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(*Dedicated to the memory of Professor Arthur Erdélyi*)

ON THE ABSOLUTE HARMONIC SUMMABILITY OF A LEGENDRE SERIES

By

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ABSTRACT

The absolute harmonic summability of a Fourier series has been studied by Varshney [2]. In this paper we study the absolute harmonic summability of a Legendre series.

1. Introduction

Let f (x) be a Lebesgue integrable in $[-1, 1]$. The Legendre series corresponding to this function is

(1.1) $$f(x) \sim \sum_{n=0}^{\infty} a_n P_n(x)$$

where

$$a_n = (n+\tfrac{1}{2}) \int_{-1}^{1} f(x) P_n(x)\, dx,$$

and $P_n(x)$ is defined by the following expansion :

(1.3) $$(1-2xz+z^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} z^n P_n(x)$$

Also let $\{S_n\}$ be the sequence of partial sums of a given infinite series Σa_n and define the sequence $\{t_n\}$ by

$$t_n = \frac{(n+1)^{-1} S_0 + n^{-1} S_1 + \ldots + S_n}{p_n}$$

$$p_n = 1 + \tfrac{1}{2} + \dots + \frac{1}{n+1}.$$

The series $\Sigma\, a_n$ is said to be absolutely harmonic summable if the series

$$\sum_{n=1}^{\infty} \left| t_n - t_{n-1} \right| \tag{1.4}$$

is convergent. It is known that this method of summability is absolutely regular and implies absolute Cesáro summability of every positive order (see [1]).

Varshney [2] has applied the method of absolute harmonic summaility to the series.

$$\tfrac{1}{2} a_0 + \sum_{n=0}^{\infty} (a_n \cos nt + b_n \sin nt) \equiv \sum_{n=0}^{\infty} U_n, \tag{1.5}$$

where a_n and b_n are the Fourier coefficients of a function $f(x)$, which is periodic with period 2π and integrable (L) over $(-\pi, \pi)$.
He proved the following theorem :

heorem A (Varshney [2, p. 589]) : *If* f (x) *is of bounded variation in* $(-\pi, \pi)$ *and satisfies*

$$| f(x+h) - f(x) | \leqslant A \log^{-1-\varepsilon}(1/h),\ h \to o\ (\varepsilon > o,\ o \leqslant x \leqslant 2\pi)$$

then the series 1.5) *is absolutely harmonic summable.*

Concerning the absolute Cesáro summability of the series (1.1) we have established the following theorem :

Theorem B (Sharma [3]) : *If* $\alpha > 1$.

$$\int_0^t |\psi(\theta, \omega)|\, d\omega = O\left(\frac{t}{(\log 1/t)\, \alpha}\right),\ t \to o$$

and

$$\int_0^h |\phi(u)|\, du = O\left(\frac{h}{(\log 1/h)^{\alpha}}\right),\ h \to o$$

then the series (1.1) *is summable* $|C, \delta|$ *for every* $\delta > 1$ *at a point* x *of the interval* $[-1+\varepsilon, 1-\varepsilon]$, ε *being a small fixed positive number.*

In this paper we prove a theorem on the absolute harmonic summbility of the series (1.1) under the condition which is weaker than that of theorem B and corresponds to that of theorem A for Fourier series. In what follows we prove the following :

Theorem C : *If* f (x) *is of bounded variation in* $[-1, 1]$ *and satisfies*

$$|f(x+h)-f(x)| = \left(\frac{A}{\log^{1+a}\left(\frac{1}{h}\right)}\right), \quad a>0,\ h\to 0,$$

then the series (1.1) *is summable* $\left|N, \frac{1}{n+1}\right|$ *in the interval* $[-1+\varepsilon, 1-\varepsilon]$, $\varepsilon>0$.

