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(*Dedicated to the memory of Professor Arthur Erdélyi*)

ON THE CONVERGENCE AND RIESZ-SUMMABILITY OF A FOUIER-BESSEL SERIES OF SPECIAL KIND

By

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SUMMARY

In this paper the authors extend a convergence theorem of F. Kito [4] regarding the Fourier-Bessel series (1.3) below. The Riesz-summability of the series (1.3) corresponding to Lebesgue-integrable functions is also proved.

1. INTRODUCTION

Let

(1.1) $$Q_\nu(\alpha, \beta) = J_\nu(\alpha)\, Y'_\nu(\beta) - J'_\nu(\beta)\, Y_\nu(\alpha),$$

where $J_\nu(t)$ and $Y_\nu(t)$ denote Bessel functions of the first and second kinds respectively for $\nu \geqslant -1/2$ and $k_1, k_2, k_3, \ldots$ etc. the positive zeros of $S(z)$ in increasing order of magnitude, where

(1.2) $$S(z) = J'_\nu(bz)\, Y'_\nu(az) - J'_\nu(az)\, Y_\nu(a), \qquad 0 < a < b.$$

The series

$$(1.3)\qquad f(x) \sim \sum_{m=1}^{\infty} p_m \, Q_\nu (xk_m, ak_m) \qquad a \leqslant x \leqslant b$$

where

$$(1.3)\qquad p_m = \left[\int_a^b t\, f(t)\, Q_\nu (tk_m, ak_m)\, dt \right] / M(k_m)$$

and

$$(1.5)\qquad M(k_m) = \int_a^b t \; Q_\nu^2 (tk_m, ak_m)\, dt$$

$$= \frac{2}{\pi^2 k_m^2} \left[\left(1 - \frac{\nu^2}{b^2 k_m^2} \right) \left(\frac{J'_\nu (ak_m)}{J'_\nu (bk_m)} \right)^2 - \left(1 - \frac{\nu^2}{a^2 k_m^2} \right) \right] .$$

corresponding to any function $f \in L[a, b]$, was first used by F. Kito [3] while studying the vibrations of a cylindrical shell immersed in water.[1]

The n-th partial sum of the series (1.3) is given by

$$(1.6)\qquad S_n(x) = \sum_{m=1}^{n} p_m Q_\nu (xk_m, ak_m) = \int_a^b t\, f(t)\, U_n(t, x)\, dt,$$

where

$$(1.7)\qquad U_n(t, x) = \sum_{m=1}^{n} \frac{Q_\nu (xk_m, ak_m)\; Q_\nu (tk_m, ak_m)}{M(k_m)} .$$

The series (1.3) is called Riesz-summable or summable (R) to a sum s, if for $k_n < L_n < k_{n+1}$,

$$\lim_{n \to \infty} \sum_{m=1}^{n} (1 - k_m/L_n)\, p_m \; Q_\nu (xk_m, ak_m) = s.$$

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[1] In [4], Dr. Kito has used $G_\nu(z)$ instead of $Y_\nu(z)$, where

$$G_\nu(z) = -(\pi/2)\, Y_\nu(z).$$

The sum in the above limit, called the Riesz-sum corresponding to (1.3), is denoted by $R_n^{(\nu)}(x, f)$. By (1.4)

$$(1.8) \qquad R_n^{(\nu)}(x, f) = \int_a^b f(t)\, \emptyset_n(t, x/R)\, dt,$$

where

$$(1.9) \qquad \emptyset_n(t, x/R) = \sum_{m=1}^{n} \left(1 - \frac{k_m}{L_n}\right) \frac{Q_\nu(xk_m, ak_m)\, Q_\nu(tk_m, ak_m)}{M(k_m)}.$$

The authors of the present paper have proved, ([1], [2]), certain properties of (1.3) regarding the order of its coefficients and convergence corresponding to functions of bounded veriation, square integrable and differentiable functions.

Kito [4] proved for a function $f \in L[a, b]$, which is also of bounded variation in $[a, b]$, that its Fourier-Bessel series (1.3), when ν is a positive integer, converges to the sum $\frac{f(x+0) + f(x-0)}{2}$.

