{n=0}^{\infty} f_n^{c+\sigma n}(x)\; t^n = \frac{(1+yw^m)^{1-c}}{1+y(1+\sigma m)w^m}\; G\left[\frac{xw}{(1+yw^m)^r}\right],$$

where

$$(2.2)\qquad w = t\,(1+y\,w^m)^{-\sigma}, \quad w(0) = 0,$$

σ is an arbitrary complex number and γ_n, the coefficient of z^n in the power series for $G[z]$, is independent of c.

To prove (2.1), we start with the function

$$(2.3) \qquad F(A_n, y, r, m, c) = \sum_{k=0}^{[n/m]} \binom{-c-nr+mrk}{k} y^k A_{n-mk}$$

where $\{A_n\}$ is an arbitrary sequence such that $\sum_{n=0}^{\infty} | A_n | < \infty$, $m \geqslant 1$ is an integer and other parameters are unrestricted in general.

For arbitrary complex values of σ, it is easy to see that

$$(2.4) \quad \sum_{n=0}^{\infty} F(A_n, y, r, m, c+\sigma n)\, t^n = \sum_{n=0}^{\infty} A_n\, t^n \sum_{k=0}^{\infty} \binom{-c-\sigma n-rn-\sigma mk}{k} (yt^m)^k.$$

On summing the inner series on the right hand side of (2.4) with the help of the following consequence of Lagrange's expansion formula [6, p. 302, Problem 216] :

$$(2.5) \quad \sum_{n=0}^{\infty} \binom{a+bn}{n} t^n = (1+v)^{a+1} [\, 1-(b-1)\, v\,]^{-1}, \quad v=t\,(1+v)^b,$$

we get after a little simplification the general formula

$$(2.6) \quad \sum_{n=0}^{\infty} F(A_n, y, r, m, c+\sigma n)\, t^n = \frac{(1+yw^m)^{1-c}}{1+y\,(1+m)w^m} \sum_{n=0}^{\infty} \left[\frac{w}{(1+yw^m)^r} \right] A_n,$$

where w is given by (2.2).

In (2.6) if we take $A_n = \gamma_n x^n$ and then make use of (1 6) and (1.5) we are immediately led to the generating function (2 1).

Alternatively, we may start with the following consequence of the generating function (1.4) :

$$(2.7) \qquad f_n^c(x) = \sum_{k=0}^{[n/m]} \frac{(c-b)_k (-y)^k}{k!} f^b_{n-mk}(x),$$

and then arrive at (2.1) by using the method illustrated by Singhai [8]. Yet another method of proving (2.1) would run parallel to that of Rekha Srivastava [9] which she employed for deriving a corresponding generating function for $g^{c+\sigma n}$ (x, r, s) wherein by putting $s=1$ we shall get the case $m=1, y=-1$ of our result (2.1).

For $\sigma=0$, (2·1) evidently reduces to (1.4), whereas the substitution $\sigma= -1/m$ transforms it to

$$\sum_{n=0}^{\infty} f_n^{c-n/m}(x)\, t^n = (1-yt^m)^{c-1}\; G\left[xt\,(1-yt^m)^{r-1/m}\right]. \tag{2.8}$$

On the other hand by putting $\sigma= -2/m$ we shall get

$$\sum_{n=0}^{\infty} f_n^{c-2n/m}(x)\, t^n = (1-4yt^m)^{-\frac{1}{2}}\left(\frac{1+\sqrt{1-4yt^m}}{2}\right)^{c} G\left[x\,t\left(\frac{1+\sqrt{1-4yt^m}}{2}\right)^{r-2/m}\right]. \tag{2.9}$$

Various other Particular cases of (2.1) can be given by particularizing $f_n^c(x)$ and assigning different values to σ.

Finally, it is to be mentioned here that the particular case (2.8) of (2.1), when expressed in the form

$$\sum_{n=0}^{\infty} f_n^{b-n/m}(x)\,(1-yt^m)^{c-b}\,t^n = (1-y\,t^m)^{c-1}\; G\left[xt\,(1-y\,t^m)^{r-1/m}\right], \tag{2.10}$$

yields the expansion formula

$$f_n^{c-n/m}(x) = \sum_{k=0}^{[n/m]} \frac{(b-c)_k}{k!}\; y^k\, f_{n-mk}^{\,b+k-n/m}(x), \tag{2 11}$$

which is analogous to the similar consequence of (1.4) given by equation (2.7) above.

3. Recurrence Relations

If we denote the left member of (1.4) by $U(x, t)$, then it is readily seen that $U(x, t)$ satisfies the differential equation function for $g^{c+\sigma n}\ (x, r, s)$ wherein by putting $s=1$ we shall get the case $m=1, y=-1$ of our result (2.1).

For $\sigma=0$, (2·1) evidently reduces to (1.4), whereas the substitution $\sigma=-1/m$ transforms it to

$$(2.8)\qquad \sum_{n=0}^{\infty} f_n^{c-n/m}(x)\, t^n = (1-yt^m)^{c-1}\, G\left[xt\,(1-yt^m)^{r-1/m}\right].$$

On the other hand by putting $\sigma=-2/m$ we shall get

$$(2.9)\qquad \sum_{n=0}^{\infty} f_n^{c-2n/m}(x)\, t^n$$

$$=(1-4yt^m)^{-\frac{1}{2}}\left(\frac{1+\sqrt{1-4yt^m}}{2}\right)^c G\left[x\,t\left(\frac{1+\sqrt{1-4yt^m}}{2}\right)^{r-2/m}\right].$$

Various other Particular cases of (2.1) can be given by particularizing $f_n^c(x)$ and assigning different values to σ.

Finally, it is to be mentioned here that the particular case (2.8) of (2.1), when expressed in the form

$$(2.10)\qquad \sum_{n=0}^{\infty} f_n^{b-n/m}(x)\,(1-yt^m)^{c-b}t^n$$

$$=(1-y\,t^m)^{c-1}\, G\left[xt\,(1-y\,t^m)^{r-1/m}\right],$$

yields the expansion formula

$$(2.11)\qquad f_n^{c-n/m}(x)=\sum_{k=0}^{[n/m]} \frac{(b-c)_k}{k\,!}\, y^k\, f_{n-mk}^{b+k-n/m}(x),$$

which is analogous to the similar consequence of (1.4) given by equation (2.7) above.

3. Recurrence Relations

If we denote the left member of (1.4) by $U(x, t)$, then it is readily s een that $U(x, t)$ satisfies the differential equation

$$(4.3)\quad \Gamma_{n-1}^{c+s}(x, y, r+s, m)=\sum_{k=0}^{[n/m]} \frac{(-r)_k\,(-y)^k}{k!}\, D_x\left\{\Gamma_{n-mk}^{c}(x, y, r+s, m)\right\},$$

$$(4.4)\quad D_x^k\left\{\Gamma_n^c(x)\right\}=\Gamma_{n-k}^{c+rk}(x);$$

$$(4.5)\quad n\,\Gamma_n^c(x)+cmy\,\Gamma_{n-m}^{c+1}(x)-x\,\Gamma_{n-1}^{c+r}(x)$$

$$+\,xymr\,\Gamma_{n-1}^{c+r+1}(x)=0,\ n\geqslant m\geqslant 1,$$

$$(4.6)\quad \Gamma_n^{c_1+\ldots+c_k}(x_1+\ldots+x_k)$$

$$=\sum_{i_1+\ldots+i_k=n} \Gamma_{i_1}^{c_1}(x_1)\ldots\Gamma_{i_k}^{c_k}(x_k),$$

$$(4.7)\quad \Gamma_n^{b_1+\ldots+b_k+1-k-n/m}(x_1+\ldots+x_k$$

$$=\sum_{i_1+\ldots+i_k=n} \Gamma_{i_1}^{b_1-i_1/m}(x_1)\ldots\Gamma_{i_k}^{b_k-i_k/m}(x_k),$$

