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(*Dedicated to the memory of Professor Arthur Erdélyi*)

A GENERAL CLASS OF UNIVARIATE DISTRIBUTIONS INVOLVING THE H-FUNCTOIN OF SEVERALS VARIABLES

By

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ABSTRACT

In this paper, an attempt has been made to present a unified theory of the classical statistical distributions associated with the generalized beta and gamma distributions of one variate. The probability density function is taken in terms of the H-function of several variables with general arguments. In particular, the characteristic function and the distribution function are investigated.

1. Introduction

In probability theory, a large number of statistical distributions have been studied from time to time by several authors. For example, Mathai and Saxena [5] introduced a general hypergeometric distribution, whose probability density functions involves a hypergeometric function ${}_2F_1$. Again, Srivastava and Singhal [7] studied another general class of distributions, whose probability density function involves the H-function of C. Fox ([4], p. 408]. It may be readily seen that the distributions, considered by Mathai and Saxena [5] and all other well-known classical statistical distributions, such as the generalized beta and gamma distributions, the exponential distribution, the generalized F-distribution, students t-distribution, the normal distribution, etc., can be derived as specialized or confluent cases of the class of distributions, considered by Srivastava and Singhal [7]. More recently, Exton [3] considered the family of distributions which have the probability density function in terms of the product of several generalized hypergeometric functions ${}_pF_q$.

In an attempt to present a further generalization of the probability distributions studied by Srivastava and Singhal [7], Exton [3], etc., here we introduce and study a general family of statistical probability distributions involving the H-function of several variables, which was defined and studied elsewhere by Srivastava and Panda (cf., e. g , [8], [9], [10] and [11]). Since the H-function of several variables includes almost all the special functions of one or more variables as its particulars cases, it can define a very general class of probability model. Thus all the classical statistical distributions, mentioned here and elsewhere, will be the special cases of our findings. The parameters of the H-function of several variables are to be resricted in such a way that the function is non-negative and finite in the region under consideration.

Following the notations explained fairly fully in the earlier papers by Srivastava and Panda [9] and [10], the H-function of n complex variables is defined in the manner given below :

$$H[z_1,\ldots,z_n]=H^{0,\lambda:(\mu',\nu');\ldots;(\mu^{(n)},\nu^{(n)})}_{A,C:[B',D'];\ldots;[B^{(n)},D^{(n)}]}\left[\begin{matrix}[(a):\theta',\ldots,\theta^{(n)}]:\\ [(c):\psi',\ldots,\psi^{(n)}];\end{matrix}\begin{matrix}[(b'):\phi'];\ldots;[(b^{(n)}):\phi'^{(n)}];\\ [(d'):\delta'];\ldots;[(d^{(n)}):\delta^{(n)}];\end{matrix}\, z_1,\ldots,z_n\right]$$

$$=\frac{1}{(2\pi\omega)^n}\int_{L_1}\cdots\int_{L_n}\Phi_1(\xi_1)\cdots\Phi_n(\xi_n)\,\psi(\xi_1,\ldots,\xi_n)\,z_1^{\xi_1}\cdots z_n^{\xi_n}\,d\xi_1\ldots d\xi_n,$$

$$\omega=\sqrt{-1} \tag{1.1}$$

here

$$\Phi_i(\xi_i)=\frac{\prod_{j=1}^{\mu^{(i)}}\Gamma[d_j^{(i)}-\delta_j^{(i)}\xi_i]\prod_{j=1}^{\nu^{(i)}}\Gamma[1-b_j^{(i)}+\phi_j^{(i)}\xi_i]}{\prod_{j=\mu^{(i)}+1}^{D^{(i)}}\Gamma[1-d_j^{(i)}+\delta_j^{(i)}\xi_i]\prod_{j=\nu^{(i)}+1}^{B^{(i)}}\Gamma[b_j^{(i)}-\phi_j^{(i)}\xi_i]}$$