In this paper we extend Kito's theorem to a general case in which $\nu \geqslant -1/2$, $\nu \neq 0$. We also establish Riesz-summability of (1.3).

Our theorems are as follows :

Theorem I. *If* $f \in L[a, b]$ *and has bounded variation in a neighbourhood* $x \in (a, b)$, *then the series (1.3) converges to the sum* $\frac{1}{2}\{f(x+0)+f(x-0)\}$.

Theorem 2. *If* $f \in L[a, b]$, $\nu \geqslant -1/2$, $\nu \neq 0$ *and if for* $x \in (a,b)$, *the limits* $f(x \pm 0)$ *exist, then the series (1.3) is summable (R) to the sum* $\frac{1}{2}\{f(x+0)+f(x-0)\}$. [1)]

2. PREREQUISITES

The following lemmas, estimating $U_n(t, x)$, $\emptyset_n(t, x/R)$ and certain integrals concerning them, are needed to prove the above theorems (K_1, K_2, K_3,...etc. denote suitable positive constants throughout the paper) :

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1) This theorem is analoguous to Fejer's theorem

Lemma I. *Let* $F(w)=\pi w \dfrac{Q_\nu(tw, aw)\, Q_\nu(xw, bw)}{S(w)}$, $a < t < x < b$.

Then on the rectangle Γ, *whose vertices are* $\pm Li$, $L_n \pm Li$ *in the w-plane, where* $k_n < L_n < k_{n+1}$, *and* L *is to be made to tend to infinity,*

$$F(w) = O\left(\frac{e^{-|v|(x-t)}}{\sqrt{xt}}\right),$$

where $w=u+iv$ *and* n *is sufficiently large.*

Proof. We know that [6 ; pp. 228-229], $\varphi(t)=\sqrt{t}\, Q_\nu(tw, aw)$ is a solution of the differential equation

$$\frac{d^2\varphi}{dt^2} + (w^2 - \frac{\nu^2-1/4}{t^2}\varphi=0.$$

Hence [5 ; p. 10, equation (1.7.3)],

$$(2.1) \qquad Q_\nu(tw, aw) = \frac{2\cos(t-a)w}{\pi w\sqrt{at}} + O\left(\frac{e^{|v|(t-a)}}{|w|}\right),$$

where $a \leqslant t \leqslant b$ and $|w| \to \infty$.

Also [5 ; p.10, equation (1.7.6)],

$$(2.2) \qquad \varphi'(t) = -\frac{2\sin(t-a)w}{\pi\sqrt{a}} + O(e^{|v|(t-a)}),$$

where $a \leqslant t \leqslant b$ and $|w| \to \infty$. By (2.1) and (2.2),

$$(2.3) \qquad S(w) = -\frac{2\sin(b-a)w}{\pi w\sqrt{ab}} + O\left(\frac{e^{|v|(b-a)}}{|w|}\right),$$

when $|w| \to \infty$. Also [5 ; p. 13],

$$(2.4) \qquad |\sin(b-a)w| > K_1 e^{|v|(b-a)},$$

on the rectangle Γ and for arbitrary real $\alpha > 0$ and $w=u+iv$, it is true that

$$(2.5) \qquad |\cos \alpha w| < \tfrac{1}{2} e^{\alpha|v|},$$

The lemma, now, follows from (2 1), (2.3) to (2.5).

