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(*Dedicated to the memory of Professor Arthur Erdélyi*)

USE OF ROUTING ALGORITHM TO OBTAIN APPROXIMATE SOLUTIONS OF VARIATIONAL PROBLEMS

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I. Introduction

The purpose of this paper is to show how the algorirhm of the routing problems may be used to obtain approximate solutions of variational problems.

In Sections 2 we give a difficult variational problem and explain why the Euler-Lagrange equation cannot be used directly. In Section 3, we show how the routing algorithm can be applied. In Section 4, we show how this could be modified to include the free boundary problem. Finally, in Section 5, we make some remarks about how a smooth approximate can be obtained.

2. An Example

Let us consider the functional

$$J(u) = \int_0^{10} (u'^2 + t^2 u^4)\, dt$$

Let us assume that u is subject to the constraint $u(0) = u(10) = \frac{1}{2}$.

Elementary use of functional analysis establishes the existance of the solution of a minimum. This minimization function satisfies the Euler-Lagrange equation $u'' - 2t^2 u^3 = 0$, $u(0) = u(10) = \frac{1}{2}$.

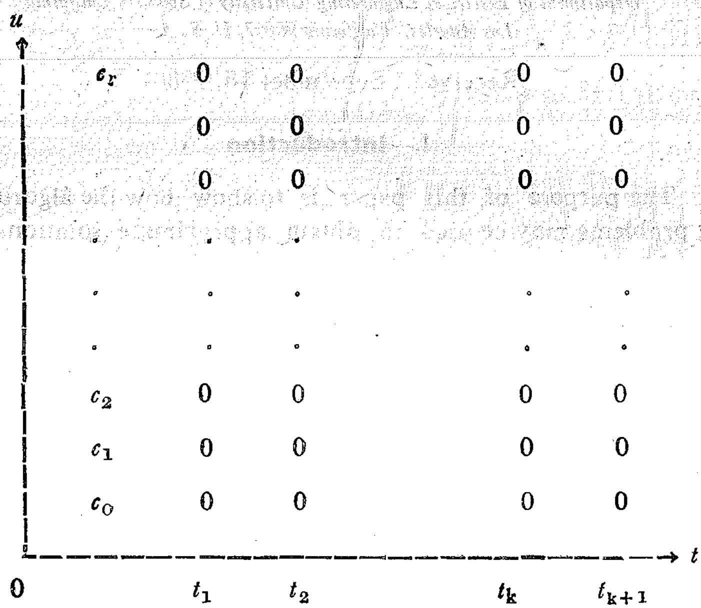
This is a non-linear differential equation subject to a two-point boundary condition. It must be solved by a method of successive approximation. Because of the length of the interval, this method may not converge.

3. Use of the Routing Algorithm

Let us add the constraint $0 \leq u \leq 1$. We know then that the desired solution lies in the rectangle $0 \leq u \leq 1$, $0 \leq t \leq 10$.

In general, either from mathematical or physical considerations, we can determine the region in which the solution must lie. Let us take this rectangle and take cross-sections at t interval of 1/10. Along each cross-section, let us take a gride of 100 points.

The number of points we take depends upon the accuracy we want and the time we are willing to devote to the problem. See the figure below.



The nodes are now the points on the cross-sections. The "time" to go from the i-th node from ohe cross-section to the j-th node on the next cross-section is given by

$$t_{ij} = \min_{u} J\,(t_k,\ t_{k+1},\ c_i,\ c_j,\ u) = \min \int_{t_k}^{t_{k+1}} (\,u^2 + t^2\ u^4)\ dt,$$

where $\quad u\,(t_k) = c_i,\ \ u\,(t_{k+1}) = c_j\,.$

From each point there are 100 possible points to go to. However in reality there are not. We can bound the slope of the solution a priori or we can ask for a smooth approximation.

These are the generalized times. The t_{ij}'s may be calculated ahead of time or as we need them. To calculate them we use a method of successive approximation. A good initial approximation will just be the straight line connecting the two modes.

Once we have calculated the t_{ij}'s, we can use the routing algorithm to determine the approximation.

4. Free boundary Problem

Let us now assume that the boundary conditions are $u\,(0)=\frac{1}{2}$, $u'\,(10)=0$. At the final step we impose the last condition to determine the t_{ij}. A good initial approximation is now a line parallel to the t-axis.

5. Conclusion

The approximation obtained in this way may be smoothed in various ways. Alternatively, it can be used as an initial approximation for another method.