$$(4.8)\quad (n+1)\,\Gamma_{n+1}^{c}(x)+y\,(n-m+cm+1)\,\Gamma_{n-m+1}^{c}(x)$$

$$=x\sum_{k=0}^{[n/m]} \frac{(r)_k\,(-y)^k}{k!}\,\Gamma_{n-mk}^{c}(x)+(1-rm)\,xy\sum_{k=0}^{[n/m]-1}\frac{(r)_k\,(-y)^k}{k!}$$

$$\Gamma_{n-mk-m}^{c}(x),\ n\geqslant m.$$

Lastly, we give the mixed generating function

$$(4{\cdot}9)\quad \sum_{n=0}^{\infty} \Gamma_n^{c+\sigma n}(x+nz, y, r, m)\, t^n$$

$$=\frac{v^{-c}\,exp\,(xu/v^r)}{1-u\,[-\sigma myu^{m-1}\,v^{-1}+zv^{-r}\,\{1-mryu^m\,v^{-1}\}]},$$

where u and v are given by

(4.10) $\quad u = tv^{-\sigma} \exp(uz/v^{r}), \; v = (1 + yu^{m})$.

Our formula (4.9) is analogous to the mixed generating function for $g^{c}_{n}(x, r, s)$ given earlier by Rekha Srivastava [9]. The proof of (4.9) would also run parallel to the proof given in [9]; we, therefore, omit the details.

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(*Dedicated to the memory of Professor Arthur Erdélyi*)

SOME INEQUALITIES FOR THE APPELL FUNCTIONS F_1 AND F_2

By

C. M. Joshi

Department of Mathematics, University of Jodhpur
Jodhpur - 342001, Rajasthan India

and

J. P. Arya

Department of Mathematics, D. A. V. College,
Muzaffarnagar - 251001, U.P., India

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Inequalities for Appell's functions F_1 and F_2 are obtained for positive real arguments and certain specified ranges of parameters, applying the results of T. M. Flett [6] and B. C. Carlson [2]. Though the inequalities obtained are not very sharp, they serve a purpose, filling certain gaps in the literature on the inequalities for these functions.

1. Introduction

One-sided inequalities for particular values of parameters have been obtained for F_2 and F_3 for complex and real arguments, respectively by Minton [8], and for F_2 in complex arguments by Erber [3], from an analytic stand point. Luke [7] has obtained some two-sided inequalities for F_1, F_2 and F_3 for special ranges of parameters and negative real arguments from a computational viewpoint. These results, by suitable application of transformation theory, lead to inequalities for rsstricted ranges of positive real arguments , *viz* $0 < x < \frac{1}{2}$, $0 < y < \frac{1}{2}$ for F_1, and $0<x+y<\frac{1}{2}$ for F_2, the corresponding restrictions on parameters being understood.

In this note we shall obtain two-sided inequalities for F_1 and F_2

for positive real arguments by appealing to the theorems of flett [6] and Carlson [2]. These inequalities, though not very sharp, may yet serve a purpose, filling certain gaps in the literature on inequalities of these functions.

We recall a theorem of Flett [6] that he obtained in estimating the Euler integral of the Gauss hypergeometric function as follows :

Theorem 1. (a) *Let* $a > c > b > 0,\ 0 < x < 1.$ *Then*

$$\frac{\Gamma(a+b-c)\,\Gamma(c)}{\Gamma(a)\,\Gamma(b)}\,(1-x)^{c-a-b} < {}_2F_1(a, b; c; x) < (1-x)^{c-a-b}. \quad \text{.. (1)}$$

(b) *If, however,* $c>a>c-b>0,\ b>0,\ 0<x<1,$ *then*

$$(1-x)^{c-a-b} < {}_2F_1(a, b; c; x) < \frac{\Gamma(a+b-c)\,\Gamma(c)}{\Gamma(a)\,\Gamma(b)}\,(1-x)^{c-a-b}. \quad \text{...(2)}$$

Another theorem that is required is that of Carlson [2, Theorem 2] which can be rewritten as :

Theorem 2. *Let* $a>c+1>0,\ c>b>0,\ -\infty<x<1,\ x\neq 0.$ *Then*

$$(1-x)^{c-a-b}\,(1-x+bx/c)^{a-c} < {}_2F_1(a, b; c; x) < (b/c)\,(1-x)^{c-a-b} + (1-b/c)\,(1-x)^{-b}. \quad \text{...(3)}$$

2. Inequalities for F_1

The function F_1, defined by

$$F_1=F_1(a; b_1, b_2; c; x, y=)\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}\,(b_1)_m\,(b_2)_n\,(x)^m\,(y)^n}{(c)_{m+n}\,m!\,n!},$$

where $|x|<1,\ |y|<1$, and c is not a negative integer or zero, can be expressed as

$$F_1=\sum_{m=0}^{\infty}\frac{(a)_m\,(b_1)_m\,(x)^m}{(c)_m\,m!}\,{}_2F_1(a+m, b_2; c+m; y). \quad \text{.. (5)}$$

With the restrictions $c>a>c-b_2>0,\ b_2>0,\ 0<y<1$, an application of Theorem 1 (b) to the inner ${}_2F_1$ of the right-hand of (5) gives

$$(1-y)^{c-a-b_2}\,{}_2F_1(a, b_1; c; x) < F_1 < \frac{\Gamma(a+b_2-c)\,\Gamma(c)}{\Gamma(a)\,\Gamma(b_2)}\,(1-y)^{c-a-b_2}\,(1-x)^{-b_1} \quad \text{...(6)}$$

Setting again $c > a > c-b_1 > 0$, $b_1 > 0$, $0 < x < 1$, a reapplication of the same theorem to (6) yields :

$$(1-x)^{c-a-b_1}(1-y)^{c-a-b_2} < F_1 < \frac{\Gamma(a+b_2-c)\,\Gamma(c)}{\Gamma(a)\,\Gamma(b_2)}(1-x)^{-b_1}(1-y)^{c-a-b_2};$$

whenever

$$c > a > c-b_1 > 0,\ c > a > c-b_2 > 0,\ b_1 > 0,\ b_2 > 0,$$
$$0 < x < 1,\ 0 < y < 1. \qquad \text{...(7)}$$

By an appeal to the symmetry property of F_1, wherein the roles of b_1 and x are changed by b_2 and y, respectively, and *vice versa*, (7) will admit the form

$$(1-x)^{c-a-b_1}(1-y)^{c-a-b_2} < F_1 < \frac{\Gamma(a+b_1-c)\,\Gamma(c)}{\Gamma(a)\,\Gamma(b_1)}(1-y)^{-b_2}(1-x)^{c-a-b_1}, \qquad \text{...(8)}$$

with the same set of conditions as before.

