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(*Dadicated to the memory of Professor Arthur Erdélyi*)

GROWTH OF THE MEAN VALUES OF ENTIRE FUNCTIONS REPRESENTED BY DIRICHLET SERIES

By

S. K. VAISH

Department of Mathematics, Sahu Jain College
Najibabad-246763, U. P., India

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1. INTRODUCTION

Let $f(s) = \sum_{n=1}^{\infty} a_n \exp. (s\lambda_n)$, where $s=\sigma+it$,

$$\lambda_{n+1} > \lambda_n, \lambda_1 \geqslant 0, \lim_{n\to\infty} \lambda_n = \infty \text{ and } \lim_{n\to\infty} \sup \frac{\log n}{\lambda_n} = 0$$

represent an entire function given by Dirichlet series. The Ritt-order ρ and lower order λ (in the sense of Ritt) of $f(s)$ are defined by [3] :

$$(1.1) \quad \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log \log M(\sigma, f)}{\sigma} = \begin{matrix}\rho\\ \lambda\end{matrix}, 0 \leqslant \lambda \leqslant \rho \leqslant \infty,$$

where $M(\sigma, f) = \sup \{ |f(\sigma+it)| : -\infty < t < \infty \}$.

Let us introduce the following mean values of $f(s)$:

$$(1.2) \quad \left\{ I_\delta(\sigma) \right\}^\delta = \left\{ I_\delta(\sigma, f) \right\}^\delta = \lim_{T\to\infty} \left\{ \frac{1}{2T} \int_{-T}^{T} |f(\sigma+it)|^\delta \, dt \right\}$$

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and

$$(1.3) \quad N_{\delta,k}(\sigma) = N_{\delta,k}(\sigma, f) = \exp.\left\{ e^{-k\sigma} \int_0^{\sigma} e^{kx} \log I_\delta(x, f)\, dx \right\}$$

where $0 < \delta < \infty$ and $0 < k < \infty$.

It is known that [2, p. 92] :

$$(1.4) \quad \lim_{\sigma\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log \log N_{\delta,k}(\sigma, f)}{\sigma} = \begin{matrix} \rho \\ \lambda \end{matrix} .$$

In this paper we have obtained the growth properties of the product of any finite number of mean values efined for entire fnnctions. Throughout this paper we shall assume that the function $f(s)$ is of finite non-zero oı der.

2. **Theorem** 1. Let $f_1(s), f_2(s)$ be two entire functions of orders ρ_1, ρ_2 and lower orders λ_1, λ_2, respectively. Then, if

$$(2.1) \log \log N_{\delta,k}(\sigma, f) \approx \log\left[\left\{ \log N_{\delta,k}(\sigma, f_1)\right\}\left\{ \log N_{\delta,k}(\sigma, f_2)\right\}\right]$$

the order ρ and lower order λ of the entire function $f(s)$ are such that

$$(2.2) \quad \lambda_1 + \lambda_2 \leqslant \lambda \leqslant \rho \leqslant \rho_1 + \rho_2,$$

and if

$$(2.3) \quad \log \log N_{\delta,k}(\sigma, f) \approx \left[\left\{\log \log N_{\delta,k}(\sigma, f_1)\right\}\left\{\log \log N_{\delta,k}(\sigma, f_2)\right\}\right]^{\frac{1}{2}},$$

then

$$(2.4) \quad (\lambda_1 \lambda_2)^{\frac{1}{2}} \leqslant \lambda \leqslant \rho \leqslant (\rho_1, \rho_2)^{\frac{1}{2}},$$

where $N_{\delta,k}(\sigma, f)$, $N_{\delta,k}(\sigma, f_1)$ and $N_{\delta,k}(\sigma, f_2)$ are the mean values of $f(s)$, $f_1(s)$ and $f_2(s)$, respectively.

