Jñānabha, Vol. 9/10, 1980

(*Dedicated to the memory of Professor Arthur Erdélyi*)

A MUTIPLE INTEGRAL INVOLVING THE PRODUCT OF THE *H*-FUNCTIONS OF ONE AND TWO VARIABLES

By

U. C. JAIN

Dopartment of Mathematics, University of Udaipur,
Udaipur - 313001, Rajasthan, India

(Received : January 25, 1979; Revised : March 28, 1979)

ABSTRACT

In this paper, a multiple integral involving the product of the *H*-functions of one and two variable has been evaluated. This integral is new and very general in nature. Some interesting integrals have been obtained as special cases of the main integral.

1. Introduction

The *H*-function of two variables occurring in this paper is a special case of the general *H*-function of two variables studied earlier by Mittal and Gupta [5]. The parameters of this function will be displayed in the following contracted notation (due essentially to Srivastava and Panda [7, p. 266, Eq. (1.5)]) which is a direct extension of that of Srivastava and Joshi [6] :

$$H_{p_1, q_1 : p_2, q_2 ; p_3, q_3}^{o, o \;:\; m_2, n_2 ; m_3, n_3}\left[\begin{matrix} x \\ y \end{matrix}\middle|\begin{matrix} (a_j ; \alpha_j, A_j)_{1, p_1} : (c_j, \epsilon_j)_{1, p_2} ; (e_j, E_j)_{1, p_3} \\ (b_j ; \beta_j, B_j)_{1, q_1} : (d_j, \delta_j)_{1, q_2} ; (f_j, F_j)_{1, q_3} \end{matrix}\right]$$

$$= H_1 [x, y] = -\frac{1}{4\pi^2} \int_{L_1} \int_{L_2} \phi(s, t)\, \theta_1(s)\, \theta_2(t)\, x^s y^t \, ds \, dt \qquad ..(1.1)$$

where

$$\phi(s, t) = \left[\prod_{j=1}^{p_1} \Gamma(a_j - \alpha_j s - A_j t) \prod_{j=1}^{q_1} \Gamma(1 - b_j + \beta_j s + B_j t) \right]^{-1} \qquad ..(1.2)$$

$$\theta_1(s) = \prod_{j=1}^{n_2} \Gamma(1-c_j+\epsilon_j s) \prod_{j=1}^{m_2} \Gamma(d_j-\delta_j s)$$

$$\left[\prod_{j=m_2+1}^{q_2} \Gamma(1-d_j+\delta_j s) \prod_{j=n_2+1}^{p_2} \Gamma(c_j-\epsilon_j s) \right]^{-1} \quad \ldots(1.3)$$

and with $\theta_2(t)$ defined analogously to $\theta_1(s)$ in terms of the parameters sets $(e_j, E_j)_{1, p_3}$ and $(f_j, F_j)_{1, q_3}$. Also $(a_j ; \alpha_j, A_j)_{1, p_1}$ abbreviates the parameter sequence $(a_1 ; \alpha_1, A_1), \ldots, (a_{p_1} ; \alpha_{p_1}, A_{p_1})$; $(c_j, \epsilon_j)_{1, p_2}$ abbreviates the parameter sequence $(c_1, \epsilon_1), \ldots, (c_{p_2}, \epsilon_{p_2})$, and so on.

The conditions on the parameters of the H-function af two variables for the convergence of the integral (1.1), the nature of contours L_1 and L_2, some of the properties, special cases and asymptotic expansions of $H_1[x, y]$ can be referred to in the paper by Goyal [2]. It will be assumed that the conditions (i)—(vi), modified appropriately, given on p. 119 in the paper by Goyal [2], are always satisfied by the various H-functions of two variables occurring in this paper.

To save space, three dots "..." appearing at a particular place in in any H-function of two variables indicate that the parameters in that position are exactly the same as that of the H-function of two variables defined by (1.1).

