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(*Dedicated to the memory of Professor Arthur Erdélyi*)

SOME INEQUALITIES FOR THE APPELL FUNCTIONS F_1 AND F_2

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Inequalities for Appell's functions F_1 and F_2 are obtained for positive real arguments and certain specified ranges of parameters, applying the results of T. M. Flett [6] and B. C. Carlson [2]. Though the inequalities obtained are not very sharp, they serve a purpose, filling certain gaps in the literature on the inequalities for these functions.

1. Introduction

One-sided inequalities for particular values of parameters have been obtained for F_2 and F_3 for complex and real arguments, respectively by Minton [8], and for F_2 in complex arguments by Erber [3], from an analytic stand point. Luke [7] has obtained some two-sided inequalities for F_1, F_2 and F_3 for special ranges of parameters and negative real arguments from a computational viewpoint. These results, by suitable application of transformation theory, lead to inequalities for rsstricted ranges of positive real arguments , *viz* $0 < x < \frac{1}{2}$, $0 < y < \frac{1}{2}$ for F_1, and $0 < x+y < \frac{1}{2}$ for F_2, the corresponding restrictions on parameters being understood.

In this note we shall obtain two-sided inequalities for F_1 and F_2

for positive real arguments by appealing to the theorems of flett [6] and Carlson [2]. These inequalities, though not very sharp, may yet serve a purpose, filling certain gaps in the literature on inequalities of these functions.

We recall a theorem of Flett [6] that he obtained in estimating the Euler integral of the Gauss hypergeometric function as follows :

Theorem 1. (a) *Let* $a > c > b > 0$, $0 < x < 1$. *Then*

$$\frac{\Gamma(a+b-c)\,\Gamma(c)}{\Gamma(a)\,\Gamma(b)}(1-x)^{c-a-b} < {}_2F_1(a, b; c; x) < (1-x)^{c-a-b}. \quad \text{..(1)}$$

(b) *If, however,* $c>a>c-b>0$, $b>0$, $0<x<1$, *then*

$$(1-x)^{c-a-b} < {}_2F_1(a, b; c; x) < \frac{\Gamma(a+b-c)\,\Gamma(c)}{\Gamma(a)\,\Gamma(b)}(1-x)^{c-a-b}. \quad \text{...(2)}$$

Another theorem that is required is that of Carlson [2, Theorem 2] which can be rewritten as :

Theorem 2. *Let* $a>c+1>0$, $c>b>0$, $-\infty<x<1$, $x\neq 0$. *Then*

$$(1-x)^{c-a-b}(1-x+bx/c)^{a-c} < {}_2F_1(a, b; c; x) < (b/c)(1-x)^{c-a-b} + (1-b/c)(1-x)^{-b}. \quad \text{...(3)}$$

2. Inequalities for F_1

The function F_1, defined by

$$F_1 = F_1(a; b_1, b_2; c; x, y=) \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b_1)_m (b_2)_n (x)^m (y)^n}{(c)_{m+n}\, m!\, n!},$$

where $|x| < 1$, $|y| < 1$, and c is not a negative integer or zero, can be expressed as

$$F_1 = \sum_{m=0}^{\infty} \frac{(a)_m (b_1)_m (x)^m}{(c)_m\, m!}\, {}_2F_1(a+m, b_2; c+m; y). \quad \text{..(5)}$$

With the restrictions $c > a > c-b_2 > 0$, $b_2 > 0$, $0 < y < 1$, an application of Theorem 1 (b) to the inner ${}_2F_1$ of the right-hand of (5) gives

$$(1-y)^{c-a-b_2}\, {}_2F_1(a, b_1; c; x) < F_1 < \frac{\Gamma(a+b_2-c)\,\Gamma(c)}{\Gamma(a)\,\Gamma(b_2)}(1-y)^{c-a-b_2}(1-x)^{-b_1} \quad \text{...(6)}$$

Setting again $c > a > c-b_1 > 0, b_1 > 0, 0 < x < 1$, a reapplication of the same theorem to (6) yields :

$$(1-x)^{c-a-b_1}(1-y)^{c-a-b_2} < F_1 < \frac{\Gamma(a+b_2-c)\,\Gamma(c)}{\Gamma(a)\,\Gamma(b_2)}(1-x)^{-b_1}(1-y)^{c-a-b_2};$$

whenever

$$c > a > c-b_1 > 0,\ c > a > c-b_2 > 0,\ b_1 > 0,\ b_2 > 0,\ 0 < x < 1,\ 0 < y < 1. \quad \ldots(7)$$

By an appeal to the symmetry property of F_1, wherein the roles of b_1 and x are changed by b_2 and y, respectively, and *vice versa*, (7) will admit the form

$$(1-x)^{c-a-b_1}(1-y)^{c-a-b_2} < F_1 < \frac{\Gamma(a+b_1-c)\,\Gamma(c)}{\Gamma(a)\,\Gamma(b_1)}(1-y)^{-b_2}(1-x)^{c-a-b_1}, \quad \ldots(8)$$

with the same set of conditions as before.