Lemma 2. *The following inequalities hold true for* $a \leqslant x \leqslant b$, $a \leqslant t \leqslant b$:

$$(2.6) \qquad | U_n (t, x) | \leqslant \frac{K_2}{| t - x |}, \; t \neq x ;$$

$$(2.7) \qquad \left| \int_a^t t^{\nu+1} U_n (t, x)\, dt \right| \leqslant \frac{K_3}{L_n} \left\{ \frac{1}{x-t} + \frac{1}{x-a} \right\}, \; x > t;$$

and

$$(2.8) \qquad \left| \int_t^b t^{\nu+1} U_n (t, x)\, dt \right| \leqslant \frac{K_4}{L_n} \left\{ \frac{1}{b-x} + \frac{1}{t-x} \right\}, \; t > x.$$

Proof. Using Watson, [7; p. 76, equation (6) and setting

$$\eta = \frac{J'_\nu (ak_m)}{J'_\nu (bk_m)} = \frac{Y'_\nu (ak_m)}{Y'_\nu (bk_m)}, \text{ we get}$$

$$S' (k_m) = b \left\{ J''_\nu (bk_m)\, Y'_\nu (ak_m) - J'_\nu (ak_m)\, Y''_\nu (bk_m) \right\} +$$

$$+ a \left\{ J'_\nu (bk_m)\, Y''_\nu (ak_m) - J''_\nu (ak_m)\, Y'_\nu (bk_m) \right\}$$

$$= -\frac{\pi k_m}{\eta} M (k_m).$$

Now F is a meromorphic function having poles at k_1, k_2, ...etc. The residue of $F(w)$ at k_m is given by

$$\frac{\pi k_m Q_\nu (tk_m, ak_m)\, Q_\nu (xk_m, bk_m)}{S' (k_m)} = -\frac{Q_\nu (tk_m, ak_m)\, Q_\nu (xk_m, ak_m)}{M (k_m)}.$$

Considering Γ to be the contour of integration, by (1.7),

$$(2.9) \qquad U_n (t, x) = \lim_{L \to \infty} \frac{1}{2 \pi i} \int_{L_n - Li}^{L_n + Li} F (w)\, dw,$$

Since by Lemma 1,

$$\lim_{L\to\infty}\int_{-Li}^{Li} F(w)\,dw = \lim_{L\to\infty}\int_{\pm Li}^{L_n\pm Li} F(w)\,dw = 0.$$

Hence by Lemma 1 and (2.9),

$$|U_n(t,x)| \leqslant \frac{2K_5}{\sqrt{xt}}\int_0^\infty e^{-v(x-t)}\,dt = \frac{K_2}{x-t},\quad x>t.$$

Similar is the case if $t > x$. This proves (2.6).

Now, by (2.9), for $x > t$,

$$\int_a^t t^{\nu+1} U_n(t,x)\,dt = \int_{L_n-\infty i}^{L_n+\infty i} \frac{w\,Q_\nu(xw,bw)}{S(w)}$$

$$\times\Big[\frac{t^{\nu+1}}{w}\Big\{J_{\nu+1}(tw)\,Y'_\nu(aw) - J'_\nu(aw)\,Y_{\nu+1}(tw)\Big\} -$$

$$-\frac{a^{\nu+1}}{w}\Big\{J_{\nu+1}(aw)\,Y'_\nu(aw) - J'_\nu(aw)\,Y_{\nu+1}(aw)\Big\}\Big]\,dw.$$

Therefore, by recurrence relations, [7, pp. 45, 66], and by (2.1) to (2.5), we obtain

$$\left|\int_a^t t^{\nu+1}\,U(t_n,x)\,dt\right| \leqslant \frac{t^{\nu+1}}{\sqrt{tx}}K_7\int_0^\infty \frac{e^{-(x-t)v}}{\sqrt{L_n^2+v^2}}\,dv +$$

$$+\frac{a^{\nu+1}}{\sqrt{at}}K_7\int_0^\infty \frac{e^{-(x-a)v}}{\sqrt{L_n^2+v^2}}\,dv$$

$$\leqslant \frac{K_3}{L_n}\left\{\frac{1}{x-t}+\frac{1}{x-a}\right\}.$$

The proof of (2.8) is similar.