We shall require the following special case of a result given by Goyal [2, p. 120, Eq. (1.2)] :

$$H^{o, o : 1, N_2 ; 1, N_3}_{P_1, Q_1 : P_2, Q_2 ; P_3, Q_3} \left[\begin{matrix} x \\ y \end{matrix} \middle| \begin{matrix} (a'_j ; \alpha'_j, A'_j)_{1, p_1} : (c'_j, \epsilon_j)_{1, p_2} ; (e'_j, E'_j)_{1, p_3} \\ (b'_j ; \beta'_j, B')_{1, Q_1} : (d'_0, \delta'_0), (d'_j, \delta'_j)_{2, Q_2} ; (f'_0, F'_0), (f'_j, F'_j)_{2, Q_3} \end{matrix} \right] = H^*_1[x, y]$$

$$= \sum_{u,v=0}^{\infty} \frac{(-1)^{u+v} \, x^{\rho_u} \, y^{\sigma_v}}{u!\, v!\, \delta_0' \, F_0'} \, \emptyset'(\rho_u, \sigma_v) \, \theta_1'(\rho_u) \, \theta_2'(\sigma_v) \tag{1.4}$$

where $\rho_u = \dfrac{d_0' + u}{\delta_0'}$, $\sigma_v = \dfrac{f_0' + v}{F_0'}$ ..(1.5)

$$\emptyset'(\rho_u, \sigma_v) = \left[\prod_{j=1}^{p_1} \Gamma(a_j' - \alpha_j' \rho_v - A_j' \sigma_v) \prod_{j=1}^{Q_1} \Gamma(1 - b_j' + \beta_j \rho_u + B_j' \sigma_v) \right]^{-1}$$ ...(1.6)

$$\theta_1'(\rho_u) = \prod_{j=1}^{N_2} \Gamma(1 - c_j' + \epsilon_j' \rho_u) \left[\prod_{j=N_2+1}^{P_2} \Gamma(c_j' - \epsilon_j' \rho_u) \prod_{j=2}^{Q_2} \Gamma(1 - d_j' + \delta_j' \rho_u) \right]^{-1}$$ (...1.7)

and with $\theta_2'(\sigma_v)$ defined analogously to $\theta_1'(\rho_u)$ in terms of the parameter sets $(e_j', E_j')_{1,P_3}$ and $(f_j', F_j')_{2,Q_3}$.

2. Main Integral—The following multiple integral has been established in this paper :

$$\int_0^{\infty} \cdots \int_0^{\infty} \prod_{i=1}^{r} \left\{ (x_i)^{k_i - 1} H_{w_i, v_i}^{u_i, o} \left[\xi_i (x_i)^{t_i} \middle| \begin{matrix} (g_j^i, G_j^i)_{1, w_i} \\ (h_j^i, H_j^i)_{1, v_i} \end{matrix} \right] \right\}$$

$$H_1^* \left[a \prod_{i=1}^{r} (x_i)^{\mu_i}, b \prod_{i=1}^{r} (x_i)^{\nu_i} \right] H_1 \left[y \prod_{i=1}^{r} (x_i)^{\eta_i}, z \prod_{i=1}^{r} (x_i)^{\lambda_i} \right]$$

$$\prod_{i=1}^{r} (dx_i)$$

$$= \sum_{u,v=o}^{\infty} \frac{(-1)^{u+v} \emptyset'(\rho_u, \sigma_v)\, \theta_1'(\rho_u)\, \theta_2'(\sigma_v)\, a^{\rho_u} b^{\sigma_v}}{u\,!\, v\,!\, \delta_o' F_o' \sum_{i=1}^{r} \left\{ t_i (\xi_i)^{(k_i+\mu_i \rho_u+\nu_i \sigma_v)/t_i} \right\}}$$

$$H^{o,\,U:\,\ldots\,;\,\ldots}_{p_1+V,\, q_1+W\,:\,\ldots\,;\,\ldots} \left[\begin{matrix} y \prod_{i=1}^{r} (\xi_i)^{-\eta/t_i} \\ z \prod_{i=1}^{r} (\xi_i)^{-\lambda_i/t_i} \end{matrix} \middle| \begin{matrix} A(u,v) : \ldots ; \ldots \\ B(u,v) : \ldots ; \end{matrix} \right] \quad \ldots(2.1)$$

where $U = \sum_{i=1}^{r} u_i$, $V = \sum_{i=1}^{r} v_i$, $W = \sum_{i=1}^{r} w_i$,

$A(u, v)$ stands for $(a_j\,;\, \alpha_j\,,\, A_j)_{1,\, p_1}$,

$$\left\{ (1-h_j^i\, H_j^1\, (k_i+\mu_i\, \rho_u+\nu_i\, \sigma_v)/t_i\; ;\; H_j^1\, \eta_i/t_i,\, H_j^1\, \lambda_i/\, t_i)_{1,\, v_r} \right\}_{i=1}^{r}$$