Equations (7) and (8), when combined, will give :

Teorem 3. (a). *If* $c > a > c-b_1 > 0$, $c > a > c-b_2 > 0$, $b_1 > 0$, $b_2 > 0$, $0 < x < 1$, $0 < y < 1$, *then*

$$(1-x)^{c-a-b_1}(1-y)^{c-a-b_2} < F_1$$

$$< \frac{\Gamma(c)}{\Gamma(a)}\min\left\{\begin{array}{l}\dfrac{\Gamma(a+b_1-c)}{\Gamma(b_1)}(1-y)^{-b_2}(1-x)^{c-a-b_1},\\[2ex] \dfrac{\Gamma(a+b_2-c)}{\Gamma(b_2)}(1-x)^{-b_1}(1-y)^{c-a-b_2}.\end{array}\right. \qquad \text{...(9)}$$

On the other hand, proceeding as before, but using Theorem 1 (a), we have :

Theorem 3 (b). *If* $a > c > b_1 > 0$, $a > c > b_2 > 0$, $0 < x < 1$, $0 < y < 1$, *then*

$$\frac{\Gamma(c)}{\Gamma(a)}\max\left\{\begin{array}{l}\dfrac{\Gamma(a+b_1-c)}{\Gamma(b_1)}(1-y)^{-b_2}(1-x)^{c-a-b_1},\\[2ex] \dfrac{\Gamma(a+b_2-c)}{\Gamma(b_2)}(1-x)^{-b_1}(1-y)^{c-a-b_2},\end{array}\right.$$

$$< F_1 < (1-x)^{c-a-b_1}(1-y)^{c-a-b_2}. \qquad \text{...(10)}$$

Yet another theorem for positive real arguments would be obtained, in the same fashion, by recourse to Theorem 2. Indeed, an application of Theorem 2 to Equation (5) leads to

$$\sum_{m=0}^{\infty} \frac{(a)_m (b_1)_m (x)^m}{(c)_m\, m!} \left(1-y+\frac{b_2 y}{c+m}\right)^{a-c} (1-y)^{c-a-b_2} < F_1$$

$$< \sum_{m=0}^{\infty} \frac{(a)_m (b_1)_m (x)^m}{(c)_m\, m!} \left\{ \frac{b_2}{c+m} (1-y)^{c-a-b_2} + \left(1-\frac{b_2}{c+m}\right) (1-y)^{-b_2} \right\}$$

whenever

$$a > c+1 > 0,\ c > b_2 > 0,\ -\infty < y < 1,\ y \neq 0, \qquad \text{...(11)}$$

and once again, as a consequence of Theorem 2, we have

$$(1-x)^{c-a-b_1} (1-y)^{-b_2} (1-x+(b_1 x/c))^{a-c} < F_1 < (1-y)^{b_2}$$

$$[1+(b_2/c)(1-y)^{c-a}]\ [(b_1/c)(1-x)^{c-a-b_1} + \{1-(b_1/c)\}(1-x)^{-b_1}],$$

$$a > c+1 > 0,\ c > b_1 > 0,\ c > b_2 > 0,\ |x| < 1,\ |y| < 1,\ x \neq 0,\ y \neq 0, \qquad \text{..(12)}$$

The symmetry in F_1 enables us to express Equation (12) in an alternative form which, in combination with (12), finally gives :

Theorem 4.

$$max \begin{cases} (1-x)^{c-a-b_1} (1-y)^{-b_2} [1-x+(b_1 x/c)]^{a-c}, \\ (1-x)^{-b_1} [1-y+(b_2 y/c)]^{a-c} (1-y)^{c-a-b_2}, \end{cases}$$

$$< F_1 < min \begin{cases} (1-y)^{-b_2}[1+(b_2/c)(1-y)^{c-a}]\ [(b_1/c)(1-x)^{c-a-b_1} + (1-(b_1/c))(1-x)^{-b_1}], \\ (1-x)^{-b_1}[1+(b_1/c)(1-x)^{c-a}][(b_2/c)(1-y)^{c-a-b_2} + \{1-(b_2/c)\}(1-y)^{-b_2}], \end{cases}$$

$$a > c+1 > 0,\ c > b_1 > 0,\ c > b_2 > 0,\ |x| < 1,\ |y| < 1,\ x \neq 0,\ y \neq 0. \qquad .\ (13)$$

For the set of values

$a=0.5$, $b_1=0.4$), $b_2=0.6$, $c=0.8$, $x=0.2$, $y=0.7$

from Theorem 3 (a) we have

$$1.467421 < F_1 < 2.07033, \qquad \ldots(14)$$

and for

$a=0.8,\ b_1=0.2,\ b_2=0.4,\ c=0.7,\ x=0.2,\ y=0.5,$

from Theorem 3 (b)

$$1{,}317449 < F_1 < 1.512126. \qquad \ldots(15)$$

Computations show that Theorem 4 gives sharper inequalities than Theorem 3 (b) in their common range of validity. In particular, for the choice

$a=1.8,\ b_1=0.4,\ b_2=0.2,\ c=0.7,\ x=0.5,\ y=0.2,$

we obtain from Theorem 4

$$2.268392 < F_1 < 2.554081,$$

while from Theorem 3 (b) we have

$$1.646811 < F_1 < 3.780315. \qquad \ldots(17)$$

3. Inequalities for F_2

We start with the single series repre entation

$$F_2 \equiv F_2\,(a;\, b_1,\, b_2;\, c_1,\, c_2;\, x, y)$$

$$\sum_{m=0}^{\infty} \frac{(a)_m\,(b_1)_m\,(x)^m}{(c_1)_m\, m\,!}\ {}_2F_1\,(a+m,\, b_2;\, c_2;\, y),$$

where $|x|+|y|<1$, and neither c_1 nor c_2 is a negative integer or zero. ...(18)

Under the restrictions

$a > a+b_2 - c_2 > c_1 > b_1 > 0,\ a > c_2 > b_2 > 0,$

$$x > 0,\ y > 0,\ x+y < 1, \qquad \ldots(19)$$

by repeated applications of Theorem 1 (b) in (18), we obtain

$$\frac{\Gamma(a+b_1+b_2-c_1-c_2)\,\Gamma(c_1)\,\Gamma(c_2)}{\Gamma(a)\,\Gamma(b_1)\,\Gamma(b_2)}\,(1-y)^{b_1-c_1}\,(1-x-y)^{c_1+c_2-a-b_1-b_2}$$

$$< F_2 < (1-y)^{c_2-c_1+b_1-b_2}\,(1-x-y)^{c_1-a-b_1}. \qquad \ldots(20)$$

Since

$$F_2\,(a;\, b_1.\, b_2;\, c_1,\, c_2;\, x;\, y) = F_2\,(a,\, b_2,\, b_1;\, c_2,\, c_1;\, y,\, x), \qquad \ldots(21)$$

it is possible to write the above inequality in a different form which, in conjunction with (20), will give :

Theorem 5. *Let* $a > a+b_2-c_2 > c_1 > b_1 > 0$, $a > a+b_1-c_1 > c_2 > b_2 > 0$, $x > 0, y > 0, x+y < 1$. *Then*

$$\frac{\Gamma(a+b_1+b_2-c_1-c_2)\ \Gamma(c_1(\ \Gamma(c_2)}{\Gamma(a)\ \Gamma(b_1)\ \Gamma(b_2)} (1-x-y)^{c_1+c_2-a-b_1-b_2}$$

$$max\left\{ (1-y)^{b_1-c_1}, (1-x)^{b_2-c_2} \right\} < F_2$$

$$< min\left\{ (1-y)^{c_2-c_1+b_1-b_2}(1-x-y)^{c_1-a-b_1}, (1-x)^{c_1-c_2+b_2-b_1}(1-x-y)^{c_2-a-b_2} \right\}. \quad ...(22)$$

Though with the choice $a>c_1+1$, $a>c_2+1$, Theorem 5 holds good but a better inequality could be obtaind by appropriate application of Theorem 2 in (18). Thus, under the conditions

$$a > max\ \{c_1+1, c_2+1\}, c_1 > b_1 > 0, c_2 > b_2 > 0,$$
$$|x| + |y| < 1,\ x \neq 0, y \neq 0, \quad ...(23)$$

we have

$$(1-y)^{c_2-a+b_1-b_2}(1-y+\frac{b^2 y}{c_2})^{a-c_2}(1-y+x\ (1-y+\frac{b_2 y}{c_2}))^{c_1-a-b_1}$$

$$(1-y-x\ (1-\frac{b_1}{c_1})\ (1-y+\frac{b_2 y}{c_2}))^{a-c_1} < F_2$$

$$< \frac{b_2}{c_2}(1-y)^{c_2-a-b_2}\left[\frac{b_1}{c_1}(1-\frac{x}{1-y})^{c_1-a-b_1}+(1-\frac{b_1}{c_1})(1-\frac{x}{1-y})^{-b_1}\right]$$

$$+(1-\frac{b_2}{c_2})(1-y)^{-b_2}\left[\frac{b_1}{c_1}(1-x)^{c_1-a-b_1}+(1-\frac{b_1}{c_1})(1-x)^{-b_1}\right]. \quad ...(24)$$