Lemma 3. *For any real ν, ν not a negative integer or zero,*

$$\lim_{n\to\infty}\int_a^b t^{\nu+1}\,U_n(t,x)\,dt = x^\nu,\quad 0<a<x<b.$$

Proof. By (1.7) and recurrence relations,

$$\int_a^b t^{\nu+1} U_n(t,x)\,dt = \sum_{m=1}^{n} \frac{2\nu\left\{ b^{\nu-1} J'_\nu(ak_m) - a^{\nu-1} J'_\nu(bk_m)\right\} Q_\nu(xk_m, ak_m)}{k_m^3 M(k_m) J'_\nu(bk_m)}$$

The function

$$G(w) = \frac{2\nu\left\{ a^{\nu-1} Q_\nu(xw, bw) - b^{\nu-1} Q_\nu(xw, aw)\right\}}{w^2 S(w)},$$

has poles at zero and at k_m, $m=1, 2, \ldots$.

Residue of $G(w)$ at $w = k_m$ is given by

$$\frac{2\nu\left\{ b^{\nu-1} J'_\nu(ak_m) - a^{\nu-1} J'_\nu(bk_m)\right\} Q_\nu(xk_m, ak_m)}{k_m^3 M(k_m) J'_\nu(bk_m)}$$

whereas at zero, the residue is $-2x^\nu$, since, for $\nu > 0$,

$$\lim_{w\to 0} \frac{J_\nu(aw)}{J_\nu(bw)} = \frac{a^\nu}{b^\nu} \text{ and } \lim_{w\to 0} \frac{Y_\nu(aw)}{Y_\nu(bw)} = \frac{b^\nu}{a^\nu},$$

when ν is not a negative integer or zero.

As in the proof of Lemma 2,

$$\int_a^b t^{\nu+1} U_n(t,x)\,dt = \frac{1}{2\pi i} \int_{L_n-\infty i}^{L_n+\infty i} G(w)\,dw) + x^\nu .$$

Also

$$\left| \frac{1}{2\pi i} \int_{L_n-\infty i}^{L_n+\infty i} G(w)\,dw \right| \leqslant \frac{\nu}{\pi}\, b^{\nu-1} K_8 \int_0^\infty \frac{e^{-v(x-a)} + e^{-v(b-x)}}{L_n^2 + v^2}\,dv$$

$$\leqslant \frac{K_9}{L_n^2} \left\{ \frac{1}{x-a} + \frac{1}{b-x} \right\}.$$

Since [1 ; Lemma 4],

(2.10) $$L_n \sim n, \text{ as } n \to \infty,$$

the lemma follows.

Lemma 4. *If* $|\lambda| < b-a$, then

$$\lim_{L\to\infty} \int_{L_n-Li}^{L_n+Li} \frac{\sin \lambda w \, dw}{w \sin (b-a) w} = O(1/n), \text{ as } n\to\infty.$$

Proof. Let $L_n = \dfrac{(2n+1)\pi}{2(b-a)}$. Then

$$\lim_{L\to\infty} \left| \int_{L_n-Li}^{L_n+Li} \frac{\sin \lambda w \, dw}{w \sin (b-a) w} \right| = \left| \int_{-\infty}^{\infty} \frac{\cos \{\lambda (L_n+iv) - L_n (b-a)\}}{(L_n+iv) \cos \{iv (b-a)\}} i \, dv \right.$$

$$\leqslant \frac{4}{L_n^2} \int_0^\infty \frac{v \sin h \lambda v}{\cos h\, v (b-a)} dv + \frac{4}{L_n} \int_0^\infty \frac{\cos h \lambda v \, dv}{\cosh v (b-a)}$$

$= O(1/n)$, as $n \to \infty$.

Lemma 5. *For any real* v, v *not a negative integer or zero and* $a<x<b$,

(2.11) $$\lim_{n\to\infty} \int_a^x t^{\nu+1} U_n(t, x)\, dt = \tfrac{1}{2} x^\nu,$$

and

(2.12) $$\lim_{n\to\infty} \int_x^b t^{\nu+1} U_n(t, x)\, dt = \tfrac{1}{2} x^\nu.$$

Proof. By (2.9),

$$\int_a^x t^{\nu+1} U_n(t, x)\, dt = \frac{1}{2i} \int_{L_n-\infty i}^{L_n+\infty i} \frac{Q_\nu(xw, bw)}{S(w)} \Big[x^{\nu+1} \cdot$$

$$\left\{ J_{\nu+1}(xw)\, Y'_\nu(aw) - J'_\nu(aw)\, Y_{\nu+1}(xw) \right\} -$$

$$- a^{\nu+1} \left\{ J_{\nu+1}(aw)\, Y'_\nu(aw) - J'_\nu(aw)\, Y_{\nu+1}(aw) \right\} \Big] dw$$

(2.13) $$= I_1 + I_2, \text{ say.}$$

As in Lemma 2, $|I_2| \to 0$, as $n \to \infty$, for $a < x$.

