Jñānabha, Vol. 9/10, 1980

(*Dedicated to the memory of Professor Arthur Erdélyi*)

SOME FINITE SERIES CONCERNING THE H-FUNCTION OF n VARIABLES

By

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(Received : September 15, 1978; Revised : March 15, 1979)

ABSTRACT

In this paper, we shall first prove some three-term recurrence relations for the H–function of n variables and then use them to generate some finite series involving the H–function of n variables.

The recurrence relations established and the series generated are believed to be new.

1. INTRODUCTION

Srivastava and Panda [3, p. 271, Eq. (4.1)] introduced the H–function of n variables by means of multiple Mellin-Barnes type contour integral in the following manner (with slight change of parameters) :

$$H^{0,\lambda:(\mu',v');\ldots;(\mu^{(n)},\nu^{(n)})}_{A,B:[C',D'];\ldots;[C^{(n)},D^{(n)}]}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(n)}]:[(c'):\phi'];\ldots;[(c^{(n)}):\phi^{(n)}];\\ [(b):\epsilon',\ldots,\epsilon^{(n)}]:[(d'):\delta'];\ldots;[(d^{(n)}):\delta^{(n)}];\end{matrix}\, z_1,\ldots,z_n\right)$$

$$=\frac{1}{(2\pi\omega)^n}\int_{L_1}\ldots\int_{L_n} U_1(s_1)\ldots U_n(s_n)\,V(s_1,\ldots,s_n)\,z_1^{s_1}\ldots z_n^{s_n}\cdot ds_1\ldots ds_n,$$

$$\omega=\sqrt{-1} \qquad \ldots(1.1)$$

where

$$V(s_1, \ldots, s_n) = \frac{\prod_{j=1}^{\lambda} \Gamma\left(1-a_j+\sum_{i=1}^{n} \theta_j^{(1)} s_i\right)}{\prod_{j=\lambda+1}^{A} \Gamma\left(a_j-\sum_{i=1}^{n} \theta_j^{(1)} s_i\right) \prod_{j=1}^{B} \Gamma\left(1-b_j+\sum_{i=1}^{n} \epsilon_j^{(1)} s_i\right)} \quad (1.2)$$

and

$$U_i(s_i) = \frac{\prod_{j=1}^{\mu^{(1)}} \Gamma\left(d_j^{(1)}-\delta_j^{(1)} s_i\right) \prod_{j=1}^{\nu^{(1)}} \Gamma\left(1-c_j^{(1)}+\emptyset_j^{(1)} s_i\right)}{\prod_{j=\mu^{(1)}+1}^{D^{(1)}} \Gamma\left(1-d_j^{(1)}+\delta_j^{(1)} s_i\right) \prod_{j=\nu^{(1)}+1}^{C^{(1)}} \Gamma\left(c_j^{(1)}-\emptyset_j^{(1)} s_i\right)}$$

$$i = 1, \ldots, n. \quad \ldots (1.3)$$

For a detailed account of conditions under which the definition (1.1) makes sense, see [3], [4] and [5]. Here we assume, following Srivastava and Panda [3], that

(a) stands for the sequence of parametrs $a_1, \ldots, a_A$;

(b) for $b_1, \ldots, b_B$; $(c^{(1)})$ for $c_1^{(1)}, \ldots, c^{(1)}_{C^{(1)}}$

$(d^{(1)})$ for $d_1^{(1)}, \ldots, d^{(1)}_{D^{(1)}}$, etc., $i=1, \ldots, n$, it being understood, for example that $c^{(1)}=c'$, $c^{(2)}=c''$, and so on.

2. NOTATIONS

Let us assume that :

$$\theta^{(1)}_{A-1} = \alpha_{A-1}\ h_i,\ \forall_i \in \{1, \ldots, n\} \quad (2.1)$$

$$\theta^{(1)}_{A} = \alpha_{A}\ h_i,\ \forall_i \in \{1, \ldots, n\} \quad \ldots (2.2)$$

$$\epsilon^{(1)}_{1} = \beta_1\ h_i,\ \forall i \in \{1, \ldots, n\} \quad \ldots (2\ 3)$$

$$\in_2^{(1)} = \beta_2 \; h_i \,, \quad \forall i \in \{1, .., n\} \qquad \ldots(2.4)$$

and $$\theta_1^{(1)} = \alpha_1 \, h_i \,, \; \forall i \in \{1,\ldots, n\} \qquad \ldots(2.5)$$