Equations (7) and (8), when combined, will give :

Teorem 3. (a). *If* $c > a > c-b_1 > 0,\ c > a > c-b_2 > 0,\ b_1 > 0,\ b_2 > 0,\ 0 < x < 1, 0 < y < 1$, *then*

$$(1-x)^{c-a-b_1}(1-y)^{c-a-b_2} < F_1$$

$$< \frac{\Gamma(c)}{\Gamma(a)}\min\left\{\begin{array}{l}\dfrac{\Gamma(a+b_1-c)}{\Gamma(b_1)}(1-y)^{-b_2}(1-x)^{c-a-b_1},\\[2ex] \dfrac{\Gamma(a+b_2-c)}{\Gamma(b_2)}(1-x)^{-b_1}(1-y)^{c-a-b_2}.\end{array}\right. \quad \ldots(9)$$

On the other hand, proceeding as before, but using Theorem 1 (a), we have :

Theorem 3 (b). *If* $a > c > b_1 > 0,\ a > c > b_2 > 0,\ 0 < x < 1,\ 0 < y < 1$, *then*

$$\frac{\Gamma(c)}{\Gamma(a)}\max\left\{\begin{array}{l}\dfrac{\Gamma(a+b_1-c)}{\Gamma(b_1)}(1-y)^{-b_2}(1-x)^{c-a-b_1},\\[2ex] \dfrac{\Gamma(a+b_2-c)}{\Gamma(b_2)}(1-x)^{-b_1}(1-y)^{c-a-b_2},\end{array}\right.$$

$$< F_1 < (1-x)^{c-a-b_1}(1-y)^{c-a-b_2}. \quad \ldots(10)$$

Yet another theorem for positive real arguments would be obtained, in the same fashion, by recourse to Theorem 2. Indeed, an application of Theorem 2 to Equation (5) leads to

$$\sum_{m=0}^{\infty} \frac{(a)_m (b_1)_m (x)^m}{(c)_m\, m\,!} \left(1-y+\frac{b_2\, y}{c+m}\right)^{a-c} (1-y)^{c-a-b_2} < F_1$$

$$< \sum_{m=0}^{\infty} \frac{(a)_m (b_1)_m (x)^m}{(c)_m\, m\,!} \left\{ \frac{b_2}{c+m} (1-y)^{c-a-b_2} + \left(1-\frac{b_2}{c+m}\right) (1-y)^{-b_2} \right\}$$

whenever

$$a > c+1 > 0,\ c > b_2 > 0,\ -\infty < y < 1,\ y \neq 0, \qquad \ldots(11)$$

and once again, as a consequence of Theorem 2, we have

$$(1-x)^{c-a-b_1} (1-y)^{-b_2} (1-x+(b_1 x/c))^{a-c} < F_1 < (1-y)^{b_2}$$

$$[1+(b_2/c)(1-y)^{c-a}]\ [(b_1/c)(1-x)^{c-a-b_1} + \{1-(b_1/c)\}(1-x)^{-b_1}],$$

$$a > c+1 > 0,\ c > b_1 > 0,\ c > b_2 > 0,\ |x| < 1,\ |y| < 1,\ x \neq 0,\ y \neq 0, \qquad \ldots(12)$$

The symmetry in F_1 enables us to express Equation (12) in an alternative form which, in combination with (12), finally gives :

Theorem 4.

$$\max \begin{cases} (1-x)^{c-a-b_1} (1-y)^{-b_2} [1-x+(b_1 x/c)]^{a-c}, \\ (1-x)^{-b_1} [1-y+(b_2 y/c)]^{a-c} (1-y)^{c-a-b_2}, \end{cases}$$

$$< F_1 < \min \begin{cases} (1-y)^{-b_2}[1+(b_2/c)(1-y)^{c-a}]\ [(b_1/c)(1-x)^{c-a-b_1} + (1-(b_1/c))(1-x)^{-b_1}], \\ (1-x)^{-b_1}[1+(b_1/c)(1-x)^{c-a}][(b_2/c)(1-y)^{c-a-b_2} + \{1-(b_2/c)\}(1-y)^{-b_2}], \end{cases}$$

$$a > c+1 > 0,\ c > b_1 > 0,\ c > b_2 > 0,\ |x| < 1,\ |y| < 1,\ x \neq 0,\ y \neq 0. \qquad .(13)$$

