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(*Dedicated to the memory of Professor Arthur Erdélyi*)

A NOTE ON A GENERATING RELATION FOR THE MULTIVARIABLE H-FUNCTION

By

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ABSTRACT

The aim of the present note is to illustrate the application of the differential operator $T_k = x\,(k+xD)$, $D \equiv d/dx$, in obtaining generating relations of a general type. Thus we have derived here an interesting known generating relation for the multivariate H-function introduced by H. M. Srivastava and Rekha Panda [4]. The corresponding generating relation for the H-function of two variables, which is also quite general in nature, follows as its particular case.

I. Introduction

Srivastava and Panda [5] defined the H-function of several complex variables $z_1, z_2, \ldots, z_r$ by means of the following multiple contour integral [4, p. 121, Eq. (1.10)]

$$H^{0,\lambda:(\mu',\nu');\ldots;(\mu^{(r)},\nu^{(r)})}_{A,C,:[B',D'];\ldots;[B^{(r)},D^{(r)}]}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(r)}]:[(b'):\phi'];\ldots;[(b^{(r)}):\phi^{(r)}];\\ [(c):\psi',\ldots,\psi^{(r)}]:[(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}];\end{matrix}\; z_1,\ldots,z_r\right)$$

$$=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\Phi_1(\zeta_1)\ldots\Phi_r(\zeta_r)\,\Psi(\zeta_1,\ldots,\zeta_r)\,z_1^{\zeta_1}\ldots z_r^{\zeta_r}\,d\zeta_1\ldots d\zeta_r,\quad \omega=\sqrt{-1}, \qquad \ldots(1.1)$$

where

$$\Phi_i(\zeta_i)=\frac{\prod_{j=1}^{\mu^{(i)}}\Gamma[d_j^{(i)}-\delta_j^{(i)}\zeta_i]\prod_{j=1}^{\nu^{(i)}}\Gamma[1-b_j^{(i)}+\phi_j^{(i)}\zeta_i]}{\prod_{j=\mu^{(i)}+1}^{D^{(i)}}\Gamma[1-d_j^{(i)}+\delta_j^{(i)}\zeta_i]\prod_{j=\nu^{(i)}+1}^{B^{(i)}}\Gamma[b_j^{(i)}-\phi_j^{(i)}\zeta_i]}$$

$$i=1,\ldots,r; \qquad \ldots(1.2)$$

$$\Psi(\zeta_1,\ldots,\zeta_r)=\frac{\prod_{j=1}^{\lambda}\Gamma[1-a_j+\sum_{i=1}^{r}\theta_j^{(i)}\zeta_i]}{\prod_{j=\lambda+1}^{A}\Gamma[a_j-\sum_{i=1}^{r}\theta_j^{(i)}\zeta_i]\prod_{j=1}^{C}\Gamma[1-c_j+\sum_{i=1}^{r}\psi_j^{(i)}\zeta_i]} \qquad \ldots(1.3)$$

where the various symbols, notations and conditions on parameters can be referred to in the papers by Srivastava and Panda ([4], [5]). Also, following Srivastava and Panda [*loc. cit.*], we shall use a contracted notation and write the first member of (1.1) in the abbreviated form

$$H^{0,\lambda:(\mu',\nu');\ldots;(\mu^{(r)},\nu^{(r)})}_{A,C:[B',D'];\ldots;[B^{(r)},D^{(r)}]}\begin{pmatrix} z_1\\ \vdots\\ z_r\end{pmatrix} \text{ or } H\begin{pmatrix} z_1\\ \vdots\\ z_r\end{pmatrix}$$

whenever no ambiguity or confusion arises.

The multiple integral in (1.1) converges absolutely if

$$|\arg(z_i)|<\tfrac{1}{2}\pi\,\Delta_i,\; i=1,\ldots,r \qquad \ldots(1.4)$$

where

$$\Delta_i\equiv-\sum_{j=\lambda+1}^{A}\theta_j^{(i)}+\sum_{j=1}^{\nu^{(i)}}\phi_j^{(i)}-\sum_{j=\nu^{(i)}+1}^{B^{(i)}}\phi_j^{(i)}$$

$$-\sum_{j=1}^{C}\psi_j^{(i)}+\sum_{j=1}^{\mu^{(i)}}\delta_j^{(i)}-\sum_{j=\mu^{(i)}+1}^{D^{(i)}}\delta_j^{(i)}>0,\; i=1,\ldots,r.$$

2. Generating relations for the *H*-function of several complex variables.

We shall make use of the following operational formula, which is essentially the same as that given by Mittal [1], in obtaining the known generating relation (2.4) given below.

