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(*Dedicated to the memory of Professor Arthur Erdélyi*)

CERTAIN THEOREMS ON BILATERAL GENERATING FUNCTIONS

By

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1. Saran [5], Shanker [6], Panda [3], Singhal and Srivastava [7], Srivastava [9], Srivastava and Singhal [10] and Srivastava and Lavoie [11] have proved a number of Theorems on bilinear as well as bilateral generating functions. The purpose of this paper is to establish two theorems for obtaining bilateral generating functions in terms of Pochhammer's double-loop type integrals. These theorems have been used to obtain a number of bilateral generating functions. The results obtained here have been further generalised to derive bilateral generating functions of polynomials of several variables.

Let the triple hypergeometric series, defined by Srivastava [8 p. 428], be represented as :

$$F\begin{matrix} K::m;n;p;r;s;t \\ L::M;N;P;R;S;T \end{matrix}\left[\begin{matrix} \{a_K\}::\{b_m\};\{b'_n\};\{b''_p\}:\{c_r\};\{c'_s\};\{c''_t\}; \\ \{d_L\}::\{e_M\};\{e'_N\};\{e''_P\}:\{f_R\};\{f'_S\};\{f''_T\}; \end{matrix} x,y,z\right]$$

$$=\sum_{u,v,w=0}^{\infty}\frac{\prod_{j=1}^{K}(a_j)_{u+v+w}\prod_{j=1}^{m}(b_j)_{u+v}\prod_{j=1}^{n}(b_j')_{v+w}\prod_{j=1}^{p}(b_j'')_{w+u}}{\prod_{j=1}^{L}(d_j)_{u+v+w}\prod_{j=1}^{M}(e_j)_{u+v}\prod_{j=1}^{N}(e'_j)_{v+w}\prod_{j=1}^{P}(e_j'')_{w+u}}$$

$$\cdot \frac{\prod_{j=1}^{r} (c_j)_u \prod_{j=1}^{s} (c'_j)_v \prod_{j=1}^{t} (c''_j)_w}{\prod_{j=1}^{R} (f_j)_u \prod_{j=1}^{S} (f'_j)_v \prod_{j=1}^{T} (f''_j)_w} \quad \frac{x^u}{u\,!} \quad \frac{y^v}{v!} \quad \frac{z^w}{w!} \qquad \ldots(1.1)$$

where $\prod_{j=1}^{k} \Gamma (a_j)_n = \prod_{j=1}^{k} \frac{\Gamma (a_j+n)}{\Gamma (a_j)}$.

We prove the following results :

Theorem 1. *If*

$$F(x, t) = \sum_{n=0}^{\infty} f_n(x)\, t^n, \qquad \ldots (1.2)$$

then

$$\frac{\Gamma(1-a)\,\Gamma(1-b)\,\Gamma(a+b)}{(2\pi i)^2} \int (-p)^{a-1} (p-1)^{b-1}\; {}_2F_1 \left[\begin{matrix} 1-b,\ c; \\ a; \end{matrix} \frac{yp}{(y-1)(p-\ )} \right]$$

$$F(x, tp)\, dp = (1-y)^c \sum_{n=0}^{\infty} \frac{(a)_n}{(a+b)_n}\; {}_2F_1 (-n, c; a; y)\, f_n (x)\, t^n \quad \ldots (1.3)$$

where the integration is taken over Pochhammer's double-loop [12, p. 256].

Theorem 2. *If*

$$G(x, t) = \sum_{n=0}^{\infty} g_n (x)\, t^n, \qquad \ldots (1.4)$$

then

$$\frac{\Gamma(1-a)\Gamma(1-b)\ (1-c)\ \ (1-d)\Gamma(a+b)\ \ (c+d)}{(2\pi i)^4}$$

$$\int\int (-p)^{a-1} (p-1)^{b-1} (-q)^{c-1} (q-1)^{d-1}$$

$${}_2F_1 \left[\begin{matrix} 1-b,\ 1-d; \\ a; \end{matrix} \frac{ypq}{(y-1)(p-1)(q-1)} \right] G\left(x, \frac{pqt}{1-y}\right) dp\, dq$$

$$= (1-y)^c \sum_{n=0}^{\infty} \frac{(a)_n\,(c)_n}{(a+b)_n\,(c+d)_n}\ {}_2F_1\,(-n;\, c+n;\, a;\, y)\, g_n\,(x)\, t^n \qquad \ldots(1.5)$$

where the path of integration is Pochhammer's double-loop type contour [12, p. 256].