For the set of values

$a=0.5$, $b_1=0.4$), $b_2=0.6$, $c=0.8$, $x=0.2$, $y=0.7$

from Theorem 3 (a) we have

$$1.467421 < F_1 < 2.07033, \qquad \ldots(14)$$

and for

$a=0.8,\ b_1=0.2,\ b_2=0.4,\ c=0.7,\ x=0.2,\ y=0.5,$

from Theorem 3 (b)

$$1.317449 < F_1 < 1.512126. \qquad \ldots(15)$$

Computations show that Theorem 4 gives sharper inequalities than Theorem 3 (b) in their common range of validity. In particular, for the choice

$a=1.8,\ b_1=0.4,\ b_2=0.2,\ c=0.7,\ x=0.5,\ y=0.2,$

we obtain from Theorem 4

$$2.268392 < F_1 < 2.554081,$$

while from Theorem 3 (b) we have

$$1.646811 < F_1 < 3.780315. \qquad \ldots(17)$$

3. Inequalities for F_2

We start with the single series representation

$$F_2 \equiv F_2\,(a;\, b_1,\, b_2;\, c_1,\, c_2;\, x,\, y)$$

$$\sum_{m=0}^{\infty} \frac{(a)_m\,(b_1)_m\,(x)^m}{(c_1)_m\, m\,!}\ {}_2F_1\,(a+m,\, b_2;\, c_2;\, y),$$

where $|x| + |y| < 1$, and neither c_1 nor c_2 is a negative integer or zero. ...(18)

Under the restrictions

$a > a+b_2 - c_2 > c_1 > b_1 > 0,\ a > c_2 > b_2 > 0,$

$x > 0,\ y > 0,\ x+y < 1,$ ...(19)

by repeated applications of Theorem 1 (b) in (18), we obtain

$$\frac{\Gamma(a+b_1+b_2-c_1-c_2)\ \Gamma(c_1)\ \Gamma(c_2)}{\Gamma(a)\ \Gamma(b_1)\ \Gamma(b_2)}\ (1-y)^{b_1-c_1}\ (1-x-y)^{c_1+c_2-a-b_1-b_2}$$

$$< F_2 < (1-y)^{c_2-c_1+b_1-b_2}\ (1-x-y)^{c_1-a-b_1}. \qquad \ldots(20)$$

Since

$$F_2\,(a;\, b_1,\, b_2;\, c_1,\, c_2;\, x;\, y) = F_2\,(a,\, b_2,\, b_1;\, c_2,\, c_1;\, y,\, x), \qquad \ldots(21)$$

it is possible to write the above inequality in a different form which, in conjunction with (20), will give :

Theorem 5. *Let* $a > a+b_2-c_2 > c_1 > b_1 > 0$, $a > a+b_1-c_1 > c_2 > b_2 > 0$, $x > 0, y > 0, x+y < 1$. *Then*

$$\frac{\Gamma(a+b_1+b_2-c_1-c_2)\,\Gamma(c_1)\,\Gamma(c_2)}{\Gamma(a)\,\Gamma(b_1)\,\Gamma(b_2)}(1-x-y)^{c_1+c_2-a-b_1-b_2}\cdot$$

$$max\left\{(1-y)^{b_1-c_1}, (1-x)^{b_2-c_2}\right\} < F_2$$

$$< min\left\{(1-y)^{c_2-c_1+b_1-b_2}(1-x-y)^{c_1-a-b_1}, (1-x)^{c_1-c_2+b_2-b_1}(1-x-y)^{c_2-a-b_2}\right\}. \quad \ldots (22)$$

Though with the choice $a>c_1+1$, $a>c_2+1$, Theorem 5 holds good but a better inequality could be obtaind by appropriate application of Theorem 2 in (18). Thus, under the conditions

$$a > max\ \{c_1+1, c_2+1\}, c_1 > b_1 > 0, c_2 > b_2 > 0,$$
$$|x| + |y| < 1,\ x \neq 0, y \neq 0, \quad \ldots (23)$$

we have

$$(1-y)^{c_2-a+b_1-b_2}\left(1-y+\frac{b_2 y}{c_2}\right)^{a-c_2}\left(1-y+x\left(1-y+\frac{b_2 y}{c_2}\right)\right)^{c_1-a-b_1}$$

$$\left(1-y-x\left(1-\frac{b_1}{c_1}\right)\left(1-y+\frac{b_2 y}{c_2}\right)\right)^{a-c_1} < F_2$$

$$< \frac{b_2}{c_2}(1-y)^{c_2-a-b_2}\left[\frac{b_1}{c_1}\left(1-\frac{x}{1-y}\right)^{c_1-a-b_1}+\left(1-\frac{b_1}{c_1}\right)\left(1-\frac{x}{1-y}\right)^{-b_1}\right]$$

$$+\left(1-\frac{b_2}{c_2}\right)(1-y)^{-b_2}\left[\frac{b_1}{c_1}(1-x)^{c_1-a-b_1}+\left(1-\frac{b_1}{c_1}\right)(1-x)^{-b_1}\right]. \quad \ldots (24)$$