As a consequence of (21), it then follows that

$$(1-x)^{c_1-a+b_2-b_1}(1-x+\frac{b_1 x}{c_1})^{a-c_1}[1-x+y\ (1-x+\frac{b_1 x}{c_1})]^{c_2-a-b_2}$$

$$[1-x-y\ (1-\frac{b_2}{c_2})\ (1-x+\frac{b_1 x}{c_1})]^{a-c_2} < F_2$$

$$< \frac{b_1}{c_1}(1-x)^{c_1-a-b_1}\left[\frac{b_2}{c_2}(1-\frac{y}{1-x})^{c_2-a-b_2}+(1-\frac{b_2}{c_2})(1-\frac{y}{1-x})^{-b_2}\right]$$

$$+(1-\frac{b_1}{c_1})(1-x)^{-b_1}\left[\frac{b_2}{c_2}(1-y)^{c_2-a-b_2}+(1-\frac{b_2}{c_2})(1-y)^{-b_2}\right], \quad ...(25)$$

under the set of conditions (23).

We, therefore, finally have

Theorem 6.

$$\max\begin{cases} (1-y)^{c_2-a+b_1-b_2}\left(1-y+\dfrac{b_2 y}{c_2}\right)^{a-c_2}\left[1-y+x\left(1-y+\dfrac{b_2 y}{c_2}\right)\right]^{c_1-a-b_1} \\ \left[1-y-x\left(1-\dfrac{b_1}{c_1}\right)\left(1-y+\dfrac{b_2 y}{c_2}\right)\right]^{a-c_1}, \\ (1-x)^{c_1-a+b_2-b_1}\left(1-x+\dfrac{b_1 x}{c_1}\right)^{a-c_1}\left[1-x+y\left(1-x+\dfrac{b_1 x}{c_1}\right)\right]^{c_2-a-b_2} \\ \left[1-x-y\left(1-\dfrac{b_2}{c_2}\right)\left(1-x+\dfrac{b_1}{c_1}\right)\right]^{a-c_2} \end{cases}$$

$$< F_2$$

$$< \min\begin{cases} \dfrac{b_2}{c_2}(1-y)^{c_2-a-b_2}\left[\dfrac{b_1}{c_1}\left(1-\dfrac{x}{1-y}\right)^{c_1-a-b_1}+\left(1-\dfrac{b_1}{c_1}\right)\left(1-\dfrac{x}{1-y}\right)^{-b_1}\right] \\ +\left(1-\dfrac{b_2}{c_2}\right)(1-y)^{-b_2}\left[\dfrac{b_1}{c_1}(1-x)^{c_1-a-b_1}+\left(1-\dfrac{b_1}{c_1}\right)(1-x)^{-b_1}\right], \\ \dfrac{b_1}{c_1}(1-x)^{c_1-a-b_1}\left[\dfrac{b_2}{c_2}\left(1-\dfrac{y}{1-x}\right)^{c_2-a-b_2}+\left(1-\dfrac{b_2}{c_2}\right)\left(1-\dfrac{y}{1-x}\right)^{-b_2}\right] \\ +\left(1-\dfrac{b_1}{c_1}\right)(1-x)^{-b_1}\left[\dfrac{b_2}{c_2}(1-y)^{c_2-a-b_2}+\left(1-\dfrac{b_2}{c_2}\right)(1-y)^{-b_2}\right], \end{cases}$$

provided

$$a > \max\{c_1+1, c_2+1\},\ c_1 > b_1 > 0,\ c_2 > b_2 > 0,\ |x|+|y|<1,$$
$$x \text{ and } y \text{ are real},\ x \neq 0,\ y \neq 0. \qquad \ldots(26)$$

In conformity with the aforementioned remark, we note as a comparison that for the set of values

$a=2$, $b_1=0.4$, $b_2=0.5$, $c_1=0.6$, $c_2=0.8$, $x=0.2$, $y=0.3$,

Theorem 5 gives

$$1.187113 < F_2 < 3.322324 \qquad (27)$$

and Theorem 6 gives

$$2.377182 < F_2 < 2.447001 \qquad \ldots(28)$$

Inequalities for F_2 could also be obtained using a result due to Buschman [1], but such inequalities will be too cumbersome to deal with from a computational standpoint.

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(*Dedicated to the memory of Professor Arthur Erdélyi*)

CERTAIN THEOREMS ON BILATERAL GENERATING FUNCTIONS

By

K. K. Baweja and S. Saran

Department of Mathematics, Punjabi University, Patiala - 147002, India

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1. Saran [5], Shanker [6], Panda [3], Singhal and Srivastava [7], Srivastava [9], Srivastava and Singhal [10] and Srivastava and Lavoie [11] have proved a number of Theorems on bilinear as well as bilateral generating functions. The purpose of this paper is to establish two theorems for obtaining bilateral generating functions in terms of Pochhammer's double-loop type integrals. These theorems have been used to obtain a number of bilateral generating functions. The results obtained here have been further generalised to derive bilateral generating functions of polynomials of several variables.

Let the triple hypergeometric series, defined by Srivastava [8 p. 428], be represented as :

$$F\begin{matrix} K::m;n;p;r;s;t \\ L::M;N;P;R;S;T \end{matrix}\left[\begin{matrix} \{a_K\}::\{b_m\};\{b'_n\};\{b''_p\}:\{c_r\};\{c'_S\};\{c''_t\}; \\ \{d_L\}::\{e_M\};\{e'_N\};\{e''_P\}:\{f_R\};\{f_S'\};\{f_T''\}; \end{matrix} x,y,z\right]$$

$$=\sum_{u,v,w=0}^{\infty}\frac{\prod_{j=1}^{K}(a_j)_{u+v+w}\prod_{j=1}^{m}(b_j)_{u+v}\prod_{j=1}^{n}(b_j')_{v+w}\prod_{j=1}^{p}(b_j'')_{w+u}}{\prod_{j=1}^{L}(d_j)_{u+v+w}\prod_{j=1}^{M}(e_j)_{u+v}\prod_{j=1}^{N}(e'_j)_{v+w}\prod_{j=1}^{P}(e_j'')_{w+u}}$$

$$\cdot \frac{\prod_{j=1}^{r} (c_j)_u \prod_{j=1}^{s} (c'_j)_v \prod_{j=1}^{t} (c''_j)_w}{\prod_{j=1}^{R} (f_j)_u \prod_{j=1}^{S} (f'_j)_v \prod_{j=1}^{T} (f''_j)_w} \quad \frac{x^u}{u\,!} \quad \frac{y^v}{v!} \quad \frac{z^w}{w!} \qquad \ldots(1.1)$$

where $\prod_{j=1}^{k} \Gamma(a_j)_n = \prod_{j=1}^{k} \frac{\Gamma(a_j+n)}{\Gamma(a_j)}$.

We prove the following results :

Theorem I. *If*

$$F(x, t) = \sum_{n=0}^{\infty} f_n(x)\, t^n, \qquad \ldots(1.2)$$

then

$$\frac{\Gamma(1-a)\,\Gamma(1-b)\,\Gamma(a+b)}{(2\pi i)^2} \int (-p)^{a-1} (p-1)^{b-1}\, {}_2F_1\left[\begin{matrix}1-b,\ c; & \dfrac{yp}{(y-1)(p-\ )}\\ a; & \end{matrix}\right]$$

$$F(x, tp)\, dp = (1-y)^c \sum_{n=0}^{\infty} \frac{(a)_n}{(a+b)_n}\, {}_2F_1(-n, c; a; y)\, f_n(x)\, t^n \quad \ldots(1.3)$$

where the integration is taken over Pochhammer's double-loop [12, p. 256].