$B(u, v)$ for $(b_j\,;\, \beta_j\,,\, B_j)_{1,\, q_1}$,

$$\left\{ (1 - g_j^1\, G_j^1\, (k_i+ \mu_i\, \rho_u + \nu_i\, \sigma_v)\; /\; t_i\;\; ;\;\; G_j^1\, \eta_i\, /\, t_i,\, G_j^1\, \lambda_i/\, t_i)_{1,\, w_r} \right\}_{i=1}^{r}$$

and the various symbols $\emptyset'(\rho_u, \sigma_v)$, $\theta_1'(\rho_u)$, $\theta_2'(\sigma_v)$, ρ_u and σ_v are explained in Section 1.

The Integral (2.1) is valid under the following conditions ;

$$\text{Re}\left\{ k_i + \mu_i\, (d_0'/\, \delta_0') + \nu_i\, (f_0'/\, F_0') \right\} + t_i \min_{1 \leqslant j \leqslant u_i} \left[\text{Re}\; (h_j^i/H_j^i) \right]$$

$$+ \eta_i \min_{1 \leqslant j \leqslant m_2} \left[Re\; (d_j/\delta_j) \right] + \lambda_i \min_{1 \leqslant j \leqslant m_3} \left[Re\; (f_j/F_j) \right] > o\,,$$

$$A = -\sum_{j=1}^{u_i} G_j^i + \sum_{j=1}^{v_i} H_j^i - \sum_{j=v_i+1}^{w_i} H_j^i > o, \; |\arg \xi_i| < (1/2)\, A_i \pi,$$

$t_i, \mu_i, \nu_i, \eta_i, \lambda_i > o, \; i = 1, ., r.$

and the double series on the right-hand side of (2.1) is absolutely convergent.

Also, $(g_j^i, G_j^i)_{1, w_i}$ abbreviates the w_i—parameter sequence $(g_1^i, G_1^i), \ldots,$

$(g_{w_i}^i, G_{w_i}^i)$, and $\left\{ (a_j^i; \alpha_j^i, A_j^i)_{1, p_i} \right\}_{i=1}^{r}$ the $(p_1 + p_2 + \ldots + p_r)$—parameter

sequence $(a_j^1; \alpha_j^1, A_j^1)_{1, p_1}, (a_j^2; \alpha_j^2, A_j^2)_{1, p_2}, \ldots, (a_j^r; \alpha_j^r, A_j^r)_{1, p_r}$.

Proof – To prove (2.1), we first use the equation (1.4) for H_1^* – function occurring in the integrand, change the order of integration and summation (which is assumed to be justified under the conditions mentioned with (2.1)), and then apply a known result of Vasishta and Goyal [8, p. 43, Eq. (2.1)].

3. Particular Cases – On account of the general nature of the integral established in (2.1), a number of other multiple integrals can be obtained by specializing the parameters of the various functions involved. To illustrate, we mention below some interesting particular cases.

(a) Taking $P_1 = Q_1 = o$, $N_2 = P_2 = Q_2 = N_3 = P_3 = 1$, $Q_3 = 2$, $\delta_0' = d_0' = E_1'$ $= F_0' = F_2' = c_1' = \epsilon_1' = 1$, $u_i = v_i = \mu_i = \nu_j = t_i = H_1^i = h_1^i = 1$, $w_i = o$, $i = 1, \ldots, r$, $e_1' = 1 - k$, $f_0' = (1/2) + n$, $f_2' = (1/2) - n$ in (2.1), and using the known results [2, p. 123, Eq. (3.4)], [3, p. 598, Eq. (4.1)], and [4, § 2.6] therein, we get the following interesting multiple integral :

$$\int_0^{\infty} \ldots \int_0^{\infty} \prod_{i=1}^{r} \left\{ (x_i)^{k_i} \exp(-\xi_i x_i) \right\} (1 + a \prod_{i=1}^{r} x_i)^{-1} \exp(-b/2 \sum_{i=1}^{r} x_i)$$

$$\cdot M_{k,n}\left(b\prod_{i=1}^{r}x_i\right) H_1\left[y\prod_{i=1}^{r}(x_i)^{\eta_i},\ z\prod_{i=1}^{r}(x_i)^{\lambda_i}\right]\prod_{i=1}^{r}(dx_i)$$