2 PROOFS

To prove Theorem 1, we write tp for t in (1.2), multiply both sides by

$$(-p)^{a-1}\,(p-1)^{b-1}\ {}_2F_1\left[\begin{matrix}1-b,\, c; \\ a;\end{matrix}\ \frac{yp}{(y-1)(p-1)}\right]$$

and integrate with respect to p along the Pochhammer's double-looptype contour, thus obtaining the left-hand side of (1.3).

On using the result [12, p. 257], viz.

$$\frac{1}{(2\pi i)^2}\int_C (-p)^{a-1}\,(p-1)^{b-1}\,dp = \frac{1}{\Gamma(1-a)\,\Gamma(1-b)\,\Gamma(a+b)} \qquad \ldots(2.1)$$

where C is Pochhammer's double-loop type contour, the right-hand side is equal to

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(a)_{n+m}\,(c)_m\,(1-b)_m\,(b)_{-m}}{(a+b)_n\,(a)_m\,m\,!}\left(\frac{-y}{y-1}\right)^m f_n\,(x)\,t^n$$

$$= \sum_{n=0}^{\infty} \frac{(a)_n}{(a+b)_n}\ {}_2F_1\,\left(a+n),\, c;\, a;\ \frac{y}{y-1}\right) f_n\,(x)\,t^n$$

This, on using the transformation

$${}_2F_1\,(a,\, b;\, c;\, x) = (1-x)^{-b}\ {}_2F_1\,(c-a,\, b;\, c;\ \frac{x}{x-1}), \qquad \ldots(2.2)$$

immediately proves that result (1.3).

Similarly, if we multiply (1.4) (with tpq for t) by

$$(-p)^{\ \ -1}\,(p-1)^{b-1}\,(-q)^{c-1}\,(q-1)^{d-1}\,{}_2F_1\left[\begin{matrix}1-b,\, 1-d; \\ a;\end{matrix}\ \frac{ypq}{(y-1)(p-1)(q-1)}\right]$$

and integrate with respect to p and q along Pochhammer's double-loop type contours, we get (1.5) after using (2.2).

3. APPLICATIONS

(i) If $f_n(x)=P_n^{(d,e)}(x)$ where $P_n^{(a,b)}(x)$ is well known Jacobi polynomial [4, p. 254 (1)], then by Brafman's result [4, p. 270], we have

$F(x, t) = F_4\,[1+e,\ 1+d : 1+d,\ 1+e;\ \tfrac{1}{2}\,t\,(x-1),\ \tfrac{1}{2}\,t\,(x+1)]$

Applying Theorem 1, we obtain

$$F\begin{matrix}1 :: 2;\,0;\,0;\,0;\,0;\,1\\0 :: 1;\,0;\,0;\,1;\,1;\,1\end{matrix}\left[\begin{matrix}a :: 1+d,\,1+e;\,-;\,-;\,-;\,-;\,c;\\-:: a+b;-;-;\,1+d;\,1+e;\,a;\end{matrix}\ \tfrac{1}{2}t(x-1),\tfrac{1}{2}t(x+1),\frac{y}{y-1}\right]$$

$$= (1-y)^c \sum_{n=0}^{\infty} \frac{(a)_n}{(a+b)_n}\ {}_2F_1\,(-n,\ c;\ a;\ y)\,P_n^{(d,e)}\,(x\ \ t^n \qquad \dots(3.1)$$

Taking $a+b = 1+d$, we get 5, p. 14 Eq. (3.2)].

(ii) If $f_n(x) = P_n^{(d-n,\ e-n)}(x)$, then, by Carlitz's result, we have

$F(x, t) = [1+\tfrac{1}{2}(x+1)\,t]^d\,[1+\tfrac{1}{2}(x-1)t\,]^e$

Applying Theorem 1, we get

$$F\begin{matrix}1 :: 0;\,0;\,0:\,1;\,1;\,1\\0 :: 1;\,0;\,0:\,0;\,0;\,1\end{matrix}\left[\begin{matrix}a :: -;\,-;\,-:\,-d;\,-e;\,c;\\- :: a+b;\,-;\,-:\,-;\,-;\,a;\end{matrix}\ \frac{-(x+1)\,t}{2},\ \frac{-(x-1)\,t}{2},\ \frac{y}{y-1}\right]$$

$$= (1-y)^c \sum_{n=0}^{\infty} \frac{(a)_n}{(a+b)_n}\ {}_2F_1\,(-n,\ c;\ a;\ y)\ P_n^{(d-n,\ e-n)}\,(x)\ t^n \qquad \dots(3.2)$$

which, on putting $a+b = f$, gives [5, p. 14, Eq, (3.1)].