Theorem 2. *If*

$$G(x, t) = \sum_{n=0}^{\infty} g_n(x)\, t^n, \qquad \ldots(1.4)$$

then

$$\frac{\Gamma(1-a)\Gamma(1-b)\ (1-c)\ \ (1-d)\Gamma(a+b)\ \ (c+d)}{(2\pi i)^4}$$

$$\int\int (-p)^{a-1} (p-1)^{b-1}\ (-q)^{c-1}\ (q-1)^{d-1}$$

$${}_2F_1\left[\begin{matrix}1-b,\ 1-d; & \dfrac{ypq}{(y-1)(p-1)(q-1)}\\ a; & \end{matrix}\right] G\left(x, \frac{pqt}{1-y}\right) dp\, dq$$

$$= (1-y)^c \sum_{n=0}^{\infty} \frac{(a)_n\,(c)_n}{(a+b)_n\,(c+d)_n}\ {}_2F_1\,(-n;\,c+n;\,a;\,y)\,g_n\,(x)\,t^n \qquad \ldots(1.5)$$

where the path of integration is Pochhammer's double-loop type contour [12, p. 256].

2 PROOFS

To prove Theorem 1, we write tp for t in (1.2); multiply both sides by

$$(-p)^{a-1}\,(p-1)^{b-1}\ {}_2F_1\left[\begin{matrix}1-b,\,c; \\ a;\end{matrix}\ \frac{yp}{(y-1)(p-1)}\right]$$

and integrate with respect to p along the Pochhammer's double-looptype contour, thus obtaining the left-hand side of (1.3).

On using the result [12, p. 257], viz.

$$\frac{1}{(2\pi i)^2}\int_C (-p)^{a-1}\,(p-1)^{b-1}\,dp = \frac{1}{\Gamma(1-a)\,\Gamma(1-b)\,\Gamma(a+b)} \qquad \ldots(2.1)$$

where C is Pochhammer's double-loop type contour, the right-hand side is equal to

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(a)_{n+m}\,(c)_m\,(1-b)_m\,(b)_{-m}}{(a+b)_n\,(a)_m\,m\,!}\left(\frac{-y}{y-1}\right)^m f_n\,(x)\,t^n$$

$$= \sum_{n=0}^{\infty} \frac{(a)_n}{(a+b)_n}\ {}_2F_1\,(a+n),\,c;\,a;\ \frac{y}{y-1})\,f_n\,(x)\,t^n$$

This, on using the transformation

$${}_2F_1\,(a,\,b;\,c;\,x) = (1-x)^{-b}\ {}_2F_1\,(c-a,\,b;\,c;\ \frac{x}{x-1}), \qquad \ldots(2.2)$$

immediately proves that result (1.3).

Similarly, if we multiply (1.4) (with tpq for t) by

$$(-p)^{\ \ -1}\,(p-1)^{b-1}\,(-q)^{c-1}\,(q-1)^{d-1}\,{}_2F_1\left[\begin{matrix}1-b,\,1-d; \\ a;\end{matrix}\ \frac{ypq}{(y-1)(p-1)(q-1)}\right]$$

and integrate with respect to p and q along Pochhammer's double-loop type contours, we get (1.5) after using (2.2).

3. APPLICATIONS

(i) If ($f_n(x) = P_n^{(d,e)}(x)$ where $P_n^{(a,b)}(x)$ is well known Jacobi polynomial [4, p. 254 (1)], then by Brafman's result [4, p. 270], we have

$F(x, t) = F_4\,[1+e, 1+d : 1+d, 1+e; \tfrac{1}{2}\,t\,(x-1), \tfrac{1}{2}\,t\,(x+1)]$

Applying Theorem 1, we obtain

$$F\begin{matrix} 1 :: 2; 0; 0; 0; 0; 1 \\ 0 :: 1; 0; 0; 1; 1; 1 \end{matrix}\left[\begin{matrix} a :: 1+d, 1+e; -; -; -; -; c; \\ - :: a+b; -; -; 1+d; 1+e; a; \end{matrix}\ \tfrac{1}{2}t(x-1), \tfrac{1}{2}t(x+1), \frac{y}{y-1}\right]$$

$$= (1-y)^c \sum_{n=0}^{\infty} \frac{(a)_n}{(a+b)_n}\ {}_2F_1(-n, c; a; y)\, P_n^{(d,e)}(x\ \ t^n \qquad \ldots (3.1)$$

Taking $a+b = 1+d$, we get 5, p. 14 Eq. (3.2)].

(ii) If $f_n(x) = P_n^{(d-n, e-n)}(x)$, then, by Carlitz's result, we have $F(x, t) = [1+\tfrac{1}{2}(x+1)\,t]^d\,[1+\tfrac{1}{2}(x-1)t\ ]^e$

Applying Theorem 1, we get

$$F\begin{matrix} 1 :: 0; 0; 0: 1; 1; 1 \\ 0 :: 1; 0; 0: 0; 0; 1 \end{matrix}\left[\begin{matrix} a :: -; -; -: -d; -e; c; \\ - :: a+b; -; -: -; -; a; \end{matrix}\ \frac{-(x+1)\,t}{2}, \frac{-(x-1)\,t}{2}, \frac{y}{y-1}\right]$$

$$= (1-y)^c \sum_{n=0}^{\infty} \frac{(a)_n}{(a+b)_n}\ {}_2F_1(-n, c; a; y)\, P_n^{(d-n, e-n)}(x)\, t^n \qquad \ldots (3.2)$$

which, on putting $a+b = f$, gives [5, p. 14, Eq, (3.1)].

(iii) If $g_n(x) = P_n^{(f,g)}(x)$, then Theorem 2, gives

$$F\begin{matrix} 2 :: 0; 2; 0: 0; 0; 0; \\ 0 :: 0; 2; 0: 1; 1; 1; \end{matrix}\left[\begin{matrix} a, c :: -; 1+f, 1+g; -: -; -; -; \\ - :: -; a+b, c+d; -: a; 1+f; 1+g; \end{matrix}\ \frac{y}{y-1}, \frac{t\,(x-1)}{2(1-y)}, \frac{t\,(x+1)}{2\,(1-y)}\right]$$

$$= (1-y)^c \sum_{n=0}^{\infty} \frac{(a)_n\,(c)_n}{(a+b)_n\,(c+d)_n}\ {}_2F_1(-n, c+n; a; y)\, P_n^{(f,g)}(x)\ t^n \qquad \ldots (3.3)$$

which, on taking $a+b=1+f$, $c+d=1+g$, yields [5, p. 16, Eq. (3.3)].

(iv) Again, if $g_n(x) = P_n^{(f-n,\, g-n)}(x)$, then Theorem 2 gives the result [5, p. 16, Eq. (3.4)] viz.

$$F\begin{matrix} 2::0;0;0:1;1;0 \\ 0::2;0;0:0;0;1 \end{matrix}\left[\begin{matrix} a, c :: -; -; -; -f, -g, -; \\ - :: a+b, c+d; -; -; -; -; a; \end{matrix} \frac{(x+1)t}{2(y-1)}, \frac{(x-1)t}{2(y-1)}, \frac{y}{y-1}\right]$$

$$=(1-y)^c \sum_{n=0}^{\infty} \frac{(a)_n\,(c)_n}{(a+b)_n\,(c+d)_n}\, {}_2F_1(-n, c+n; a; y)\, P_n^{(f-n,\, g-n)}(x)t^n \quad \ldots(3.4)$$

Srivastava and Singhal [10, p., 357, Eq. (14)] have obtained an expansion formula by using the series iteration method, from which also the above results (3 1) to (3.4) can be obtained as specialised cases.

