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(Dedicated to the memory of Professor Arthur Erdélyi)

ON A UNIFICATION OF THE GENERALIZED HUMBERT AND LAGUERRE POLYNOMIALS

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ABSTRACT

The object of this paper is to initiate a systematic study of a new class of polynomials which unifies (as well as provides extensions of) several known sysstems of polynomials which are embraced by the generalized Humbert polynomials of H. W. Gould [3] and the recent generalization of Laguerre, Jacobi, Hermite, and other polynomials that was introduced into analysis by R. Panda [5] (see also [1], [4] and [7]).

1. INTRODUCTION

Recently, having been motivated by the earlier works of Rainville [7, p. 137, Theorem 48], Chandel ([1], [2]) and Jain [4], Rekha Panda [5] introduced an elegant generalization of several known polynomial systems belonging to (or providing extensions of) the families of the classical Jacobi Hermite and Laguerre polynomials poynomials by means of the generating relation

$$(1.1)\qquad (1-t)^{-c}\ G\left[\frac{xt^{s}}{(1-t)^{r}}\right] = \sum_{n=0}^{\infty} g^{c}_{n}\,(x, r, s)\, t^{n},$$

where

$$G[z] = \sum_{n=0}^{\infty} \gamma_n z^n, \quad \gamma_0 \neq 0, \tag{1.2}$$

c is an arbitrary complex number, r is any integer, positive or negative, and $s=1, 2, 3$ . .

A comparison of (1.1) with the generating function [3, p. 697]

$$(C-mxt+yt^m)^p = \sum_{n=0}^{\infty} P_n(m, x, y, p, C)\, t^n \tag{1.3}$$

for the generalized Humbert polynomials of Gould [3] suggests that it would be interesting and worthwhile to study a new class of polynomials $\{ f^c_n(x, y, r, m) \mid n=0, 1, 2, \ldots \}$ defined by the generating relation

$$(1+yt^m)^{-c}\, G\left[\frac{xt}{(1+yt^m)^r}\right] = \sum_{n=0}^{\infty} f_n^c(x, y, r, m)\, t^n, \tag{1.4}$$

where, as in (1.1) and (1.2),

$$G[z] = \sum_{n=0}^{\infty} \gamma_n z^n, \quad \gamma_0 \neq 0, \tag{1.4}$$

$m \geqslant 1$ is an integer and other parameters are unrestricted in general.

From (1.4) and (1.5) it is easy to deduce that $f_n^a(x, y, r, m)$ is a a polynomial of degree n in x with its explicit representation as

$$f_n^c(x, y, r, m) = \sum_{k=0}^{[n/m]} \binom{-c-nr+mrk}{k} y^k \gamma_{n-mk} x^{n-mk}. \tag{1 6}$$

On choosing γ_n appropriately and restricting r to integral values, the right member of (1.6) may also be put in hypergeometric form.

It is obvious that when $y = -1$, and $m=1$, (1·4) would correspon d to the special case $s=1$ of (1.1), whereas on taking $\gamma_n = (-m)^n \binom{p}{n}$, $c = -p$

and $r=1$, (1.4) would transform into (1.3) with $C=1$. Thus the class of polynomials $\{ f_n^c (x, y, r, m) \mid n=0, 1, 2, \ldots \}$ defined by (1.4) and (1.5) provides an interesting unification as well as generalization of the various polynomials included $g_n^c (x, r, 1)$ and the generalized Humbert polynomial $P_n (m, x, y, p, 1)$ which it self is a generalization of several known polynomials including those of Legendre, Gegenbauer, Humbert, Tchebycheff, Princherle, and many others. For the different conditions on the parameters of $f_n^c (x, y, r, m)$ under which it reduces to the polynomials mentioned above and many others, e. g., the polynomials of Sister Celine, Jacobi, Rice reference may be made to [3], [4] and [5].

Being motivated by the observations mentioned in the above paragraphs, we initiate with this paper a systematic study of ihe polynomials $f_n^c (x, y, r, m)$. In Section 2 of this paper we derive a generating function for $f_n^{c+\sigma n} (x, y, r, m)$ and discuss its various particular cases and their applications in the derivations of some expansion formulae. Section 3 incorporates a number of recurrence relations for $f_n^c (x, y, r, m)$. In Section 4 we give some additional results for the particular case $\gamma_n = 1/n\,!$ of $f_n^c (x, y, r, m)$ which has been denoted by $\Gamma_n^c (x, y, r, m)$.