(iii) If $g_n(x) = P_n^{(f,g)}(x)$, then Theorem 2, gives

$$F\begin{matrix}2 :: 0;\,2;\,0:\,0;\,0;\,0;\\0 :: 0;\,2;\,0:\,1;\,1;\,1;\end{matrix}\left[\begin{matrix}a,\,c :: -;\ 1+f,\ 1+g;\ -:\ -;\ -;\ -;\\- ::\ \ -;\ a+b,\ c+d;\ -:\ a;\ 1+f;\ 1+g;\end{matrix}\ \frac{y}{y-1},\ \frac{t\,(x-1)}{2(1-y)},\ \frac{t\,(x+1)}{2\,(1-y)}\right]$$

$$= (1-y)^c \sum_{n=0}^{\infty} \frac{(a)_n\,(c)_n}{(a+b)_n\,(c+d)_n}\ {}_2F_1\,(-n,\ c+n;\ a;\ y)\ P_n^{(f,g)}(x)\ t^n \qquad \dots(3.3)$$

which, on taking $a+b=1+f$, $c+d=1+g$, yields [5, p. 16, Eq. (3.3)].

(iv) Again, if $g_n(x) = P_n^{(f-n,\, g-n)}(x)$, then Theorem 2 gives the result [5, p. 16, Eq. (3.4)] viz.

$$F\,{}^{2\,::\,0;\,0;\,0:\,1;\,1;\,0}_{0\,::\,2;\,0;\,0:\,0;\,0;\,1}\left[\begin{matrix} a,\, c \,::\, -;\, -;\, -;\, -f,\, -g,\, -; \\ -\,::\, a+b,\, c+d;\, -;\, -;\, -;\, -;\, a; \end{matrix}\; \frac{(x+1)t}{2(y-1)},\, \frac{(x-1)t}{2(y-1)},\, \frac{y}{y-1}\right]$$

$$=(1-y)^c \sum_{n=0}^{\infty} \frac{(a)_n\,(c)_n}{(a+b)_n\,(c+d)_n}\, {}_2F_1(-n,\, c+n;\, a;\, y)\, P_n^{(f-n,\, g-n)}(x)\, t^n \quad \ldots(3.4)$$

Srivastava and Singhal [10, p., 357, Eq. (14)] have obtained an expansion formula by using the series iteration method, from which also the above results (3 1) to (3.4) can be obtained as specialised cases.

4. Extension To Polynomials In Two And More Variables

If

$$F(x_1, x_2, t_1, t_2) = \sum_{m,n=0}^{\infty} f_{m,n}(x_1, x_2)\, t_1^m\, t_2^n,$$

then an analogous theorem corresponding to (1.3) can be written as :

$$\frac{\Gamma(1-a_1)\Gamma(1-b_1)\Gamma(1-b_2)\Gamma(1-b_2)\,\Gamma(a_1+b_1)\Gamma(a_2+b_2)}{(2\pi i)^4}$$

$$\int\!\!\int (-p_1)^{a_1-1}(p_1-1)^{b_1-1}$$

$$(-p_2)^{a_2-1}(p_2-1)^{b_2-1}\; {}_2F_1\left[\begin{matrix} 1-b_1,\, c_1; \\ a_1; \end{matrix}\; \frac{y_1\, p_1}{(y_1-1)(p_1-1)}\right]$$

$${}_2F_1\left[\begin{matrix} 1-b_2,\, c_2; \\ a_2; \end{matrix}\; \frac{y_2\, p_2}{(y_2-1)(p_2-1)}\right] F(x_1, x_2, p_1 t_1, p_2 t_2)\, dp_1\, dp_2$$

$$=(1-y_1)^{c_1}\,(1-y_2)^{c_2} \sum_{m,n=0}^{\infty} \frac{(a_1)_m\,(a_2)_n}{(a_1+b_1)_m\,(a_2+b_2)_n}$$

$${}_2F_1(-m,\, c_1;\, a_1;\, y_1)\; {}_2F_1(-n,\, c_2;\, a_2;\, y_2)\, f_{m,n}(x_1, x_2)\, t_1^m\, t_2^n \quad \ldots(4.1)$$

where the integration is taken along Pochhammer's double-loop type contour.