4. Extension To Polynomials In Two And More Variables

If

$$F(x_1, x_2, t_1, t_2) = \sum_{m,n=0}^{\infty} f_{m,n}(x_1, x_2)\, t_1^m\, t_2^n,$$

then an analogous theorem corresponding to (1.3) can be written as :

$$\frac{\Gamma(1-a_1)\Gamma(1-b_1)\Gamma(1-b_2)\Gamma(1-b_2)\,\Gamma(a_1+b_1)\Gamma(a_2+b_2)}{(2\pi i)^4}$$

$$\int\!\!\int (-p_1)^{a_1-1}(p_1-1)^{b_1-1}$$

$$(-p_2)^{a_2-1}(p_2-1)^{b_2-1}\, {}_2F_1\left[\begin{matrix} 1-b_1, c_1; \\ a_1; \end{matrix} \frac{y_1\, p_1}{(y_1-1)(p_1-1)}\right]$$

$${}_2F_1\left[\begin{matrix} 1-b_2, c_2; \\ a_2; \end{matrix} \frac{y_2\, p_2}{(y_2-1)(p_2-1)}\right] F(x_1, x_2, p_1 t_1, p_2 t_2)\, dp_1\, dp_2$$

$$=(1-y_1)^{c_1}(1-y_2)^{c_2} \sum_{m,n=0}^{\infty} \frac{(a_1)_m\,(a_2)_n}{(a_1+b_1)_m\,(a_2+b_2)_n}$$

$${}_2F_1(-m, c_1; a_1; y_1)\, {}_2F_1(-n, c_2; a_2; y_2)\, f_{m,n}(x_1, x_2)\, t_1^m\, t_2^n \quad \ldots(4.1)$$

where the integration is taken along Pochhammer's double-loop type contour.

Similarly, analogous to (1.5), we have the following results :

Theorem. 3. *If*

$$G(x_1, x_2, t_1, t_2) = \sum_{m,n=0}^{\infty} g_{m,n}(x_1, x_2)\, t_1^m\, t_2^n,$$

then

$$\frac{\Gamma(1-a_1)\,\Gamma(1-a_2)\,\Gamma(1-b_1)\Gamma(1-b_2)\,\Gamma(1-c_1)\Gamma(1-c_2)\Gamma(1-d_1)\,\Gamma(1-d_2)}{(2\pi i)^8}$$

$$\cdot\Gamma(a_1+b_1)\Gamma(a_2+b_2)\Gamma(c_1+d_1)\,\Gamma(c_2+d_2)\int\int\int\int(-p_1)^{a_1-1}(p_1-1)^{b_1-1}$$

$$\cdot q_1^{c_1-1}(q_1-1)^{d_1-1}(-p_2)^{a_2-1}(p_2-1)^{b_2-1}(-q_1)^{c_1-1}(q_1-1)^{d_1-1}(-q_2)^{c_2-1}$$

$$(q_2-1)^{d_2-1}$$

$$\cdot{}_2F_1\left[\begin{matrix}1-b_1, 1-d_1; \\ a_1;\end{matrix}\ \frac{y_1\, p_1\, q_1}{(y_1-1)(q_1-1)(q_1-1)}\right]$$

$$\cdot{}_2F_1\left[\begin{matrix}1-b_2, 1-d_2; \\ a_2;\end{matrix}\ \frac{y_2\, p_2\, q_2}{(y_2-1)(p_2-1)(q_2-1)}\right]$$

$$G\left(x_1, x_2, \frac{t_1\, p_1\, q_1}{1-y_1}, \frac{t_2\, p_2\, q_2}{1-y_2}\right) dp_1\, dp_2\, dq_1\, dq_2$$

$$= (1-y_1)^{c_1}(1-y_2)^{c_2} \sum_{m,n=0}^{\infty} \frac{(a_1)_m\,(c_1)_m\,(a_2)_n\,(c_2)_n}{(a_1+b_1)_m(c_1+d_1)_m(a_2+b_2)_n(c_2+d_2)_n}$$

$$\cdot{}_2F_1(-m, c_1+m; a_1; y_1)\ {}_2F_1(-n, c_2+n; a_2; y_2)\ g_{m,n}(x_1, x_2)\, t_1^m\, t_2^n \quad \ldots(4.2)$$

the integration being taken along the Pochhammer's double-loop type contour.

5. APPLICATIONS

(i) If we apply (4.1) to the generating function

$${}_0F_1(-; f_1; -xt)\ {}_0F_1(-; f_2; -y\rho)\ {}_0F_1[-; 1+e-f_1-f_2; (t+\rho)(1-x-y)]$$

$$= \sum_{m,n=0}^{\infty} \frac{t^m \rho^n}{(1+e-f_1-f_2)_{m+n}} \mathcal{J}_{m,n}(e, f_1, f_2; x, y) \qquad \ldots(5.1)$$

where $\mathcal{J}_{m,n}$ is a Jacobi polynomial of two variables [1, p. 100], we obtain

$$\sum_{r,s=0}^{\infty} \frac{(a_1)_r (a_2)_s [t(1-x-y)]^r [\rho(1-x-y)]^s}{(a_1+b_1)_r (a_2+b_2)_s (1+e-f_1-f_2)_{r+s} r! s!}$$

$$. F\begin{matrix} 1:1;0 \\ 0:1;2 \end{matrix}\left[\begin{matrix} a_1+r : c_1 ; - ; \\ -: a_1; a_1+b_1+r; f_1 ; \end{matrix} \frac{y_1}{y_1-1}, -xt \right]$$

$$. F\begin{matrix} 1:1;0 \\ 0:1;2 \end{matrix}\left[\begin{matrix} a_2+s : c_2; - ; \\ -: a_2; a_2+b_2+s, f_2 ; \end{matrix} \frac{y_2}{y_2-1}, -y\rho \right]$$

$$= (1-y_1)^{c_1} (1-y_2)^{c_2} \sum_{m,n=0}^{\infty} \frac{(a_1)_m (a_2)_n}{(a_1+b_1)_m (a_2+b_2)_n (1+e-f_1-f_2)_{m+n}}$$

$$. {}_2F_1(-m, c_1; a_1; y_1)\ {}_2F_1(-n, c_2; a_2; y_2)\ \mathcal{J}_{m,n}(e, f_1, f_2; x, y)\ t^m \rho^n \qquad \ldots(5.2)$$

where

$$F\begin{matrix} g:h,H \\ j:k;K \end{matrix}\left[\begin{matrix} \{a_g\} : \{b_h\} ; \{B_H\} ; \\ \{d_j\} : \{c_k\} ; \{C_K\} ; \end{matrix} x, y \right]$$

$$= \sum_{m,n=0}^{\infty} \frac{\prod_{p=1}^{g} (a_p)_{m+n} \prod_{p=1}^{h} (b_p)_m \prod_{p=1}^{H} (B_p)_n}{\prod_{p=1}^{j} (d_p)_{m+n} \prod_{p=1}^{k} (c_p)_m \prod_{p=1}^{K} (C_p)_n} \frac{x^m}{m!} \frac{y^n}{n!} \qquad \ldots(5.3)$$

is a generalised hypergeometric function of two variables [2, p. 112]. In (5.2), replacing t by tk_1 and ρ by ρk_2; multiplying both sides by $e^{-k_1-k_2} k_1^{a_1+b_1-1} k_2^{a_2+b_2-1}$ and integrating with respect to k_1 and k_2 from 0 to ∞, we get [5, p. 19, Eq. (6.1)].