In the proof of the theorem we require following lemma :

Lemma 1 (Sansone [4, p. 208]) :

(1.6) $$P_n(\cos\theta) = n^{-\frac{1}{2}} k(\theta)\left[\cos\left\{(n+\tfrac{1}{2})\theta-\frac{\pi}{4}\right\}\left(1-\frac{1}{4n}\right)\right.$$

$$\left.+\frac{1}{8n}\cot\theta\sin\left\{(n+\tfrac{1}{2})\theta-\frac{\pi}{4}\right\}\right]+O(n^{-5/2})$$

$$k(\theta) = \sqrt{\frac{2}{\pi\sin\theta}} \text{ for } \varepsilon' \leqslant \theta \leqslant \pi-\varepsilon',\ 0<\varepsilon'<\frac{\pi}{2}.$$

2. **Proof of Theorem C** : We have :

$$t_n - t_{n-1} = \frac{1}{p_n\, p_{n-1}} \int_0^{\pi} \phi(\theta,\omega)\left[\sum_{k=0}^{n-1}\left(\frac{p_n}{k+1}-\frac{p_k}{n+1}\right)(n-k+\tfrac{1}{2})\, P_{n-k}(\cos\theta)\right.$$

$$\left.\times P_{n-k}(\cos\omega)\right] \sin\omega\, d\omega$$

where

$$\phi(\theta,\omega) = \{f(\cos\omega)-f(\cos\theta)\}$$

We denote the integral in the above expansion with the following:

$$t_n - t_{n-1} = \frac{1}{p_n\, p_{n-1}} \left[\int_0^{c/n} + \int_{c/n}^{\epsilon/2} + \int_{\varepsilon/2}^{\theta} + \int_{\theta}^{\pi-\varepsilon/2} + \int_{\pi-\epsilon/2}^{\pi-c/n} + \int_{\pi-c/n}^{\pi} \right]$$

$$= L_1 + L_2 + L_3 + L_4 + L_5 + L_6 \text{, say.}$$

Now

$$\text{(2.1)}\quad L_1 = \frac{1}{p_n\, p_{n-1}} \int_0^{c/n} \phi(\theta,\omega) \sum_{k=0}^{n-1} \left(\frac{p_n}{k+1} - \frac{p_k}{n+1} \right) (n-k+\tfrac{1}{2})\, P_{n-k}(\cos\theta)\, P_{n-k}(\cos\omega) \sin\omega\, d\omega.$$

$= O(n^{-3/2})$; $\phi(\theta, \omega)$ is bounded.

Similarly

$$\text{(2.2)}\qquad L_6 = O(n^{-3/2})$$

Now using the first part of lemma 1, we have

$$\text{(2.3)}\quad L_3 = \frac{1}{p_n\, p_{n-1}} \int_{\epsilon/2}^{\theta} \phi(\theta,\omega) \sum_{k=0}^{n-1} \left(\frac{p_n}{k+1} - \frac{p_k}{n+1} \right) k(\theta)\, k(\omega) \frac{1}{4(n-k)}$$

$$\times \cos\left\{ (n-k+\tfrac{1}{2})\,\theta - \frac{\pi}{4} \right\} \cos\left\{ (n-k+\tfrac{1}{2})\,\omega - \frac{\pi}{4} \right\} \sin\omega\, d\omega$$

$$+ \frac{1}{p_n\, p_{n-1}} \int_{\epsilon/2}^{\theta} \phi(\theta,\omega) \sum_{k=0}^{n-1} \left(\frac{p_n}{k+1} - \frac{p_k}{n+1} \right) k(\theta)\, k(\omega) \frac{1}{8(n-k)}$$

$$\times \cos\left\{ (n-k+\tfrac{1}{2})\,\theta - \frac{\pi}{4} \right\} \sin\left\{ (n-k+\tfrac{1}{2})\,\omega - \frac{\pi}{4} \right\} \sin\omega\, d\omega$$

$$+ \frac{1}{p_n\, p_{n-1}} O\left[\sum_{k=0}^{n-1} \left(\frac{p_n}{k+1} - \frac{p_k}{n+1} \right) \frac{1}{(n-k)^2} \right]$$

$$= L_{3\cdot1} + L_{3\cdot2} + L_{3\cdot3} + L_{3\cdot4} \text{, say.}$$

$$(2.4)\quad L_{3\cdot1}=\frac{1}{2p_n p_{n-1}}\int_{\varepsilon/2}^{\theta}\phi(\theta,\omega)\sum_{k=o}^{n-1}\left(\frac{p_n}{k+1}-\frac{p_k}{n+1}\right)k(\theta)\,k(\omega)$$