$(i=1,\ldots,n)$;

$$\psi(\xi_1,\ldots,\xi_n)=\frac{\prod_{j=1}^{\lambda}\Gamma[1-a_j+\sum_{i=1}^{n}\theta_j^{(i)}\xi_i]}{\prod_{j=\lambda+1}^{A}\Gamma[1-a_j+\sum_{i=1}^{n}\theta_j^{(i)}\xi_i]\prod_{j=1}^{C}\Gamma[1-c_j+\sum_{i=1}^{n}\psi_j^{(i)}\xi_i]} \quad \ldots(1\cdot 3)$$

Also, let the associated positive numbers

$$\left.\begin{array}{l}\theta_j^{(i)}, j=1,\ldots,A\,;\ \phi_j^{(i)}, j=1,\ldots,B^{(i)}\,;\\ \psi_j^{(i)}, j=1,\ldots,C\,;\ \delta_j^{(i)}, j=1,\ldots,D^{(i)}\,;\ i=1,\ldots,n\end{array}\right\} \quad \ldots(1.4)$$

be constrained by the inequalities :

$$\Lambda_i \equiv -\sum_{j=\lambda\ 1}^{A}\theta_j^{(i)}+\sum_{j=1}^{\nu^{(i)}}\phi_j^{(i)}-\sum_{j=\nu^{(i)}+1}^{B^{(i)}}\phi_j^{(i)}-\sum_{j=1}^{C}\psi_j^{(i)}$$

$$+\sum_{j=1}^{\mu^{(i)}}\delta_j^{(i)}-\sum_{j=\mu^{(i)}+1}^{D^{(i)}}\delta_j^{(i)}>0, \quad \ldots(1.5)$$

$$\Omega_i \equiv \sum_{j=1}^{A}\theta_j^{(i)}+\sum_{j=1}^{B^{(i)}}\phi_j^{(i)}-\sum_{j=1}^{C}\psi_j^{(i)}-\sum_{j=1}^{D^{(i)}}\delta_j^{(i)}<0, \quad \ldots(1.6)$$

Then it is known that the multiple Mellin-Barnes contour integral defining the H-function of several variables (1.1) would converge absolutely when

$$|\arg(z_i)|<\tfrac{1}{2}\Lambda_i\pi, \quad i=1,\ldots,n, \quad \ldots(1.7)$$

Furthermore, from the known asymptotic expansion ([9], p. 131, Eq. (1.9)), we have

$$H[z_1,\ldots,z_n]=\begin{cases}O(|z_1|^{\alpha}1\ldots|z_n|^{\alpha}),\ \max\{|z_1|,\ldots,|z_n|\}\to 0,\\ O(|z_1|^{\beta_1}\ldots|z_n|^{\beta_n}),\ \lambda\equiv 0,\ \min\{|z_1|,\ldots,|z_n|\}\to\infty\end{cases}$$

where, with $i=1,\ldots, n$,

$$\begin{cases} \alpha_i = \min \{ \operatorname{Re} (d_j^{(i)}) / \delta_j^{(i)} \}, \; j = 1, \ldots, \mu^{(i)}, \\ \beta_i = \max \{ \operatorname{Re} (b_j^{(i)} - 1) / \emptyset_j^{(i)} \}, \; j = 1, \ldots, \nu^{(i)}, \end{cases} \qquad \ldots (1.9)$$