(ii) If (4.2) is applied to the generating function (5.1), we get

$$\sum_{r,s=0}^{\infty} \frac{(a_1)_r\,(c_1)_r\,(a_2)_s\,(c_2)_s}{(a_1+b_1)_r\,(c_1+d_1)_r\,(a_2+b_2)_s\,(c_2+d_2)_s}\;\frac{[t(1-x-y)]^r}{r!}\;\frac{[\rho(1-x-y)]^s}{s!}$$

$$.F\begin{matrix}2:0;0\\0:1;3\end{matrix}\left[\begin{matrix}a_1+r,\,c_1+r:-;-;\\ -:a_1;\,f_1;\,a_1+b_1+r,\,c_1+d_1+r;\end{matrix}\;\frac{y_1}{y_1-1},\;\frac{-xt}{1-y_1}\right]$$

$$.F\begin{matrix}2:0;0\\0:1;3\end{matrix}\left[\begin{matrix}a_2+s,\,c_2+s:-;-;\\ -:a_2;f_2,\,a_2+b_2+s,\,c_2+d_2+s;\end{matrix}\;\frac{y_2}{y_2-1},\;\frac{-y\rho}{1-y_2}\right]$$

$$=(1-y_1)^{c_1}(1-y_2)^{c_2}\sum_{m,n=0}^{\infty}\frac{(a_1)_m\,(c_1)_m\,(a_2)_n\,(c_2)_n\,t^m\,\rho^n}{(a_1+b_1)_m(c_1+d_1)_m(a_2+b_2)_n(c_2+d_2)_n(1+e-f_1-f_2)_{m+n}}$$

$$.{}_2F_1\,(-m,\,c_1+m;\,a_1;\,y_1)\;{}_1F_2\,(-n,\,c_2+n;\,a_2;\,y_2)\;J_{m\,n}\,(e,\,f_1,\,f_2;\,x,\,y) \quad \ldots (5.4)$$

Replacing t by $tk_1\,k_2$ and ρ bp $\rho\;k_3\;k_4$, multiplying both sides by

$$e^{-k_1-k_2-k_3-k_4}k^{a_1+b_2-1}k^{a_2+b_2-1}k^{c_1+d_1-1}\;k^{c_2+d_2-1},$$

integrating with respect to k_1, k_2, k_3 and k_4 from 0 to ∞, we obtain [5, p. 19, Eq. (6.2)].

The results (4.1) and (4.2) suggest the possibility of their extension to any finite number of variables.

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(Dedicated to the memory of Professor Arthur Erdélyi)

FLUCTUATING FLOW OF A VISCO–ELASTIC FLUID PAST AN INFINITE PLANE, POROUS WALL WITH FLUCTUATING SUCTION IN SLIP FLOW REGIME

By

B. SIDDAPPA and G.S. PATEL

*Department of Mathematics, Karanatak University Post-Graduate, Centre, Gulbarga--***585105**, *India*

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ABSTRACT

This paper considers the Rivlin-Ericksen fluid flow past an infinite plane porous wall with fluctuating suction in slip flow conditions. It is observed that:

(i) The velocity profile is effected by the rarefaction parameter h_1 in the slip flow, but this is not the case in no slip flow.

(ii) If h_1 increases, the magnitude of the mean velocity increases for a fixed viscoelastic parameter K.

(iii) The main stream fluctuations cause the fluctuations in the skin friction.

(iv) As h_1 increases, the skin friction decreases for a fixed A.

(v) As A increases, the amplitude of the skin friction $|H|$ increases.

(vi) For large A a change in h_1 produces larger variations in $|H|$

(vii) For fixed h_1, as K increases, the magnitude of the mean velocity increases too.

I. Introduction

Lighthill [1] has studied the effects of fluctuations of the main stream velocity on the flow of an incompressible fluid past two dimensional bodies. Stuart [2] has obtained the exact solution of the Navier-Stokes equations for such an oscillatory flow over an infinite plane porous wall wiih constant suction. Siddappa [4] has extended Stuart's work for Rivlin-Ericksen visco-elastic fluid. Siddappa and Chetty [5] studied how Siddappa's results get modified when no slip boundary conditions are replaced by the velocity slip conditions. Here we extend this problem for the fluctuating suction.

2. Basic Equations

We consider the flow due to fluctuating main stream of Rivlin-Ericksen fluid flow past an infinite plane, porous wall with fluctuating suction at the surface.

Let $y=0$ be the wall. Let u and v be the velocity components along and normal to the wall.

The visco-elastic equations to the problem are

$$\frac{\partial v}{\partial y}=0 \tag{2.1}$$

$$\frac{\partial u}{\partial t}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial P}{\partial x}+\alpha\frac{\partial^2 u}{\partial y^2}+\beta\frac{\partial^2}{\partial y^2}\left(\frac{\partial u}{\partial t}+v\frac{\partial u}{\partial y}\right) \tag{2.2}$$

$$\frac{1}{\rho}\frac{\partial P}{\partial y}+2\left(2\beta+\gamma\right)\left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial^2 u}{\partial y^2}\right)=0 \tag{2.3}$$

where α is the kinematic viscosity, β is the kinematic visco elasticity, γ is the kinematic cross-viscoelasticity.

3. Boundary Conditions

Th first order velocity slip condition is

(i) $$u=\left(\frac{2-f_1}{f_1}\right)L\left(\frac{\partial u}{\partial y}\right)=L_1\left(\frac{\partial u}{\partial y}\right) \text{ at } y=0 \tag{3.1}$$

(ii) $u=U(t)$

(iii) $\dfrac{\partial u}{\partial y}=0$

$$\left.\begin{aligned} &u=U(t)\\ &\frac{\partial u}{\partial y}=0\end{aligned}\right\}\text{ as } y\to\infty \tag{3.2}$$

which arises because of the symmetry about the axis of the pipe through the point at infinity.

Here f_1 is Maxwell's reflexion coefficient.

$L=\mu(\pi/2P\rho)^{\frac{1}{2}}$ is the mean free path, and is constant for an incompressible fluid, and $L_1=\left(\frac{2-f_1}{f_1}\right)L$

4. Mathematical Analysis

Considering (2.2) as $y\rightarrow\infty$, $u\rightarrow U(t)$ and the partial derivatives of u w.r.t. y tend to zero.

Therefore we have

$$\frac{dU}{dt}=-\frac{1}{\rho}\frac{dP}{dx}=-\frac{1}{\rho}\frac{dg}{dx} \tag{4.1}$$

and so by integrating we have

$$g(x,t)=-\rho x\frac{dU}{dt}+h(t) \tag{4.2}$$

where h (t) is at most a function of t.

Thus the pressure is given by

$$P(x,y,t)=\rho(2\beta+\gamma)\left(\frac{\partial u}{\partial y}\right)^2-\rho x\frac{dU}{dt}+h(t) \tag{4.3}$$

It is clear from (4.3) that the pressure of the fluid depends on y in addition to x and t, unlike in the classical viscous case. This is because of the viscoelastic nature of the fluid.