Proof—Since the entire functions $f_1(s)$ and $f_2(s)$ are of orders ρ_1 and ρ_2, therefore from (1.4), we have

$$\lim_{\sigma\to\infty} \sup \frac{\log\log N_{\delta,k}(\sigma,f_1)}{\sigma} = \rho_1,$$

and

$$\lim_{\sigma\to\infty} \sup \frac{\log\log N_{\delta,k}(\sigma,f_2)}{\sigma} = \rho_2.$$

Hence, for any $\epsilon > 0$ and $\sigma > \sigma_0$,

$$\text{(2.5)} \quad \frac{\log\log N_{\delta,k}(\sigma,f_1)}{\sigma} < \rho_1 + \frac{\epsilon}{2}.$$

and

$$\text{(2.6)} \quad \frac{\log\log N_{\delta,k}(\sigma,f_2)}{\sigma} > \rho_2 + \frac{\epsilon}{2}.$$

Adding (2.5) and (2.6), we get

$$\text{(2.7)} \quad \frac{\log\left[\left\{\log N_{\delta,k}(\sigma,f_1)\right\}\left\{\log N_{\delta,k}(\sigma,f_2)\right\}\right]}{\sigma} < \rho_i + \rho_2 + \epsilon$$

Proceeding as above for the limit inferior, we find

$$\text{(2.8)} \quad \frac{\log\left[\left\{\log N_{\delta,k}(\sigma,f_1)\right\}\left\{\log N_{\delta,k}(\sigma,f_2)\right\}\right]}{\sigma} > \lambda_1 + \lambda_2 - \epsilon.$$

Now, if (2.1) holds, then from (2.7) and (2.8), we have, for any $\epsilon < o$ and sufficiently large σ,

$$\lambda_1 + \lambda_2 - \epsilon < \frac{\log\log N_{\delta,k}(\sigma,f)}{\sigma} < \rho_1 + \rho_2 + \epsilon.$$

Taking limits and using (1.4), it leads to (2.2).

Again, multiplying (2.5) and (2.6), we get

$$\text{(2.9)} \quad \frac{\left\{\log\log N_{\delta,k}(\sigma,f_1)\right\}\left\{\log\log N_{\delta,k}(\sigma,f_2)\right\}}{\sigma^2} < \left(\rho_1 + \frac{\epsilon}{2}\right)\left(\rho_2 + \frac{\epsilon}{2}\right),$$

for any $\epsilon > 0$ and sufficiently large σ.

Similarly, we have

$$(2.10)\quad \frac{\{\log\log \mathcal{N}_{\delta,k}(\sigma, f_1)\}\{\log\log \mathcal{N}_{\delta,k}(\sigma, f_2)\}}{\sigma^2} > (\lambda_1 - \frac{\epsilon}{2})(\lambda_2 - \frac{\epsilon}{2}),$$

for any $\epsilon > 0$ and sufficiently large σ.

Further, if (2.3) holds, then from (2.9) and (2.10), on taking limits, (2.4) follows.

Corollary I. If $f_\xi(s)$, $(\xi = 1, 2, \ldots, n)$ are n entire functions of orders $\rho_1, \rho_2, \ldots \rho_n$ and lower orders $\lambda_1, \lambda_2, \ldots, \lambda_n$ and having the mean values $\mathcal{N}_{\delta,k}(\sigma, f_1), \mathcal{N}_{\delta,k}(\sigma, f_2), \ldots, \mathcal{N}_{\delta,k}(\sigma, f_n)$, respectively. Then, if

$$\log\log \mathcal{N}_{\delta,k}(\sigma, f) \approx \log\Big[\{\log \mathcal{N}_{\delta,k}(\sigma, f_1)\}\{\log \mathcal{N}_{\delta,k}(\sigma, f_2)\} \ldots \ldots\{\log \mathcal{N}_{\delta,k}(\sigma, f_n)\}\Big],$$

the order ρ and lower order λ of $f(s)$ having the mean value $\mathcal{N}_{\delta,k}(\sigma, f)$ are such that