Under the restrictions (2.2) and (2.3), let the H-function of n variables defined by (1.1) be briefly represented by $H^+ [a_A , b_1]$ and furthermore by $H_0^+ [a_A , b_1]$ in case $\theta_A^{(1)} = \in_1^{(1)}$. And under the same restrictions and replacing a_A by $a_A - 1$, we shall denote the H-function of n variables defined by (1.1) as $H^+ [a_A - 1, b_1]$ and, furthermore, as $H_0^+ [a_A - 1, b_1$] provided $\theta_A^{(1)} = \in_1^{(1)}$, and so on. Likewise, $H^+ [b_1, b_2]$ will mean the function of n variables defined by (1.1) under the restrictions (2.3) and (2.4), and so on.

Also, for the sake of notational convenience, the

determinants $\begin{vmatrix} a_A - r & b_1 \\ \alpha_A & \beta_1 \end{vmatrix}$, $\begin{vmatrix} a_A - r & b_1 \\ 1 & 1 \end{vmatrix}$ will be

represented by $\Delta (a_A - r, b_1)$, $\Delta' (a_A - r, b_1)$, respectively, and so on.

3. The Recurrence Relations

(1) $$\beta_1 H^+ [a_A - 1, b_1] + \alpha_A H^+ [a_A , b_1 + 1] = \Delta (a_A - 1, b_1) \, H^+ [a_A , b_1] \qquad \ldots(3.1),$$

(2) $$\beta_1 H^+ [a_1, b_2 + 1] - \beta_2 H^+ [b_1 + 1, b_2] = -\Delta \; (b_2, b_1) \, H^+ [b_1, b_2] \qquad \ldots(3.2),$$

(3) $$\alpha_{A-1} \; H^+ [a_{A-1} , a_A - 1] - \alpha_A H^+ [a_{A-1} - 1, a_A] = \Delta (a_A - 1, a_{A-1} - 1) \, H^+ [a_{A-1} , a_A] \qquad \ldots(3.\cdot),$$

$$(4)\quad a_A H^+[a_1-1, a_A] + a_1 H^+[a_1, a_A-1] = -\Delta(a_1-1, a_A-1)\, H^+[a_1, a_A] \qquad ...(3.4)$$

To develop the proof of the recurrence relation (3.1), we shall extend the method of Buschman, see [1, p. 41] and [2, p. 1419]. We note from the definition of the H-function of n veriables, with the restrictions (2.2) and (2.3), that the replacement of a_A by a_A-1 introduces the multiplier $a_A-1-a_A \sum_{i=1}^{n} h_i s_i$ into the contour integral for $H^+[a_A, b_1]$ Similarly replacement of b_1 by b_1+1 introduces the factor $-b_1+\beta_1 \sum_{i=1}^{n} h_i s_i$. Consequently, we can form the three-term recurrence formula involving the undetermined coefficients L, M, N:

$$L\, H^+[a_A-1, b_1] + M\, H^+[a_A, b_1+1] = N\, H^+[a_A, b_1], \qquad ...(3.5)$$

and then require that

$$L\left(a_A-1-a_A \sum_{i=1}^{n} h_i s_i\right) + M\left(-b_1+\beta_1 \sum_{i=1}^{n} h_i s_i\right) = N \qquad ..(3.6)$$

be an identy in $s_1,\ldots, s_n$. Hence L, M, N can be evaluated. Substituting for L, M, N in (3.3) and after a little simplification we obtain the desired result (3.1). The proofs of recurrence relations (3.2) — (3.4) can be developed in a similar manner.

4. Finite Series

$$(1)\quad \frac{\alpha_A}{\beta_1} \sum_{k=1}^{m} \frac{(-1)^k H^+[a_A-k+1, b_1+1]}{\Gamma\left[\frac{\alpha_A}{\beta_1} b_1-(a_A-1)+k\right]}$$

$$= \frac{(-1)^{m+1} H^+[a_A-m, b_1]}{\Gamma\left[\frac{\alpha_A}{\beta_1} b_1-(a_A-1)+\right]} + \frac{H^+[a_A, b_1]}{\Gamma\left[\frac{\alpha_A}{\beta_1} b_1-(a_A-1\right]} \qquad (4.1),$$