$$=\sum_{u,v=0}^{\infty}\frac{(-1)^{u+v}(\frac{1}{2}+k+n)_v\ a^u\ b^{v+n+\frac{1}{2}}}{v!\ (2n+1)_v}\prod_{i=1}^{r}\left\{(\xi_i)^{-(k_i+v+u+n+\frac{5}{2})}\right\}$$

$$H^{o,r:\ldots\ldots;\ \ldots}_{p_1+r,q_1:\ldots;\ \ldots}\left[\begin{array}{c|c} y\prod_{i=1}^{r}(\xi_i)^{-\eta_i} & (-k_i-u-v-n-\frac{1}{2};\ \eta_i,\lambda_i)_{1,r}:\ldots;\ldots \\ z\prod_{i=1}^{r}(\xi_i)^{-\lambda_i} & \ldots\qquad :\ldots;\ldots \end{array}\right] \quad \ldots(3.1)$$

where $M_{k,n}(x)$ is the whittaker function [4, p. 44, Eq. (2.3.4a)] and $\mathrm{Re}\,(k_i+n+3/2)+\eta_i \min_{1\leq j\leq m_2}[\mathrm{Re}\,(d_j/\delta_j)]+\lambda_i \min_{1\leq j\leq m_3}[\mathrm{Re}\,(f_j/F_j)]>o$, and $\eta_i.\ \lambda_i>o$, $i=1,\ldots,r$.

(b) Putting $p_1=q_1=n_3=p_3=o$, $m_3=q_3=1, f_1=o$, $F_1=1$ and letting $z\to o$, $\lambda_i=1$, $i=1, \ \ , r$ in (3.1), using the result [2, p. 123, Eq. (3.5)] therein, we get

$$\int_0^{\infty}\int_0^{\infty}\prod_{i=1}^{r}\left\{(x_i)^{k_i}\exp(-\xi_i x_i)\right\}\exp\left(-b/2\sum_{i=1}^{r}x_i\right) M_{k,n}\left(b\prod_{i=1}^{r}x_i\right)$$

$$\cdot H^{m_2,n_2}_{p_2,q_2}\left[y\prod_{i=1}^{r}(x_i)^{\eta_i}\ \middle|\ \begin{array}{l}(c_j,\varepsilon_j)_{1,p_2}\\(d_j,\delta_j)_{1,q_2}\end{array}\right]\prod_{i=1}^{r}(dx_i)$$

$$=\sum_{u,v=0}^{\infty}\frac{(-1)^{u+v}(\frac{1}{2}+k+n)_v a^u b^{v+n+\frac{1}{2}}}{v!\ (2n+1)_v}\prod_{i=1}^{r}\left\{(\xi_i)^{-(k_i+u+v+n+\frac{5}{2})}\right\}$$

$$\cdot H^{m_2, n_2 + r}_{p_2 + r, q_2}\left[y \prod_{i=1}^{r} (\xi_i)^{-\eta_i} \middle| \begin{matrix} (-k_i-u-v-n-\frac{1}{2}, \eta_i)_{1,r}, (c_j, \epsilon_j)_{1,p_2} \\ (d_j, \delta_j)_{1,q_2} \end{matrix} \right] \quad \ldots(3.2)$$

where $\text{Re}\,(k_i+n+3/2)+\eta_i \min\limits_{1\leq j\leq m_2} [\,\text{Re}\,(d_j/\delta_j)\,] > 0,\ \eta_i > 0,\ i=1,\ldots,r,$

$$A = \sum_{j=1}^{m_2} \delta_j - \sum_{j=m_2+1}^{q_2} \delta_j + \sum_{j=1}^{n_2} \epsilon_j - \sum_{j=n_2+1}^{p_2} \epsilon_j > 0, \text{ and}$$

$$|\arg y| < (1/2)\, A\, \pi,$$

(c) Again taking $p_1=q_1=n_3=p_2=n_3=o$, $m_2=q_2=m_3=q_3=2$, $p_3=1$, $\delta_1=\delta_2=E_1=F_1=F_2=1$, $d_1=\nu/2$, $d_2=-\nu/2$, $e_1=1-k'$, $f_1=n'+1/2$, $f_2=-n'+1/2$, $\lambda_j=\eta_j=1$, $i=1,\ldots r$, replacing y by $y^2/4$ in (3.1), using the known results [2, p. 123, Eq. (3.5)], [3, p. 598, Eq. (4.1)] and [4, p. 53, § 2.6] therein, we get the following integral :