$$\lambda_1 + \lambda_2 + \ldots + \lambda_n \leqslant \lambda \leqslant \rho \leqslant \rho_1 + \rho_2 + \ldots \rho_n,$$

and if

$$\log\log \mathcal{N}_{\delta,k}(\sigma, f) \approx \Big[\{\log\log \mathcal{N}_{\delta,k}(\sigma, f_1)\}\{\log\log \mathcal{N}_{\delta,k}(\sigma, f_2)\} \ldots \ldots\{\log\log \mathcal{N}_{\delta,k}(\sigma, f_n)\}\Big]^{1/n},$$

then

$$(\lambda_1 \lambda_2 \ldots \lambda_n)^{1/n} \leqslant \lambda \leqslant \rho \leqslant (\rho_1 \rho_2 \ldots \rho_n)^{1/n}.$$

Corollary 2. If $f_\xi(s)$, $(\xi = 1, 2, \ldots, n)$ are n entire functions of regular growth, of orders $\rho_1, \rho_2, \ldots, \rho_n$, respectively, then so is of $f(s)$, of order ρ, and

$$\rho = \rho_1 + \rho_2 + \ldots + \rho_n.$$

3. Let $\mu(\sigma, f) = \max_{n \geqslant 1} \{|a_n| \exp(\sigma \lambda_n)\}$ be the maximum term of $f(s)$ and $\nu(\sigma, f) = \max\{n : \mu(\sigma, f) = |a_n| \exp(\sigma \lambda_n)\}$ be its rank, then it is known that [1, p. 20] :

(3.1) $\log I_\delta(\sigma, f) \approx \log \mu(\sigma, f)$.

Theorem 2. Let $f(s)$ be an entire function of order ρ and lower order λ, $0 < \lambda < \rho < \infty$, then

$$(3.2)\quad \limsup_{\sigma \to \infty} \frac{\log N_{\delta, k}(\sigma, f)}{\sigma \lambda_{\nu(\sigma, f)}} \leqslant 1/k\,(1 - \lambda/\rho)$$

and

$$(3.3)\quad \limsup_{\sigma \to \infty} \frac{\log N_{\delta, k}(\sigma, f)}{\lambda_{\nu(\sigma, f)} \log \lambda_{\nu(\sigma, f)}} \leqslant 1/k\,(1/\lambda - 1/\rho).$$

Proof. From (1.3), we have

$$\log N_{\delta, k}(\sigma, f) = e^{-k\sigma} \int_0^\sigma e^{kx} \log I_\delta(x, f)\, dx \leqslant 1/k \log I_\delta(\sigma, f).$$

Hence,

$$(3.4)\quad \limsup_{\sigma \to \infty} \frac{\log N_{\delta, k}(\sigma, f)}{\sigma \lambda_{\nu(\sigma, f)}} \leqslant 1/k \limsup_{\sigma \to \infty} \frac{\log I_\delta(\sigma, f)}{\sigma \lambda_{\nu(\sigma, f)}}$$

$$= 1/k \limsup_{\sigma \to \infty} \frac{\log \mu(\sigma, f)}{\sigma \lambda_{\nu(\sigma, f)}}.$$

in view of (3.1).

We know [4, p. 84] that for $0 < \rho < \infty$,

$$(3.5)\quad \limsup_{\sigma \to \infty} \frac{\log \mu(\sigma, f)}{\sigma \lambda_{\nu(\sigma, f)}} \leqslant 1 - \lambda/\rho,$$

and

$$(3.6)\quad \limsup_{\sigma \to \infty} \frac{\log \mu(\sigma, f)}{\lambda_{\nu(\sigma, f)} \log \lambda_{\nu(\sigma, f)}} \leqslant 1/\lambda - 1/\rho.$$

Thus (3.2) follows (3.4) and (3.5).

Similarly, we can easily derive (3.3), if we use (3.6) instead of (3.5).

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