Similarly, analogous to (1.5), we have the following results :

Theorem. 3. *If*

$$G(x_1, x_2, t_1, t_2) = \sum_{m,n=0}^{\infty} g_{m,n}(x_1, x_2)\, t_1^{m}\, t_2^{n},$$

then

$$\frac{\Gamma(1-a_1)\ (1-a_2)\ (1-b_1)\Gamma(1-b_2)\ (1-c_1)\Gamma(1-c_2)\Gamma(1-d_1)\ (1-d_2)}{(2\pi i)^8}$$

$$\cdot\Gamma(a_1+b_1)\Gamma(a_2+b_2)\Gamma(c_1+d_1)\ (c_2+d_2) \int\!\!\int\!\!\int\!\!\int (-p_1)^{a_1-1}(p_1-1)^{b_1-1}$$

$$\cdot q_1^{c_1-1}(q_1-1)^{d_1-1}(-p_2)^{a_2-1}(p_2-1)^{b_2-1}(-q_1)^{c_1-1}(q_1-1)^{d_1-1}(-q_2)^{c_2-1}$$

$$(q_2-1)^{d_2-1}$$

$$\cdot {}_2F_1\left[\begin{matrix}1-b_1,\ 1-d_1; \\ a_1;\end{matrix}\ \frac{y_1\, p_1\, q_1}{(y_1-1)(q_1-1)(q_1-1)}\right]$$

$$\cdot {}_2F_1\left[\begin{matrix}1-b_2,\ 1-d_2; \\ a_2;\end{matrix}\ \frac{y_2\, p_2\, q_2}{(y_2-1)(p_2-1)(q_2-1)}\right]$$

$$G\left(x_1, x_2, \frac{t_1\, p_1\, q_1}{1-y_1}, \frac{t_2\, p_2\, q_2}{1-y_2}\right) dp_1\, dp_2\, dq_1\, dq_2$$

$$= (1-y_1)^{c_1}(1-y_2)^{c_2} \sum_{m,n=0}^{\infty} \frac{(a_1)_m\,(c_1)_m\,(a_2)_n\,(c_2)_n}{(a_1+b_1)_m(c_1+d_1)_m(a_2+b_2)_n(c_2+d_2)_n}$$

$$\cdot {}_2F_1(-m, c_1+m; a_1; y_1)\ {}_2F_1(-n, c_2+n; a_2; y_2)\ g_{m,n}(x_1, x_2)\, t_1^{m}\, t_2^{n} \quad \ldots(4.2)$$

the integration being taken along the Pochhammer's double-loop type contour.

5. APPLICATIONS

(i) If we apply (4.1) to the generating function

$${}_0F_1(-; f_1; -xt)\ {}_0F_1(-; f_2; -y\rho)\ {}_0F_1\left[-; 1+e-f_1-f_2; (t+\rho)(1-x-y)\right]$$

$$= \sum_{m,n=0}^{\infty} \frac{t^m \rho^n}{(1+e-f_1-f_2)_{m+n}} \mathcal{J}_{m,n}(e, f_1, f_2; x, y) \qquad \ldots(5.1)$$

where $\mathcal{J}_{m,n}$ is a Jacobi polynomial of two variables [1, p. 100], we obtain

$$\sum_{r,s=0}^{\infty} \frac{(a_1)_r (a_2)_s [t(1-x-y)]^r [\rho(1-x-y)]^s}{(a_1+b_1)_r (a_2+b_2)_s (1+e-f_1-f_2)_{r+s}\, r!\, s!}$$

$$.F\begin{matrix} 1:1;0 \\ 0:1;2 \end{matrix}\left[\begin{matrix} a_1+r : c_1 ; - ; \\ -: a_1; a_1+b_1+r; f_1 ; \end{matrix} \frac{y_1}{y_1-1}, -xt\right]$$