$$\times\cos\left\{(n-k+\tfrac{1}{2})(\theta+\omega)-\frac{\pi}{2}\right\}\sin\omega\,d\omega$$

$$+\frac{1}{2p_n p_{n-1}}\int_{\varepsilon/2}^{\theta}\phi(\theta,\omega)\sum_{k=o}^{n-1}\left(\frac{p_n}{k+1}-\frac{p_k}{n+1}\right)k(\theta)\,k(\omega)$$

$$\cos\{(n-k+\tfrac{1}{2})(\theta-\omega)\}\times\sin\omega\,d\omega$$

$$=\ L_{3\cdot1\cdot1}+L_{3\cdot1\cdot2}\text{ , say.}$$

Putting $\theta-\omega=t$, we obtain

$$(2.5)\quad L_{3\cdot1\cdot2}\leqslant\frac{1}{2p_{n-1}}\left|\int_{o}^{\theta-\varepsilon/2}\phi(t)\,k(\theta)\,k(\theta-t)\sum_{k=o}^{\infty}\frac{\cos(n-k+\frac{1}{2})t}{k+1}\right.$$

$$\left.\sin(\theta-t)\,dt\right|$$

$$+\frac{1}{2p_{n-1}}\left|\int_{o}^{\theta-\varepsilon/2}\phi(t)\left\{\sum_{k=n}^{\infty}\frac{\cos(n-k+\frac{1}{2})t}{k+1}+\sum_{k=o}^{n-1}\frac{p_k\cos(n-k+\frac{1}{2})t}{(n+1)p_n}\right\}\right.$$

$$\left.\times\,k(\theta)\,k(\theta-t)\sin(\theta-t)\,dt\right|$$

$$=\ S_1+S_2+S_3+S_4\text{ , say.}$$

By Abel's transformation, we have on the line of Varshney [2].

$$\sum_{k=n}^{\infty}\frac{\cos(n-k+\frac{1}{2})t}{k+1}=\frac{1}{2(n+1)}\cos t/2+\frac{1}{2(n+1)}\sin t/2+O(n^{-2}t^{-2})$$

$$\text{for } t\geqslant\frac{1}{n}$$

$$\sum_{k=o}^{n-1}p_k\cos(n-k)t=O\left\{\left(1+\log\frac{1}{t}\right)t^{-1}\right\}-\tfrac{1}{2}p_{n-1}$$

$$(2.4)\quad L_{3\cdot 1}=\frac{1}{2\,p_n\,p_{n-1}}\int_{\varepsilon/2}^{\theta}\phi(\theta,\omega)\sum_{k=0}^{n-1}\left(\frac{p_n}{k+1}-\frac{p_k}{n+1}\right)k(\theta)\,k(\omega)$$

$$\times\cos\left\{(n-k+\tfrac{1}{2})(\theta+\omega)-\frac{\pi}{2}\right\}\sin\omega\,d\omega$$

$$+\frac{1}{2\,p_n\,p_{n-1}}\int_{\varepsilon/2}^{\theta}\phi(\theta,\omega)\sum_{k=0}^{n-1}\left(\frac{p_n}{k+1}-\frac{p_k}{n+1}\right)k(\theta)\,k(\omega)$$

$$\cos\{(n-k+\tfrac{1}{2})(\theta-\omega)\}\times\sin\omega\,d\omega$$

$$=L_{3\cdot1\cdot1}+L_{3\cdot1\cdot2}\text{, say.}$$

Putting $\theta-\omega=t$, we obtain

$$(2.5)\quad L_{3\cdot1\cdot2}\leqslant\frac{1}{2\,p_{n-1}}\left|\int_0^{\theta-\varepsilon/2}\phi(t)\,k(\theta)\,k(\theta-t)\sum_{k=0}^{\infty}\frac{\cos(n-k+\frac{1}{2})t}{k+1}\sin(\theta-t)\,dt\right|$$

$$+\frac{1}{2\,p_{n-1}}\left|\int_0^{\theta-\varepsilon/2}\phi(t)\left\{\sum_{k=n}^{\infty}\frac{\cos(n-k+\frac{1}{2})\,t}{k+1}+\sum_{k=0}^{n-1}\frac{p_k\cos(n-k+\frac{1}{2})\,t}{(n+1)\,p_n}\right\}\times k(\theta)\,k(\theta-t)\sin(\theta-t)\,dt\right|$$

$$=S_1+S_2+S_3+S_4\text{, say.}$$

By Abel's transformation, we have on the line of Varshney [2].