Similarly, using (2.1), (2.3), the recurrence relations and the asymptotic expressions [7; p. 199], we obtain,

$$I_1 = \frac{x^{\nu}}{2\pi i} \int_{L_n-\infty i}^{L_n+\infty i} \left\{ \frac{2\cos(b-x)w \sin(x-a)w}{w \sin(b-a)w} + O\left(\frac{1}{|w|^2}\right) \right\} dw$$

$$= \frac{x^{\nu}}{2\pi i} \int_{L_n-\infty i}^{L_n+\infty i} \left\{ \frac{1}{w} - \frac{\sin(b+a-2x)w}{w \sin(b-a)w} \right. dw + O(1/L_n)$$

(2.14) $= \frac{1}{2} x^{\nu} + O(1/n)$, as $n \to \infty$, by (2.10) and Lemma 4.

The lemma follows from (2.13), (2.14) and Lemma 3.

Lemma 6. *For* $a < t \leqslant b$, $a < x \leqslant b$, *we have,*

$$\int_a^t t^{\nu+1} U_n(t, x)\, dt = O(1/n), \text{ as } n \to \infty.$$

Proof. From the proofs of Lemmas 2 and 5,

(2.15) $\int_a^t t^{\nu+1} U_n(t, x)\, dt =$

$$\lim_{L\to\infty} \frac{1}{2i}\left[\int_{L_n-Li}^{L_n+Li} + \int_{-Li}^{L_n-Li} + \int_{Li}^{L_n+Li}\right] H(w)\, dw,$$

where

$$H(w) = \frac{2t^{\nu+1/2}}{\pi\sqrt{x}} \left\{ \frac{\sin(b+t-a-x)w}{w \sin(b-a)w} - \frac{\sin(b+a-t-x)w}{w \sin(b-a)w} \right\}.$$

By (2.4), the second and third limits in (2.15) are each zero. Hence the Lemma follows.

By inequalities of Lemma 2, the following Lemma is true [7; pp. 589-591],

Lemma 7. *If* $f \in L[a, b]$, $a \leqslant A < B \leqslant b$, *and if* $x \notin (A, B)$, *then*

$$\lim_{n\to\infty} \int_A^B t f(t)\, U_n(t, x)\, dt = 0.$$

The following lemma can be proved on the lines of Lemma 2 :

Lemma 8. *The following inequalities hold true for* $a < x < b$, $a \leqslant t \leqslant b$:

$$| \phi_n (t, x/R) | < \frac{K_{11}}{L_n (t-x)^2}, \; \textit{if } x \neq t,$$

$$| \phi_n (t, x/R) | < K_{12} \, L_n, \quad \textit{for all } x \textit{ and } t.$$

Lemma 9. *For any real* v, v *not a negative integer or zero and* $a<x<b$,

$$\lim_{n \to \infty} \int_a^b t^{v+1} \, \phi_n (t, x/R) \, dt = x^v \; ;$$

$$\lim_{n \to \infty} \int_a^x t^{v+1} \, \phi_n (t, x/R) \, dt = \frac{x^v}{2} \; ;$$

$$\lim_{n \to \infty} \int_a^b t^{v+1} \, \phi_n (t, x/R) \, dt = \frac{x^v}{2}$$

This lemma follows from Lemmas 4 and 5 and the fact that convergence implies Riesz-summability.