$$.F\begin{matrix} 1:1;0 \\ 0:1;2 \end{matrix}\left[\begin{matrix} a_2+s : c_2 ; - ; \\ -: a_2; a_2+b_2+s, f_2 ; \end{matrix} \frac{y_2}{y_2-1}, -y\rho\right]$$

$$= (1-y_1)^{c_1} (1-y_2)^{c_2} \sum_{m,n=0}^{\infty} \frac{(a_1)_m (a_2)_n}{(a_1+b_1)_m (a_2+b_2)_n (1+e-f_1-f_2)_{m+n}}$$

$$.{}_2F_1(-m, c_1; a_1; y_1)\ {}_2F_1(-n, c_2; a_2; y_2)\, \mathcal{J}_{m,n}(e, f_1, f_2; x, y)\, t^m \rho^n \qquad \ldots(5.2)$$

where

$$F\begin{matrix} g:h,H \\ j:k;K \end{matrix}\left[\begin{matrix} \{a_g\} : \{b_h\} ; \{B_H\} ; \\ \{d_j\} : \{c_k\} ; \{C_K\} ; \end{matrix} x, y\right]$$

$$= \sum_{m,n=0}^{\infty} \frac{\prod_{p=1}^{g} (a_p)_{m+n} \prod_{p=1}^{h} (b_p)_m \prod_{p=1}^{H} (B_p)_n}{\prod_{p=1}^{j} (d_p)_{m+n} \prod_{p=1}^{k} (c_p)_m \prod_{p=1}^{K} (C_p)_n} \frac{x^m}{m!} \frac{y^n}{n!} \qquad \ldots(5.3)$$

is a generalised hypergeometric function of two variables [2, p. 112]. In (5.2), replacing t by tk_1 and ρ by ρk_2; multiplying both sides by $e^{-k_1-k_2} k_1^{a_1+b_1-1} k_2^{a_2+b_2-1}$ and integrating with respect to k_1 and k_2 from 0 to ∞, we get [5, p. 19, Eq. (6.1)].

(ii) If (4.2) is applied to the generating function (5.1), we get

$$\sum_{r,s=0}^{\infty} \frac{(a_1)_r (c_1)_r (a_2)_s (c_2)_s}{(a_1+b_1)_r (c_1+d_1)_r (a_2+b_2)_s (c_2+d_2)_s} \frac{[t(1-x-y)]^r}{r!} \frac{[\rho(1-x-y)]^s}{s!}$$

$$.F\begin{matrix} 2:0;0 \\ 0:1;3 \end{matrix}\left[\begin{matrix} a_1+r, c_1+r : -; -; \\ -: a_1; f_1; a_1+b_1+r, c_1+d_1+r; \end{matrix} \frac{y_1}{y_1-1}, \frac{-xt}{1-y_1}\right]$$

$$.F\begin{matrix} 2:0;0 \\ 0:1;3 \end{matrix}\left[\begin{matrix} a_2+s, c_2+s : -; -; \\ -: a_2; f_2, a_2+b_2+s, c_2+d_2+s; \end{matrix} \frac{y_2}{y_2-1}, \frac{-y\rho}{1-y_2}\right]$$

$$=(1-y_1)^{c_1}(1-y_2)^{c_2} \sum_{m,n=0}^{\infty} \frac{(a_1)_m (c_1)_m (a_2)_n (c_2)_n t^m \rho^n}{(a_1+b_1)_m (c_1+d_1)_m (a_2+b_2)_n (c_2+d_2)_n (1+e-f_1-f_2)_{m+n}}$$

$$. {}_2F_1(-m, c_1+m; a_1; y_1)\, {}_1F_2(-n, c_2+n; a_2; y_2)\, J_{m\,n}(e, f_1, f_2; x, y) \quad .. (5.4)$$

Replacing t by $tk_1 k_2$ and ρ bp $\rho\, k_3\, k_4$, multiplying both sides by

$$e^{-k_1-k_2-k_3-k_4} k^{a_1+b_2-1} k^{a_2+b_2-1} k^{c_1+d_1-1}\ k^{c_2+d_2-1},$$

integrating with respect to k_1, k_2, k_3 and k_4 from 0 to ∞, we obtain [5, p. 19, Eq. (6.2)].

The results (4.1) and (4.2) suggest the possibility of their extension to any finite number of variables.

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