$$\sum_{k=n}^{\infty}\frac{\cos(n-k+\frac{1}{2})\,t}{k+1}=\frac{1}{2\,(n+1)}\cos t/2+\frac{1}{2\,(n+1)}\sin t/2+O\,(n^{-2}\,t^{-2})$$

$$\text{for } t\geqslant\frac{1}{n}$$

$$\sum_{k=0}^{n-1}p_k\cos(n-k)\,t=O\left\{\left(1+\log\frac{1}{t}\right)t^{-1}\right\}-\tfrac{1}{2}\,p_{n-1}$$

Similarly

$$\sum_{k=o}^{n-1} p_k \sin (n-k)\, t \;=\; O\left\{ (1+\log \frac{1}{t})\, t^{-1} \right\} - \tfrac{1}{2}\, p_{n-1}$$

From these resul s, we obtain

$$\sum S_2 \leqslant A \sum \frac{1}{p_{n-1}} \int_{c/n}^{\theta-\varepsilon/2} |\phi(t)|\, n^{-2}\, t^{-2}\, dt + A \sum \frac{1}{(n+1)\, p_n\, p_{n-1}}$$

$$\int_{c/n}^{\theta-\varepsilon/2} |\phi(t)| \frac{\log 1/t}{t}\, dt$$

$$+ \sum \frac{1}{p_{n-1}} \left| \int_{c/n}^{\theta-\epsilon/2} \phi(t)\, k(\theta)\, k(\theta-t) \sin(\theta-t) \left[\frac{1}{n+1} \cos t/2 \right.\right.$$

$$\left.\left. + \tfrac{1}{2} \frac{p_{n-1}}{(n+1)\, p_n} \sin t/2 - \frac{p_{n-1}}{2\,(n+1)\, p_n} \cos t/2 \right] dt \right|$$

$$= O\left[\sum \frac{1}{n\, (\log n)^{1+a}} \right]$$

$$= O(1).$$

$$\sum S_1 \leqslant \sum \frac{1}{2\, p_{n-1}} \left| \int_0^{\theta-\epsilon/2} \phi(t) \left[\left\{ a(t) \cos t/2 + \beta(t) \sin t/2 \right\} \right.\right.$$

$$\left. k(\theta)\, k(\theta-t) \sin(\theta-t) \right]$$

$$\left. \times \sin t\, dt \right| + \sum \frac{1}{2\, p_{n-1}} \left| \int_0^{\theta-\epsilon/2} \phi(t) \left[\left\{ \beta(t) \cos t/2 - a(t) \sin t/2 \right\} \right.\right.$$

$$\left.\left. k(\theta)\, k(\theta-t) \sin(\theta-t) \right] \times \sin t\, dt \right|$$

where

$$a(t) = \sum_{k=o}^{\infty} \frac{\cos kt}{k+1} \quad ; \quad \beta(t) = \sum_{k=o}^{\infty} \frac{\sin kt}{k+1}$$

Now we write

$$\sum S_1 = \sum \frac{1}{2p_{n-1}} \left| \int_0^{\theta-\epsilon/2} \phi(t)\, p(t) \cos nt\, di \right| + \sum \frac{1}{2p_{n-1}} \left| \int_0^{\theta-\epsilon/2} \phi(t)\, q(t) \sin nt\, dt \right|$$

$$= \sum \frac{|a_n| + |b_n|}{2p_{n-1}}$$

where

$$p(t) = k(\theta)\, k(\theta-t) \sin(\theta-t) \{ \alpha(t) \cos t/2 + \beta(t) \sin t/2 \} \text{ and}$$
$$q(t) = k(\theta)\, k(\theta-t) \sin(\theta-t) \{ \beta(t) \cos t/2 - \alpha(t) \sin t/2 \}.$$

Since $\alpha(t)$ and $\beta(t)$ are continuous for $n \leqslant t \leqslant$ and for $0 < t \leqslant n$, their absolute values are each less than $A \log 1/t$. Hence $p(t)$ and $q(t)$ are also continuous and $|p(t)|$, $|q(t)|$ both are less than $A \log 1/t$.