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} T^n_{\alpha+1+\beta n}\, f(x) = \frac{(1+v)^{\alpha+1}}{1-\beta v}\, f[x(1+v)] \qquad \ldots(2.1)$$

where $v = xz\,(1+v)^{\beta+1}$, β being a constant, $f(x)$ admits a formal power series in x, $T_k = x\,(k+x\,D)$, $D \equiv d/dx$, and $T_k{}^n$ means that the operator T_k is repeated n times. Taking

$$f(x) = H\begin{pmatrix} y_1/x^{\sigma_1} \\ \vdots \\ y_r/x^{\sigma_r} \end{pmatrix}$$

in (2.1) and assuming that T_k operates on x alone, the left-hand side of (2.1) is given by

$$\sum_{n=0}^{\infty} \frac{z^n}{n!}\, T^n_{\alpha+1+\beta n} H\begin{pmatrix} y_1/x^{\sigma_1} \\ \vdots \\ y_r/x^{\sigma_r} \end{pmatrix}$$

$$= \sum_{n=0}^{\infty} \frac{z^n}{n!} \frac{1}{(2\pi\omega)^r} \int_{L_1} \cdots \int_{L_r} \Phi_1(\zeta_1) \cdots \Phi_r(\zeta_r)$$

$$\cdot\, \Psi(\zeta_1, \ldots, \zeta_r)\, y_1^{\zeta_1} \ldots y_r^{\zeta_r}\, T^n_{\alpha+1+\beta n}\left(x^{\sum_{i=1}^{r} \sigma_i \zeta_i}\right) d\zeta_1 \ldots d\zeta_r \qquad \ldots(2.2)$$

The interchange of the order of integration and differentiation is easily seen to be permissible under the conditions stated. Now evaluate

$$T_k^n\, x^\gamma = (k+\gamma)_n\, x^{\gamma+n} \qquad \ldots(2.3)$$

and substitute the values in (2.1) from (2.2) and (2.3). Further, putting $z = t/x$, $y_1 = z_1\, x^{\sigma_1}, \ldots, y_r = z_r\, x^{\sigma_r}$, and interpreting the result by means of (1.1) and (1.3), we obtain the following known generating relation for the multivariate H-function.

$$\sum_{n=0}^{\infty} (-1)^n \frac{t^n}{n!} H^{0,\lambda+1:(\mu',\nu');\dots;(\mu^{(r)},\nu^{(r)})}_{A+1,C+1:[B',D'];\dots;[B^{(r)},D^{(r)}]}\left(\begin{matrix}[1+\alpha+\beta n:\sigma_1,\dots,\sigma_r]:[(a):\theta',\dots,\theta^{(r)}]:[(b'):\phi'];\dots;[(b^{(r)}):\phi^{(r)}];\\ [1+\alpha+\beta n+n:\sigma_1,\dots,\sigma_r]:[(c):\psi',\dots,\psi^{(r)}]:[(d'):\delta'];\dots;[(d^{(r)}):\delta^{(r)}];\end{matrix} z_1,\dots,z_r\right)$$

$$=\frac{(1+v)^{\alpha+1}}{1-\beta v} H^{0,\lambda:(\mu',\nu');\dots;(\mu^{(r)},\nu^{(r)})}_{A,C:[B',D'];\dots;[B^{(r)},D^{(r)}]}\left(\begin{matrix} z_1/(1+v)^{\sigma_1}\\ \vdots \\ z_r/(1+v)^{\sigma_r}\end{matrix}\right) \quad \dots(2.4)$$

where $v=t\,(1+v)^{\beta+1}$, $\sigma_i>0$, $i=1,\dots,r$ and the conditions (1.4) and (1.5) are also satisfied.

Particular cases

(1) For $r=2$, we have the following generating relation for the H function of two variables

$$\sum_{n=0}^{\infty} (-1)^n \frac{t^n}{n!} H^{0,n_1+1:m_2,n_2;m_3,n_3}_{p_1+1,q_1+1,:p_2,q_2;p_3,q_3}\left(\begin{matrix}(1+\alpha+\beta n;\sigma_1,\sigma_2);(a_j;\alpha_j,A_j)_{1,p_1}:(c_j:\gamma_j)_{1,p_2};(e_j:E_j)_{1,p_3}\\ (1+\alpha+\beta n+n;\sigma_1,\sigma_2);(b_j;\beta_j,B_j)_{1,q_1}:(d_j:\delta_j)_{1,q_2};(f_j:F_j)_{1,q_3}\end{matrix}\middle|\begin{matrix}x\\ y\end{matrix}\right)$$

$$=\frac{(1+v)^{\alpha+1}}{1-\beta v} H^{0,n_1:m_2,n_2;m_3,n_3}_{p_1,q_1:p_2,q_2;p_3,q_3}\left(\begin{matrix}(a_j:\alpha_j,A_j)_{1,p_1}:(c_j:\gamma_j)_{1,p_2};(e_j:E_j)_{1,p_3}\\ (b_j:\beta_j,B_j)_{1,q_1}:(d_j:\delta_j)_{1,q_2};(f_j:F_j)_{1,q_3}\end{matrix}\middle|\begin{matrix}x/(1+v)^{\sigma_1}\\ y/(1+v)^{\sigma_2}\end{matrix}\right) \quad \dots(2.5)$$

where the contracted notation for the (Mittal-Gupta [2]) H-function of two variables, used in (2.5), is due to Srivastava and Panda [5, p. 266, Eq. (1.5) *et seq.*].

(2) Taking all $\alpha's$, $\beta's$, $\gamma's$, $\delta's$, $A's$, $B's$, $E's$, $F's$, σ_1, and σ_2 equal to unity in (2.5), we obtain the corresponding generating relation for G-function of two variables which is recently being derived by Shukla [3].

The result (2.4) was obtained earlier by Srivastava and Panda [4, p. 130, Eq. (4.3)] adopting a different method. The present note shows the importance and utility of the differential operator T_k in obtaining the generating relation. A number of other generating relations involving different special functions can also be obtained by this method, but we omit the details.

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