In what follows, for the sake of brevity, we shall abbreviate $f_n^c (x, y, r, m)$ by $f_n^c (x)$ and $\Gamma^c_n (x, y, r, m)$ by $\Gamma_n^c (x)$ unless there is any ambiguity regarding other parameters.

2. Generating Function for $f_n^{c+\sigma n} (x)$

The generating function that we propose to derive in this section is

$$(2.1) \qquad \sum_{n=0}^{\infty} f_n^{c+\sigma n} (x)\; t^n = \frac{(1+y w^m)^{1-c}}{1+y(1+\sigma m) w^m}\; G\left[\frac{xw}{(1+y w^m)^r} \right],$$

where

$$(2.2) \qquad w = t\,(1+y\,w^m)^{-\sigma}, \; w\,(0) = 0,$$

σ is an arbitrary complex number and γ_n, the coefficient of z^n in the power series for $G\,[z]$, is independent of c.

To prove (2.1), we start with the function

$$(2.3)\qquad F(A_n, y, r, m, c) = \sum_{k=0}^{[n/m]} \binom{-c-nr+mrk}{k} y^k A_{n-mk}$$

where $\{A_n\}$ is an arbitrary sequence such that $\sum_{n=0}^{\infty} |A_n| < \infty$, $m \geqslant 1$ is an integer and other parameters are unrestricted in general.

For arbitrary complex values of σ, it is easy to see that

$$(2.4)\qquad \sum_{n=0}^{\infty} F(A_n, y, r, m, c+\sigma n)\, t^n = \sum_{n=0}^{\infty} A_n\, t^n \sum_{k=0}^{\infty} \binom{-c-\sigma n-rn-\sigma mk}{k} (yt^m)^k.$$

On summing the inner series on the right hand side of (2.4) with the help of the following consequence of Lagrange's expansion formula [6, p. 302, Problem 216] :

$$(2.5)\qquad \sum_{n=0}^{\infty} \binom{a+bn}{n} t^n = (1+v)^{a+1} [\, 1-(b-1)\, v\,]^{-1}, \quad v=t\,(1+v)^b,$$

we get after a little simplification the general formula

$$(2.6)\qquad \sum_{n=0}^{\infty} F(A_n, y, r, m, c+\sigma n)\, t^n = \frac{(1+yw^m)^{1-c}}{1+y\,(1+m)w^m} \sum_{n=0}^{\infty} \left[\frac{w}{(1+yw^m)^r} \right]^n A_n,$$

where w is given by (2.2).

In (2.6) if we take $A_n = \gamma_n x^n$ and then make use of (1 6) and (1.5) we are immediately led to the generating function (2 1).

Alternatively, we may start with the following consequence of the generating function (1.4) :

$$(2.7)\qquad f_n^{c}(x) = \sum_{k=0}^{[n/m]} \frac{(c-b)_k\, (-y)^k}{k!}\, f^{b}_{n-mk}(x),$$

and then arrive at (2.1) by using the method illustrated by Singhai [8]. Yet another method of proving (2.1) would run parallel to that of Rekha Srivastava [9] which she employed for deriving a corresponding generating function for $g^{c+\sigma n}(x, r, s)$ wherein by putting $s=1$ we shall get the case $m=1, y=-1$ of our result (2.1).

For $\sigma=0$, (2·1) evidently reduces to (1.4), whereas the substitution $\sigma=-1/m$ transforms it to

$$(2.8)\qquad \sum_{n=0}^{\infty} f_n^{c-n/m}(x)\, t^n = (1-yt^m)^{c-1}\; G\left[xt\,(1-yt^m)^{r-1/m}\right].$$

On the other hand by putting $\sigma=-2/m$ we shall get

$$(2.9)\qquad \sum_{n=0}^{\infty} f_n^{c-2n/m}(x)\, t^n$$

$$=(1-4yt^m)^{-\frac{1}{2}}\left(\frac{1+\sqrt{1-4yt^m}}{2}\right)^{c} G\left[x\,t\left(\frac{1+\sqrt{1-4yt^m}}{2}\right)^{r-2/m}\right].$$

Various other Particular cases of (2.1) can be given by particularizing $f_n^c(x)$ and assigning different values to σ.