Thus the constants a_n and b_n are Fourier coefficients of even and odd functions respectively and each of these function belongs to L^2 class. We define

$$\psi(t) = \phi(t) \text{ in } [0, \theta-\epsilon/2]$$
$$= 0 \text{ in } [\theta-\epsilon/2, \pi]$$

Therefore, from Parsvall's theorem, we have

$$\sum_{n=1}^{\infty} a_n^2 \sin^2 nh \leqslant A \int_0^{\pi} \{\psi(t+h)\, p(t+h) - \psi(t-h)\, p(t-h)\}^2\, dt$$

$$\leqslant 2A \{I_1(h) + I_2(h)\} \text{ where}$$

$$I_1(h, \int_0^{\pi} \{p(t+h)\}^2 \{\psi(t+h) - \psi(t-h)\}^2\, dt$$

$$I_2(h) = \int_0^{\pi} \{\psi(t-h)\}^2 \{p(t+h) - p(t-h)\}^2\, dt$$

Now from the hypothesis we observe that

$$|\psi(t)| = |f \cos(\theta-t) - f(\cos\theta)| = \left(\frac{A}{(\log 1/t)^{1+\alpha}}\right)$$

Then for a positive h, we have

$$\psi(t+h)-\psi(t-h) \leqslant \left(\frac{A}{(\log 1/h)^{1+a}}\right).$$

Hence

$$I_1(h) \leqslant \left(\frac{A}{\log^{2a+2} 1/h}\right)\int_0^{\pi} \log^2 1/t \leqslant \left(\frac{A\,h}{\log^{2a}(1/h)}\right).$$

On the line of McFadden [1], we have

$$(2.6)\; I_2(h) \leqslant A\int_{-h}^{h} \psi^2(t)\, p^2(t+2h)\, dt + A\int_{-h}^{h} \psi^2(t)\, p^2(t) + A$$

$$\int_{-h}^{\pi-h} \psi^2(t)\, \{p(t+2h)-p(t)\}^2\, dt$$

$$= I_{2\cdot1} + I_{2\cdot2} + I_{2\cdot3},$$

$$(2.7)\quad I_{2\cdot1} \leqslant A\int_{-h}^{h} \frac{1}{\log^{2a+2} 1/t} \log^2 \frac{1}{t+2h}\, dt \leqslant \left(\frac{A\,h}{\log^{2a} 1/h}\right).$$

Similarly

$$(2.8)\quad I_{2\cdot2} \leqslant \left(\frac{Ah}{\log^{2a} 1/h}\right).$$

Finally, we know that

$$\left|\frac{d}{dt}\left(\sum_{k=0}^{\infty} \frac{\cos kt}{k+1}\right)\right| = \left|-\sum_{k=0}^{\infty} (\sin kt) - \sum_{k=0}^{\infty} \frac{\sin kt}{k+1}\right| \leqslant \frac{A}{t} + A \log 1/t \leqslant \frac{A}{t}$$

By mean value theorem we have

$$(2.9)\quad I_{2\cdot3} = \int_{-h}^{\pi-h} \psi^2(t)\, \{p(t+2h)-p(t)\}^2\, dt \leqslant \left(\frac{Ah}{\log^{2a} 1/h}\right)$$

Now setting $h=\left(\frac{\pi}{2N}\right)$ and collecting (2.6), (2.7), (2 8) and (2.9) we find that

(2.10) $I_2(h) = \dfrac{A}{\mathcal{N}\log^{2\alpha}\mathcal{N}}$

Since f is continuous and of bounded variation it is clear that ψ is also continuous and of bounded variation. Let $\omega(\delta)$ be the modulus of continuity of ψ and V the total variation of ψ over $(0, 2\pi)$.