3. Proof of Theorem 1

By (1.6) and Lemma 5,

$$\lim_{n \to \infty} \left\{ S_n (x) - \tfrac{1}{2} \{ f (x+0) + f (x-0) \} \right\} =$$

$$= \lim_{n \to \infty} \int_a^x tv+1 \left\{ t^{-v} f (t) - x^{-v} f (x-0) \right\} U_n (t, x) \, dt +$$

$$+ \lim_{n \to \infty} \int_x^b t^{v+1} \left\{ t^{-v} f (t) - x^{-v} f (x+0) \right\} U_n (t, x) \, dt$$

$$= \lim_{n \to \infty} (A+B), \text{ say.}$$

By hypothesis, $t^{-v} f(t) - x^{-v} f(x+0)$ is of bounded variation in a neighbourhood, say $(x-\eta, x+\eta)$, of x. Hence there are bounded increasing positive functions g_1 and g_2 of $t \in (x, v+\eta)$ such that

$t^{-v} f(t) - x^{-v} f(x+0) = g_1 (t) - g_2 (t), \;\; t \in (x, x+\eta)$, and $g_1 (x+0) = g_2 (x+0) = 0$.

Therefore, for any $\varepsilon > 0$, there is $\delta > 0$, $\delta \leqslant \eta$, such that $0 \leqslant g_1 (t) < \varepsilon$, $0 \leqslant g_2 (t) < \epsilon$, *for* $x \leqslant t \leqslant x+\delta$.

Now,

$$B=\int_{x}^{x+\delta} t^{\nu+1} g_1(t)\, U_n(t,x)\, dt - \int_{x}^{x+\delta} t^{\nu+1} g_2(t)\, U_n(t,x)\, dt +$$

$$+\int_{x+\delta}^{b} t^{\nu+1}\left\{ t^{-\nu} f(t) - x^{-\nu} f(x+0)\right\} U_n(t,x)\, dt$$

$= B_1 + B_2 + B_3$, say.

By the second mean value theorem and Lemma 6, B_1 and B_2 are both equal to $O(1/n)$, as $n\to\infty$, and by Lemma 7, $B_3=o(1)$, as $n\to\infty$.

Hence, $B=o(1)$, as $n\to\infty$.

In a similar way, $A=o(1)$, as $n\to\infty$, which proves the theorem.

4. Proof of Theorem 2

As in the proof of Theorem 1, by (1.8) and Lemmas 5 and 9,

$$\lim_{n\to\infty}\left[R_n^{(\nu)}(x,f) - \tfrac{1}{2}\{f(x+0)+f(x-0)\}\right] =$$

$$\lim_{n\to\infty}\int_{a}^{x} t^{\nu+1}\left\{ t^{-\nu} f(t) - x^{-\nu} f(x-0)\right\} \phi_n(t, x/R)\, dt +$$

$$\lim_{n\to\infty}\int_{x}^{b} t^{\nu+1}\left\{ t^{-\nu} f(t) - x^{-\nu} f(x+0)\right\} \phi_n(t, x/R)\, dt$$

$= \lim_{n\to\infty} (I + I')$, say.

Given $\epsilon > 0$, let us choose $\delta > 0$ such that

$$(4.1)\qquad \left| t^{-\nu} f(t) - x^{-\nu} f(x-0)\right| < \epsilon,$$

for $x-\delta < t < x$. If n is so large that $\delta > 1/L_n$, then

$$I = \int_{a}^{x-\delta} + \int_{x-\delta}^{x-1/L_n} + \int_{x-1/L_n}^{x} = I_1 + I_2 + I_3, \text{ say.}$$

By Lemma 8,

$$|I_1| \leqslant \frac{K_{11}}{L_n\,\delta^2}\int_{a}^{x-\delta}\left| t f(t) - t^{\nu+1} x^{-\nu} f(x+0)\right| dt = o(1), \text{ as } n\to\infty.$$

Also by (4.1),

$|I_2| \leqslant K_{11}\, \epsilon$, and $|I_3| \leqslant K_{13}\, \epsilon$.

Thus, $I = o\,(1)$, as $n \to \infty$. Similarly, I' is established and the theorem follows.

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