(2) $$\frac{\beta_1}{\alpha_A} \sum_{k=1}^{m} \frac{(-1)^k H^+ [a_A - 1, b_1 + k - 1]}{\Gamma\left[b_1 + k - \frac{\beta_1}{\alpha_A}(a_A - 1)\right]}$$

$$= \frac{(-1)^{m+1} H^+ [a_A, b_1 + m]}{\Gamma\left[b_1 + m - \frac{\beta_1}{\alpha_A}(a_A - 1)\right]} + \frac{H^+ [a_A, b_1]}{\Gamma\left[b_1 - \frac{\beta_1}{\alpha_A}(a_A - 1)\right]} \quad .. (4.2),$$

(3) $$\sum_{k=1}^{m} (-1)^k \alpha_A^{k-1} \beta_1^{m-k} \Delta (a_A + k - 1, b_1 + k) H^+ [a_A + k, b_1 + k]$$

$$= (-1)^m \alpha_A^m H^+ [a_A + m, b_1 + m + 1] - \beta_1^m H^+ [a_A, b_1 + 1] \quad ....(4.3),$$

(4) $$\sum_{k=0}^{m} \binom{m}{k} \overset{+}{H}_0 [a_A - k, b_1 + m - k]$$

$$= \frac{(-1)^m \Gamma (b_1 - a_A + m + 1)}{\Gamma (b_1 - a_A + 1)} \overset{+}{H}_0 [a_A, b_1] \quad .. (4.4),$$

(5) $$\frac{\beta_1}{\beta_2} \sum_{k=1}^{m} \frac{(-1)^k H^+ [b_1 + k - 1, b_2 + 1]}{\Gamma\left[b_1 + k - \frac{\beta_1}{\beta_2} b_2\right]}$$

$$= \frac{(-1)^m H^+ [b_1 + m, b_2]}{\Gamma\left[b_1 + m - \frac{\beta_1}{\beta_2} b_2\right]} - \frac{H^+ [b_1, b_2]}{\Gamma\left[b_1 - \frac{\beta_1}{\beta_2} b_2\right]} \quad ...(4.5),$$

(6) $$\frac{\beta_2}{\beta_1} \sum_{k=1}^{m} \frac{(-1)^k H^+ [b_1 + 1, b_2 + k + 1]}{\Gamma\left[b_2 + k - \frac{\beta_2}{\beta_1} b_1\right]}$$

$$= \frac{(-1)^m H^+ [b_1, b_2 + m]}{\Gamma\left[b_2 + m - \frac{\beta_2}{\beta_1} b_1\right]} - \frac{H^+ [b_1, b_2]}{\Gamma\left[b_2 - \frac{\beta_2}{\beta_1} b_1\right]} \quad ...(4.6)$$

(7)
$$\frac{a_{A-1}}{a_A} \sum_{k=1}^{m} \frac{(-1)^k H^{+} (a_{A-1} - k + 1, a_A - 1]}{\Gamma\left[\frac{a_{A-1}}{a_A}(a_A - 1) - (a_{A-1} - 1) + k\right]}$$

$$= \frac{(-1)^m [a_{A-1} - m, a_A]}{\Gamma\left[\frac{a_{A-1}}{a_A}(a_A - 1) - (a_{A-1} - 1) + m\right]} - \frac{H^{+} [a_{A-1}, a_A]}{\Gamma\left[\frac{a_{A-1}}{a_A}(a_A - 1) - a_{A-1} - 1)\right]}$$

...(4.7)

(8)
$$\frac{a_A}{a_{A-1}} \sum_{k=1}^{m} \frac{(-1)^k H^{+} [a_{A-1} - 1, a_A - k + 1]}{\Gamma\left[\frac{a_A}{a_{A-1}}(a_{A-1} - 1) - (a_A - 1) + k\right]}$$

$$= \frac{(-1)^m H^{+} [a_{A-1}, a_A - m]}{\Gamma\left[\frac{a_A}{a_{A-1}}(a_{A-1} - 1) - (a_A - 1) + m\right]} - \frac{H^{+} [a_{A-1}, a_A]}{\Gamma\left[\frac{a_A}{a_{A-1}}(a_{A-1} - 1) - (a_A - 1)\right]}$$