The equation of our interest is

$$\frac{\partial u}{\partial t}+v\frac{\partial u}{\partial y}=\frac{dU}{dt}+\alpha\frac{\partial^2 u}{\partial y^2}+\beta\frac{\partial^2}{\partial y^2}\left\{\frac{\partial u}{\partial t}+v\frac{\partial u}{\partial y}\right\} \tag{4.4}$$

Integrating (2.1) following Messiha [3] we have

$$V=-V_0\left(1+eAe^{i\omega t}\right) \tag{4.5}$$

negative sign indicates that the suction velocity is directed towards the plate. We look for the solution of the equation (4.4) following Lighthill [1] of the form

$$u = U_0[\phi_0(y) + \epsilon\phi_1(y)\, e^{i\omega t}] \tag{4.6}$$

The free stream velocity is

$$u = U(t) = U_0(1 + \epsilon e^{i\omega t}) \text{ as } y \to \infty \tag{4.7}$$

substituting the expressions for u, U and v into (4.4)and equating the harmonic and the nonharmonic terms of the equation we get

$$-V_0 \frac{d\phi_0}{dy} = \alpha \frac{d^2\phi_0}{dy^2} - \beta V_0 \frac{d^3\phi_0}{dy^3} \tag{4.8}$$

and

$$i\omega\phi_1 - V_0 \frac{d\phi_1}{dy} = i\omega + \alpha \frac{d^2\phi_1}{d\ ^2} + \beta i\omega \frac{d^2\phi_1}{dy^2} - \beta V_0 \frac{d^3\phi_1}{dy^3} - \alpha A \frac{d^2\phi_0}{dy^2} \tag{4.9}$$

with the boundary conditions

$$\left.\begin{array}{l} \left.\begin{array}{l} \phi_0 = L_1 \dfrac{d\phi_0}{dy} \\ \phi_1 = L_1 \dfrac{d\phi_1}{dy} \end{array}\right\} \text{at } y=0 \\ \text{and} \\ \left.\begin{array}{l} \phi_0 = \phi_1 = 1 \\ \dfrac{d\phi_0}{dy} = \dfrac{d\phi_1}{dy} = 0 \end{array}\right] \text{at } y \to \infty \end{array}\right\} \tag{4.10}$$

putting $\eta = \dfrac{V_0}{\alpha} y$. Equation (4.8), (4.9) and the boundary conditions (4.10) reduce to

$$K \frac{d^3\phi_0}{d\eta^3} + \frac{d^2\phi_0}{d\eta^2} + \frac{d\phi_0}{d\eta} = 0 \tag{4.11}$$

$$K \frac{d^3\phi_1}{d\eta^3} + (1) - i\lambda k) \frac{d^2\phi_1}{d\eta^2} + \frac{d\phi_1}{d\eta} - i\lambda\phi_1 = i\lambda + A \frac{d^2\phi_0}{d\eta^2} \tag{4.12}$$

and

$$\left.\begin{aligned} &\phi_0=h_1\frac{d\phi_0}{d\eta},\ \phi_1=h_1\frac{d\phi_1}{d\eta}\ \text{at}\ \eta=0 \\ &\phi_0=\phi_1=1,\ \frac{d\phi_0}{d\eta}=\frac{d\phi_1}{d\eta}=0\ \text{at}\ \eta\to\infty \end{aligned}\right\} \tag{4.13}$$

where $\lambda=\frac{\omega a}{V^2_0}$, $K=-\beta\frac{V_0^2}{a^2}$ and

$h_1=L_1\frac{V_0}{a}$ is the rarefaction parameter.

The solution of the equation (4.11) subject to the conditions (4.13) is

$$\phi_0=1-\frac{e^{-a_1\eta}}{1+a_1h_1} \tag{4.14}$$

where $a_1=\frac{1-\sqrt{1-4K}}{2K}$ for $K\leqslant 0$

and the other solution is

$$\phi_0=1-\frac{e^{-a_2\eta}}{1+a_2h_1} \tag{4.15}$$

where $a_2=\frac{1+\sqrt{1-4K}}{2K}$ for $0<K\leqslant\frac{1}{4}$.

The solution of (4.12) subject to the conditions (4.13) is

$$\phi_1=1-\frac{e^{-h\eta}}{1+hh_1}-\frac{Aa^2e^{-a\eta}}{(1+ah_1)\,f(-a)} \tag{4.16}$$

where $h=h_r+ih_1$ is given by the equation

$$f(h)=Kh^3+(1-i\lambda K)h^2+h-i\lambda=0 \tag{4.17}$$

and $a=a_1$ for $K\leqslant 0$; $a=a_2$ for $0<K\leqslant\frac{1}{4}$.

The equation (4.17) is solved numerically [4].

When K=0, (4,14) and (4.15) become the asymptotic suction profile in the Newtonian fluid.

The unperturbed thickness δ is given by

$$\delta = \int_{y=0}^{\infty}\left(1-\frac{u}{U}\right) dy, \text{ where U is the value of u at } y=\infty \cdot$$

$$\text{Thus } \delta = \frac{a}{V_0} \int_{\eta=0}^{\infty}\left[1 - \left\{1 - \frac{e^{-a\eta}}{1+ah_1}\right\}\right] d\eta,$$

whence

$$\delta = \frac{a}{V_0(1+ah_1)a} = \sqrt{\left(\frac{\lambda a}{wa^2(1+a\,h_1)^2}\right)}$$

As h_1 increases, thickness decreases for $K \leqslant 0$, and also for $0 < K \leqslant \frac{1}{4}$; and when $h_1=0$, it reduces to the result of Siddappa [4].

The total velocity component parallel to the wall is given by

$$u=U_0\left[1-\frac{e^{-a\eta}}{1+ah_1}+\varepsilon\left(1-\frac{e^{-h\eta}}{1+hh_1}-\frac{Aa^2 \; e^{-a\eta}}{(1+ah_1)f(-a)}\right)e^{iwt}\right] \quad (1.18)$$

Thus the velocity profile is effected by the rarefaction parameter h in the slip flow. This is not valid in the case of no slip flow since $h_1=0$.

If $h_1 \to \infty$, the velocity (4·18) become $u=U_0\ [1+\varepsilon e^{iwt}]$ which is the main stream velocity. This does not happen in the case of no slip flow.

In slip flow, because of the boundary conditions, the shear stress at the wall is proportionol to the slip velocity at the wall and given by

$$\tau_\omega = \rho\alpha\left(\frac{\partial u}{\partial y}\right)_{y=0} = \frac{\rho\alpha}{L_1}(u)_{y=0}$$

$$=\rho V_0 U_0\left[1+\frac{a}{1+ah_1}+\varepsilon\left(\frac{h}{1+hh_1}-\frac{Aa^2}{h_1(1+ah_1)f(-\alpha)}\right)e^{iwt}\right.$$

$$=\rho V_0 U_0\left[\frac{a}{1+ah_1}+a\,\varepsilon\,|\,H\,|\,e^{i(\omega t+\theta)}\right] \quad (4\ 19)$$

The non-dimensional skin friction is given by

where $\tau_0 = \dfrac{a}{1+ah_1}$

$$\text{mean friction} \frac{\tau}{V_0 U_0 \rho} = \tau_0 + \epsilon \mid H \mid e^{i(\omega t+\theta)} \tag{4.20}$$

$$\mid H \mid = \sqrt{H_r^2 + H_i^2} \tag{4.21}$$

$$\theta = \tan^{-1} (H_i/H_r) \tag{4.22}$$

$$H_r = \frac{h_r + h_i\left(h_r^2 + h_i^2\right)}{\left(1+h_1h_r\right)^2 + h_1^2 h_r^2} + \frac{Aa^2(1+ah_1)}{h_1^2\left(1+ah_1\right) + a^2\lambda^2}$$

$$H_i = \frac{h_i}{\left(1+h_1h_r\right)^2 + h_1^2 h_i^2} - \frac{Aa^3\lambda}{h_1^2\left(1+ah_1\right) + a^2\lambda^2}.$$

DISCUSSION

It is observed from (4.20) that the mean friction τ_0 decreases with the increasing values of slip parameter h_1. We observe from equation (4.19) that the main stream fluctuations cause the fluctuations in the skin friction.

For a fixed h_l it is observed that $\mid H \mid$ increase with increase in the suction parameter. Also as $h_1 \to \infty$, $\mid H \mid$ vanishes asymptotically, for a fixed A, $\mid H \mid$ decreases for the increase in h_1. It is observed that:

(i) As K increases for a fixed h_l velocity increases.

(ii) For fixed K as h_1 increases velocity increases.

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