In terms of Fox's H-function, we find it worthwhile to record here the interesting relationship :

$$\lim_{z_2,\ldots,z_n \to 0} H^{0,\lambda : (\mu',\nu') ; (1, B'') ; \ldots ; (1, B^{(n)})}_{A, C : [B', D'] ; [B'', D''+1] ; \ldots [B^{(n)}, D^{(n)}+1]}$$

$$\left[\begin{matrix} [(a) : \theta', 1,\ldots, 1] : [(b') : \emptyset'] ; [(b'') : 1] ;\ldots; [(b^{(n)}) : 1] ; \\ [(c) : \psi', 1,\ldots, 1] : [(d') : \delta'] ; (0,1), [(d'') : 1] ;\ldots; (0, 1), [(d^{(n)}) : 1] ; \end{matrix} \; z_1, \ldots, z_n \right]$$

$$= \prod_{i=2}^{n} \left[\prod_{j=1}^{B^{(i)}} \Gamma (1 - b_j^{(i)}) \left\{ \prod_{j=1}^{D^{(i)}} \Gamma (1 - d_j^{(i)}) \right\}^{-1} \right]$$

$$H^{\mu', \nu' + \lambda}_{B'+A, D'+C} \left[z_1 \middle| \begin{matrix} (a_j, \theta_j')_{1, \lambda}, (b_j', \emptyset_j')_{1, B'}, (a_j, \theta_j')_{\lambda+1, A} \\ (d_j', \delta_j')_{1, D'}, (c_j, \psi_j')_{1, C} \end{matrix} \right] \qquad \ldots (1.10)$$

Throught the present work, *we shall assume that the convergence (and existance) conditions given by (1.5), (1.6) and (1.7) are satisfied by each of the various H-functions involved.*

To simplify the space problem, we specify the parameters of the H-function of several variables in the following manner, throught the paper. Thus

$$H \left[\begin{matrix} [k : k_1 ,\ldots, k_n] ; \\ [u : u_1 ,\ldots, u_n] ; \end{matrix} \; z_1 ,\ldots, z_n \right]$$

would mean

$$H^{0, \lambda+1 : (\mu', \nu') ;\ldots; (\mu^{(n)}, \nu^{(n)})}_{A+1, C+1 : [B' D'] ;\ldots; [B^{(n)}, D^{(n)}]} \left[\begin{matrix} [k : k_1,\ldots, k_n], [(a) : \theta',\ldots, \theta^{(n)}] : \\ [(c) : \psi',\ldots, \psi^{(n)}], [u : u_1, \ldots, u_n] : \end{matrix} \right.$$

$$\left. \begin{matrix} [(b') : \emptyset'] ; \ldots ; [(b^{(n)}) ; \emptyset^{(n)}] ; \\ [(d') : \delta'] ; \ldots ; [(d^{(n)}) : \delta^{(n)}] ; \end{matrix} \; z_1 ,\ldots, z_n \right] \qquad \ldots (1.11)$$

2. Probability Density Functions

This paper deals with certain classical statistical distributions associated with beta (or finite) and gamma (or infinite) distributions of one variate. The probability density function is taken in terms of the H-function of several variables, defined by (1.1). with generals arguments. First we find the probability density function.

Let a density function be defined by

$$f(x) = K\, x^{\sigma-1} (1-x)^{\rho-1}\, H[z_1 x^{\sigma_1} (1-x)^{\rho_1}, \ldots, z_n x^{\sigma_n} (1-x)]^{\rho_n},\ 0 \leqslant x \leqslant 1, \quad \ldots(2.1)$$

for *finite distribution or generalized beta distribution*, and $f(x) = 0$, elsewhere. If $f(x)$ is a probability density function, then it should satisfy the relation

$$\int_{-\infty}^{\infty} f(x)\, dx = \int_{0}^{1} f(x)\, dx \equiv 1. \quad \ldots(2.2)$$

Putting the value of $f(x)$ given by (2.1) in (2.2) and evaluating the resulting integral with the help of Mellin-Barnes contour integral for the H-function of several variables, given by (1.1) and the well-known definition of beta function (see, e. g. [2], p 9, Eq. (1)), we find that

$$K^{-1} = H\left[\begin{matrix} [1-\sigma : \sigma_1, \ldots, \sigma_n]\ [1-\rho : \rho_1, \ldots, \rho_n]; \\ [1-\sigma-\rho : \sigma_1+\rho_1, \ldots, \sigma_n+\rho_n]\ ; \end{matrix}\ z_1, \ldots, z_n\right] \quad \ldots(2.3)$$

provided

$$\left.\begin{matrix} \mathrm{Re}(\sigma) + \sigma_i \alpha_i > 0, \text{ and} \\ \mathrm{Re}(\rho) + \rho_i \alpha_i < \forall\, i \in \{1, \ldots, n\} \end{matrix}\right\} \quad \ldots(2\ 4)$$

where α_i is given by (1.9).