...(4.8),

(9)
$$\frac{a_1}{a_A} \sum_{k=1}^{m} \frac{H^{+} [a_1 - k + 1, a_A - 1]}{\Gamma\left[\frac{a_1}{a_A}(a_A - 1) - (a_1 - 1) + k\right]}$$

$$= \frac{H^{+} [a_1, a_A]}{\Gamma\left[\frac{a_1}{a_A}(a_A - 1) - (a_1 - 1)\right]} - \frac{H^{+} [a_1 - m, a_A]}{\Gamma\left[\frac{a_1}{a_A}(a_A - 1) - (a_1 - 1) + m\right]}$$

.....(4.9),

and

(10)
$$\frac{a_A}{a_1} \sum_{k=1}^{m} \frac{(-1)^k H^{+} [a_1 - 1, a_A - k + 1]}{\Gamma\left[\frac{a_A}{a_1}(a_1 - 1) - (a_A - 1) + k\right]}$$

$$= \frac{(-1)^{m+1} H^{+}[a_1, a_A - m]}{\Gamma\left[\frac{a_A}{a_1}(a_1-1)-(a_A-1)+m\right]} + \frac{H^{+}[a_1, a_A]}{\Gamma\left[\frac{a_A}{a_1}(a_1-1)-(a_A-1)\right]} \quad \ldots(4.10)$$

Formulas (4.1) to (4.3) will be derived from recurrence relation (3.1) by the formulation of collapsing series whereas (4.4) by iteration of (3.1) in case $\theta_A^{(1)} = \in_1^{(1)}$.

Similarly formulas (4.5) and (4.6), (4.7) and (4.8), and (4.9) and (4.10), can be obtained from recurrences (3.2), (3.3) and (3.4), respectively, by the formation of collapsing series.

(i) Pairing terms involving b_1 and replacing a_A by $a_A - k+1$, (3.1) can be rewritten as :

$$a_A H^{+}[a_A - k+1, b_1+1] = -\beta_1 H^{+}(a_A - k, b_1] + \triangle(a_A - k, b_1) H^{+}[a_A - k+1, b_1]$$

and then noting that

$$\triangle(a_A - k, b_1) = -\beta_1 \frac{\Gamma\left[\frac{a_A}{\beta_1} b_1 - (a_A - 1) + k\right]}{\Gamma\left[\frac{a_A}{\beta_1} b_1 - (a_A - 1) + k - 1\right]},$$

We can sum on k and collapse the resulting series on the the right. Consequently, we obtain (4.1).

(ii) If however, we pair terms involving a , replace b_1 by b_1+k-1, rewrite (3.1), sum on k and collapse the resulting series on the right, we arrive at (4.2).

(iii) If we replace a_A by $a_A + k$, b_1 by $b_1 + k$ and rewrite (3.1) after multiplying both sides by $(-1)^k a_A^{k-1} \beta_1^{m-k}$, we can sum on k and collapse the resulting se ies on rhe right. Consequently we get (4.3).

(iv) If we take $\theta_A^{(i)} = \Theta_1^{(1)}$, (3.1) becomes

$$\Delta' (a_A - 1, b_1) \overset{+}{H}_0 [a_A, b_1] = \overset{+}{H}_0 [a_A, b_1 + 1] + 1 \overset{+}{H}_0 [a_A - 1, b_1].$$

Now multiply both sides by $\Delta' (a_A - 2, b_1) \Delta' (a_A - 1, b_1 + 1)$ and consequently expend each term on right hand side by iteration of this same relation, we get, after a little simplification :

$$(-1)^2 \frac{\Gamma (b_1 - a_A + 3]}{\Gamma (b_1 - a_A + 1)} \overset{+}{H}_0 [a_A, b_1]$$

$$\sum_{k=0}^{2} \binom{2}{k} \overset{+}{H}_0 [a_A - k, b_1 + 2 - k]$$

Further iterations will finaly yield :

$$(-1)^m \frac{\Gamma (b_1 - a_A + m + 1)}{\Gamma (b_1 - a_A + 1)} \overset{+}{H}_0 [a_A ; b_1]$$

$$= \sum_{k=0}^{m} \binom{m}{k} \overset{+}{H}_0 [a_A - k, b_1 + m - k].$$

ACKNOWLEDGEMENTS

The author wishes to express his grateful thanks to Professor H. M. Srivastava, Department of Mathematics, University of Victoria, Canada, for his generous help in the preparation of this paper.

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