$$\int_o^\infty \cdots \int^\infty \prod_{i=1}^{r} \left\{ (x_i)^{k_i} \exp(-\xi_i x_i) \right\} \exp\left(-\frac{b+z}{2} \sum_{i=1}^{r} x_i\right)$$

$$\cdot\ M_{k,n}\left(b \prod_{i=1}^{r} x_i\right) W_{k',n'}\left(z \prod_{i=1}^{r} x_i\right) K_\nu\left(y \prod_{i=1}^{r} x_i\right) \prod_{i=1}^{r} (dx_i)$$

$$= \sum_{u,v=o}^{\infty} \frac{(-1)^{u+v} (\frac{1}{2}+k+n)_v\, (a^u\, b^{v+n+\frac{1}{2}}}{2\ v!\ (2n+1)_v} \prod_{i=1}^{r} \left\{ (\xi_i)^{-(k_i+u+v+n+\frac{5}{2})} \right)$$

$$H^{o, r : 2, o ; 2, o}_{r, o : o, 2 ; 1, 2}\left[\begin{matrix} \frac{y^2}{4} \prod_{i=1}^{r} (\xi_i)^{-1} \\ z \prod_{i=1}^{r} (\xi_i)^{-1} \end{matrix} \middle| \begin{matrix} (-k_i-u-v-n-\frac{1}{2})_{1,r} : \ - \ ; \ 1-k' \\ : \nu/2,\ \nu/2; n'+\frac{1}{2}, -n'+\frac{1}{2} \end{matrix} \right] \quad (3.3)$$

where $\mathrm{Re}\,(k_i + n \pm n' \pm \frac{\nu}{2} + 2) > o$, $i=1,\ldots,r$.

In (3.3), $G\begin{bmatrix} x \\ y \end{bmatrix}$ stands for the G-function of two variables introduced by Agarwal [1], $K_\nu(x)$ is the modified Bessel function, and $W_{k,n}(x)$ is the well-known Whittaker function.

Lastly, if we put $N_2=P_2=N_3=P_3=P_1=Q_1=o$, $Q_2=Q_3=1$, $\mu_i=\nu_i=1$. $i=1,\ldots,r$ in (2.1) and let a, $b\to o$ in it, we get a recent result of Vasishta and Goyal [8, p. 43, Eq. (2.1)] which is turn reduces to the result by Mittal and Gupta [5, p. 121] by taking r=1 in the result of Vasishta and Goyal [8].

Acknowledgements

The author is grateful to Professor H. M. Srivastava (of the University of Victoria, Canada) for his valuable suggestions regarding the improvement of the paper. The author is also thankful to Dr. K. C. Gupta and Dr. S. P. Goyal for useful discussions during the preparation of this paper.

REFERENCES

1. R. P. Agarwal, An extension of Meijer's G-function, Proc. Nat. Inst Sci. India Part A **31** (1965), 536-546.

2. S. P. Goyal, The H-function of two variables, Kyungpook Math. J. **15** (1975), 117-131.

3. K. C. Gupta and U. C. Jain, The H-function. II, Proc. Nat. Acad. Sci. India Sect. A **36** (1966), 594-609.

4. A. M. Mathai and R. K. Saxena, Generalized Hypergeometric Function with Applications in Statistics and Physical Sceinces, Lecture Notes in Math., No. 348, Springer-Verlag, 1973.

5. P. K. Mittal and K. C Gupta, An integral involving generalized function of two variables, Proc. Indian Acad Sci. Sect. A 75 (1972), 117-123.

6. H. M. Srivastava and C. M. Joshi, Integration of certain products associated with a generalized Meijer function, Proc. Cambridge Philos. Soc. **65** (1969), 471-477.

7. H.M. Srivastava and R. Panda, Some bilateral generating functions for a class of generalized hypergeometric polynomials, J. Reine Angew. Math. **283/284** (1976), 265-274.

8. S. K. Vasishta and S. P. Goyal, Multiple integrals involving the H-function of two variables, Proc. Indian Acad. Sci. Sect. A **83** (1976), 41-49.