Again let

$$f(x) = Q\, e^{-sx}\, x^{\sigma-1}\, H[z_1 x^{\sigma_1}, \ldots, z_n x^{\sigma_n}] \quad \ldots(2.5)$$

where $0<x<\infty$, $\mathrm{Re}(s) > 0$, $\mathrm{Re}(\sigma) > 0$, $\mathrm{Re}(\sigma) + \sigma_i \alpha_i > 0$, and

(α_i is given by (1.9))

$$Q^{-1} = s^{-\sigma}\, H\left[\begin{matrix} [1-\sigma : \sigma_1, \ldots, \sigma_n]; \\ \text{———}\ ; \end{matrix}\ z_1 s^{-\sigma_1}, \ldots, z_n s^{-\sigma_n}\right] \quad \ldots(2.6)$$

then $f(x)$ will be a probability density function for *infinite distribution or generalized gamma distribution*. By virtue of (2.2), we easily get

$$\int_0^\infty e^{-sx} \; x^{\sigma-1} H\left[z_1 x^{\sigma_1}, ..., z_n x^{\sigma_n} \right] dx = 1. \qquad ...(2.7)$$

Now, on evaluating the above integral with the help of (1.1) and the definition of gamma function (see, e. g. [2], p. 1, Eq (1)), we obtain the expression for Q given by (2.6).

Remark I. Complex values of the parameters hold little interest for statistics, but (2.1) and (2.5) can still define probability models ([1], p. 59 ; see also [6], p. 85, Eq. (4 2.2)).

Remark 2. Since Re (s) $>$ 0 in (2.5), the convergence of the integral (2.7) at its upper limit of integration can be guaranteed under the conditions stated already with the help of the following asymptototic expansion (see, e. g., 11], p. 357, Eq. (1.17)).

$$H[z_1, ..., z_n] = O\left(|z_1|^{\gamma_1} ... |z_n|^{\gamma_n} \right), \lambda \neq 0, \min\{ |z_1|, ..., |z_n| \} \to \infty, \qquad ...(2.8)$$

for some $\gamma_1, .., \gamma_n$; which would evidently complement the asymptotic expansions given by (1.8).

3. The Distribution function

The distribution function or the cumulative probability function $F(t)$ corresponding to a probability density function $f(x)$ is defined as

$$F(t) = \int_{-\infty}^{t} f(x)\, dx. \qquad ...(3.1)$$

We obtain here the distribution function for beta and gamma distributions separately.

If $f(x)$ is defined by (2.1) (with $\rho=1, \rho_1=\rho_2=...=\rho_n=0$), the distribution function for *finite distribution* will be given by

$$F(t) = K\, t^{\sigma} \; H\left[\begin{matrix} [1-\sigma : \sigma_1, .., \sigma_n] ; \\ [-\sigma : \sigma_1, ..., \sigma_n] ; \end{matrix} \; z_1 t^{\sigma_1}, ..., z_n t^{\sigma_n} \right] \qquad ...(3.2)$$

where

$$K^{-1} = H\left[\begin{matrix}[1-\sigma : \sigma_1, \ldots, \sigma_n] ; \\ [-\sigma : \sigma_1, \ldots, \sigma_n] ;\end{matrix}\; z_1, \ldots, z_n\right], \qquad \ldots(3.3)$$

and $\operatorname{Re}(\sigma) + \sigma_i a_i > 0$, a_i is given by (1.9), $i=1, \ldots, n$,

where, as already assumed in Section 1, the function $f(x) = 0$ for negative values of x.