We start from the inequality

$$\sum_{h=1}^{2\mathcal{N}} \left\{ p\left(t+\frac{k\pi}{\mathcal{N}}\right)\right\}^2 \left[\psi\left(t+\frac{k\pi}{\mathcal{N}}\right)-\psi\left(t+(k-1)\frac{\pi}{\mathcal{N}}\right)\right]^2$$

$$\leqslant A\log^2 1/t \frac{V}{\log^{1+\alpha}\mathcal{N}}$$

which we integrate over $(0, \pi)$. On account of the periodicity, replacing x by $x+\xi$, does not effect the value of the integral, and so all integrals formed from the left hand side are equal. Hence we have

(2.11) $I_1\left(\dfrac{\pi}{2\mathcal{N}}\right) \leqslant \dfrac{A}{\mathcal{N}\log^{1+\alpha}\mathcal{N}}$

Combining (2.10) and (2.11) we find that

$$\sum_{n=1}^{\infty} a_n^2 \sin^2\left(\frac{n\pi}{2\mathcal{N}}\right) \leqslant \frac{A}{\mathcal{N}\log^{2\alpha}\mathcal{N}} \quad (\alpha < 1)$$

Taking $\mathcal{N}=2^\nu$, we get

$$\sum_{n=2^{\nu-1}+1}^{2^\nu} a_n^2 \leqslant 2 \sum_{n=2^{\nu-1}+1}^{2^\nu} a_n^2 \sin^2 \frac{n\pi}{2^\nu+1} \leqslant A\,(2^{-\nu}\cdot \nu^{-2\alpha}$$

Therefore applying Schwartz's inequality we get

$$\sum_{n=2^{\nu-1}+1}^{2^\nu} |a_n| \log^{-1} n \leqslant \left\{ \sum_{n=2^{\nu-1}+1}^{2^\nu} a_n \right\}^{\frac{1}{2}} \left\{ \sum_{n=2^{\nu-1}+1}^{2^\nu} \log^{-2} n \right\}^2$$

$$\leqslant \frac{A}{\nu^{1+\alpha}}.$$

A similar relation holds in the case of b_n

$$\sum_{n=2^{\nu-1}+1}^{2^\nu} |b_n| \log^{-1} n \leqslant \frac{A}{\nu^{1+\alpha}}.$$

Hence

$$\sum_{n=1}^{\infty} \frac{a_n |+|b_n|}{p_{n-1}} \leqslant A \sum_{\nu=1}^{\infty} \sum_{n=2^{\nu-1}+1}^{2^\nu} \frac{|a_n|}{\log n} + A \sum_{\nu=1}^{\infty}$$

$$\sum_{n=2^{\nu-1}+1}^{2^\nu} \frac{|b_n|}{\log n} \leqslant A \sum_{\nu=1}^{\infty} \frac{1}{\nu^{1+\alpha}} = O(1).$$

Since

$$\sum_{k=n}^{\infty} \frac{\cos (n-k)\, t}{k+1} \leqslant \frac{4}{(n+1)\,|\,1-e^{-it}\,|}$$

We have

$$\sum S_3 \leqslant A \sum \frac{1}{p_{n-1}} \int_0^{c/n} |\phi(t)| \frac{1}{(n+1)\,t}\, dt = \sum \frac{1}{n(\log n)^{1+\alpha}}$$

$$= O(1)$$

Also

$$\sum S_4 = \sum \frac{A}{(n+1)\, p_n\, p_{n-1}} \left| \int_0^{c/n} \phi(t) \left(\sum_{k=o}^{n-1} p_k \cos(n-k)\, t \right) dt \right|$$

$$= O\left(\sum \frac{1}{n\,(\log n)^{1+\alpha}} \right)$$

$$= O(1).$$

Therefore

$$\sum L_{3\cdot 1\cdot 2} = O(1).$$

In the same way

$$\sum L_{3\cdot 1\cdot 1} = O(1).$$

Also it is evident that the absolute values of $L_{3\cdot 2}$, $L_{3\cdot 3}$, $L_{3\cdot 4}$ are always less than that of $L_{3\cdot 1}$. Therefore we get

$$\sum L_3 = O(1).$$

and finalising $\sum L_2 = O(1)$.

L_2 can also be treated in a similar manner.

Thus the theorem is completely proved.

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