As a consequence of (21), it then follows that

$$(1-x)^{c_1-a+b_2-b_1}\left(1-x+\frac{b_1 x}{c_1}\right)^{a-c_1}\left[1-x+y\left(1-x+\frac{b_1 x}{c_1}\right)\right]^{c_2-a-b_2}$$

$$\left[1-x-y\left(1-\frac{b_2}{c_2}\right)\left(1-x+\frac{b_1 x}{c_1}\right)\right]^{a-c_2} < F_2$$

$$< \frac{b_1}{c_1}(1-x)^{c_1-a-b_1}\left[\frac{b_2}{c_2}\left(1-\frac{y}{1-x}\right)^{c_2-a-b_2}+\left(1-\frac{b_2}{c_2}\right)\left(1-\frac{y}{1-x}\right)^{-b_2}\right]$$

$$+\left(1-\frac{b_1}{c_1}\right)(1-x)^{-b_1}\left[\frac{b_2}{c_2}(1-y)^{c_2-a-b_2}+\left(1-\frac{b_2}{c_2}\right)(1-y)^{-b_2}\right], \quad \ldots (25)$$

under the set of conditions (23).

We, therefore, finally have

Theorem 6.

$$
\max\begin{cases}(1-y)^{c_2-a+b_1-b_2}\left(1-y+\frac{b_2 y}{c_2}\right)^{a-c_2}\left[1-y+x\left(1-y+\frac{b_2 y}{c_2}\right)\right]^{c_1-a-b_1} \\ \left[1-y-x\left(1-\frac{b_1}{c_1}\right)\left(1-y+\frac{b_2 y}{c_2}\right)\right]^{a-c_1}, \\ (1-x)^{c_1-a+b_2-b_1}\left(1-x+\frac{b_1 x}{c_1}\right)^{a-c_1}\left[1-x+y\left(1-x+\frac{b_1 x}{c_1}\right)\right]^{c_2-a-b_2} \\ \left[1-x-y\left(1-\frac{b_2}{c_2}\right)\left(1-x+\frac{b_1}{c_1}\right)\right]^{a-c_2}\end{cases}
$$

$$
< F_2
$$

$$
< \min\begin{cases}\frac{b_2}{c_2}(1-y)^{c_2-a-b_2}\left[\frac{b_1}{c_1}\left(1-\frac{x}{1-y}\right)^{c_1-a-b_1}+\left(1-\frac{b_1}{c_1}\right)\left(1-\frac{x}{1-y}\right)^{-b_1}\right] \\ +\left(1-\frac{b_2}{c_2}\right)(1-y)^{-b_2}\left[\frac{b_1}{c_1}(1-x)^{c_1-a-b_1}+\left(1-\frac{b_1}{c_1}\right)(1-x)^{-b_1}\right], \\ \frac{b_1}{c_1}(1-x)^{c_1-a-b_1}\left[\frac{b_2}{c_2}\left(1-\frac{y}{1-x}\right)^{c_2-a-b_2}+\left(1-\frac{b_2}{c_2}\right)\left(1-\frac{y}{1-x}\right)^{-b_2}\right] \\ +\left(1-\frac{b_1}{c_1}\right)(1-x)^{-b_1}\left[\frac{b_2}{c_2}(1-y)^{c_2-a-b_2}+\left(1-\frac{b_2}{c_2}\right)(1-y)^{-b_2}\right],\end{cases}
$$

provided

$a > \max\{c_1+1, c_2+1\}$, $c_1 > b_1 > 0$, $c_2 > b_2 > 0$, $|x|+|y|<1$,

x and y are real, $x \neq 0$, $y \neq 0$. ...(26)

In conformity with the aforementioned remark, we note as a comparison that for the set of values

$a=2$, $b_1=0.4$, $b_2=0.5$, $c_1=0.6$, $c_2=0.8$, $x=0.2$, $y=0.3$,

Theorem 5 gives

$$1.187113 < F_2 < 3.322324 \qquad (27)$$

and Theorem 6 gives

$$2.377182 < F_2 < 2.447001 \qquad \ldots(28)$$

Inequalities for F_2 could also be obtained using a result due to Buschman [1], but such inequalities will be too cumbersome to deal with from a computational standpoint.

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