Similarly, for *infinite distribution,* the distribution function $F(t)$ defined by (3.1) can also be obtained for (2.5), which is given below :

$$F(t) = Q \sum_{r=0}^{\infty} \frac{(-s)^r (t)^{\sigma+r}}{r!} H\left[\begin{matrix}[1-\sigma-r : \sigma_1, \ldots, \sigma_n] ; \\ [-\sigma-r : \sigma_1, \ldots, \sigma_n] ;\end{matrix}\; z_1 t^{\sigma_1}, \ldots, z_n t^{\sigma_n}\right] \qquad \ldots(3.4)$$

where Q is given by (2.6).

We remark in passing that the distribution function, obtained by Srivastava and Singhal ([7], p. 6, Eq. (14)), can be deduced as a particular case of our function (3.2). This can be varified, if we put in (3.2), $\lambda=A=C=0$, $\mu^{(j)}=1$, $\nu^{(j)}=B^{(j)}$, $d_1^{(j)}=0$, $\delta_1^{(j)}=1$, ($j=2, 3, .., n$), take $z_2, z_3, \ldots, z_n \to 0$, and make a use of relationship given by (1.10). Also, the distribution function, recently given by Exton ([3], p 129, Eq. (7.1.1.5)) can alternatively be deduced as a particular case of (3.2). Indeed, if in (3.2) we put $\lambda=A=C=0$, $\mu^{(i)}=1$, $\nu^{(i)} = B^{(i)}$, $d_1^{(i)}=0$, $\delta_1^{(i)}=1$, replace $D^{(i)}$ by $D^{(i)}+1$ ($i=1, \ldots, n$), make suitable changes in parameters and appeal to the known relationships ([6], p. 151) and ([8], p. 272, Eq (4.7)).

4. The Characteristic Function

The characteristic function denoted by $\emptyset(t)$ may be represented as $< e^{itx} >$, where the angle brackets denote "mathematical expectation" We may thus write the characteristic function as

$$\emptyset(t) = < e^{itx} > = \int_{-\infty}^{\infty} e^{itx} f(x)\, dx \qquad \ldots(4.1)$$

The characteristic function for *finite distribution*, when $f(x)$ is given by (2.1), is

$$\phi(t)=K\sum_{r=0}^{\infty}\frac{(it)^r}{r!}H\left[\begin{matrix}[1-\sigma-r:\sigma_1,\ldots,\sigma_n],[1-\rho:\rho_1,\ \ ,\rho_n]; \\ [1-\sigma-\rho-r:\sigma_1+\rho_1,\ldots,\sigma_n+\rho_n]\quad ;\end{matrix}\ z_1,\ldots,z_n\right] \qquad \ldots(4.2)$$

where K is given by (2.3).

Also, the characteristic function for *infinite distribution*, when $f(x)$ is given by (2.5), is

$$\phi(t)=Q\,(s-it)^{-\sigma}H\left[\begin{matrix}[1-\sigma:\sigma_1,\ldots,\sigma_n]; \\ \text{———}\quad ;\end{matrix}\ z_1(s-it)^{-\sigma_1},\ldots,z_n(s-it)^{-\sigma_n}\right] \qquad \ldots(4.3)$$

where Q is given by (2.6).

On reduction of the H-function of several variables occurring in (4.3) in to Fox's H-function, with the help of relationship (1.10), we get the characteristic function considered by Srivastava and Singhal ([7], p. 5, Eq. (2)). Finally, by suitably specializing the parameters of the H-function of several variables and invoking the known result ([8], p. 272 Eq. (4.7)), the expression (4.3) will reduce to the characteristic function studied earlier by Exton ([3], p. 129, Eq. (7.1.1.5)).

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