Finally, it is to be mentioned here that the particular case (2.8) of (2.1), when expressed in the form

$$(2.10)\qquad \sum_{n=0}^{\infty} f_n^{b-n/m}(x)\,(1-yt^m)^{c-b}\,t^n$$

$$=(1-y\,t^m)^{c-1}\; G\left[xt\,(1-y\,t^m)^{r-1/m}\right],$$

yields the expansion formula

$$(2.11)\qquad f_n^{c-n/m}(x)=\sum_{k=0}^{[n/m]} \frac{(b-c)_k}{k\,!}\; y^k\, f_{n-mk}^{\,b+k-n/m}(x),$$

which is analogous to the similar consequence of (1.4) given by equation (2.7) above.

3. Recurrence Relations

If we denote the left member of (1.4) by $U(x, t)$, then it is readily seen that $U(x, t)$ satisfies the differential equation function for $g^{c+\sigma n}(x, r, s)$ wherein by putting $s=1$ we shall get the case $m=1, y=-1$ of our result (2.1).

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which is analogous to the similar consequence of (1.4) given by equation (2.7) above.

3. Recurrence Relations

If we denote the left member of (1.4) by $U(x, t)$, then it is readily seen that $U(x, t)$ satisfies the differential equation

(4.3) $$\Gamma_{n-1}^{c+s}(x, y, r+s, m)=\sum_{k=0}^{[n/m]} \frac{(-r)_k\,(-y)^k}{k!}\, D_x \left\{ \Gamma_{n-mk}^{c}(x, y, r+s, m)\right\},$$

(4.4) $$D_x^k \left\{ \Gamma_n^c(x) \right\} = \Gamma_{n-k}^{c+rk}(x);$$

(4.5) $$n\,\Gamma_n^c(x) + cmy\,\Gamma_{n-m}^{c+1}(x) - x\,\Gamma_{n-1}^{c+r}(x) + xymr\,\Gamma_{n-1}^{c+r+1}(x) = 0,\quad n \geqslant m \geqslant 1,$$

(4.6) $$\Gamma_n^{c_1+\ldots+c_k}(x_1+\ldots+x_k) = \sum_{i_1+\ldots+i_k=n} \Gamma_{i_1}^{c_1}(x_1)\ldots\Gamma_{i_k}^{c_k}(x_k),$$

(4.7) $$\Gamma_n^{b_1+\ldots+b_k+1-k-n/m}(x_1+\ldots+x_k) = \sum_{i_1+\ldots+i_k=n} \Gamma_{i_1}^{b_1-i_1/m}(x_1)\ldots\Gamma_{i_k}^{b_k-i_k/m}(x_k),$$

(4.8) $$(n+1)\,\Gamma_{n+1}^{c}(x) + y\,(n-m+cm+1)\,\Gamma_{n-m+1}^{c}(x) = x\sum_{k=0}^{[n/m]} \frac{(r)_k\,(-y)^k}{k!}\,\Gamma_{n-mk}^{c}(x) + (1-rm)\,xy \sum_{k=0}^{[n/m]-1} \frac{(r)_k\,(-y)^k}{k!}\,\Gamma_{n-mk-m}^{c}(x),\quad n \geqslant m.$$

Lastly, we give the mixed generating function

(4.9) $$\sum_{n=0}^{\infty} \Gamma_n^{c+\sigma n}(x+nz, y, r, m)\, t^n = \frac{v^{-c}\exp(xu/v^r)}{1-u\left[-\sigma myu^{m-1}v^{-1}+zv^{-r}\{1-mryu^m v^{-1}\}\right]},$$

where u and v are given by

(4.10) $u = tv^{-\sigma} \exp(uz/v^r)$, $v = (1 + yu^m)$.

Our formula (4.9) is analogous to the mixed generating function for $g^c_n(x, r, s)$ given earlier by Rekha Srivastava [9]. The proof of (4.9) would also run parallel to the proof given in [9]; we, therefore, omit the details.

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