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(*Dedicated to the memory of Professor Arthur Erdélyi*)

ON LAPLACE TRANSFORMS AND THEIR APPLICATIONS

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ABSTRACT

In this paper a theorem concerning the Laplace transform of the product of the *H*-function of Fox and a suitably chosen function F(x,t) is established. This theorem is then used to establish anothor theorem concerning certain fractional integrals. Applications of these theorems are capable of yielding a number of known and new results by choosing the various functions and parameters suitably.

1. Introduction

The Laplace transform of a function f(z) is defined as

$$(1.1)\qquad L\left[f(z)\,;p \right] = \int_0^\infty e^{-pz}\, f(z)dz, \quad Re\,(p) > 0.$$

The Riemann–Liouville fractional integral of order q is given by

$$(1.2)\qquad I_y^q \left[f(z) \right] \frac{1}{\Gamma(q)} \int_0^y (y-z)^{q-1} f(z)\, dz,\ Re\,(q) > 0.$$

The *H*-function of Fox [4, p. 408] is defined and represented in the following form :

$$(1.3)\qquad H_{P,Q}^{M,N}\left[z \,\middle|\, \begin{matrix} (a_j, \alpha_j)_{1,P} \\ (b_j, \beta_j)_{1,Q} \end{matrix} \right] = \frac{1}{2\pi\omega} \int_L \theta\,(s)\, z^s ds,$$

with $\omega = \sqrt{-1}$ and

(1.4) $$\theta(s) = \frac{\prod_{j=1}^{M} \Gamma(b_j - \beta_j s) \prod_{j=1}^{N} \Gamma(1 - a_j + \alpha_j s)}{\prod_{j=M+1}^{Q} \Gamma(1 - b_j + \beta_j s) \prod_{j=N+1}^{P} \Gamma(a_j - \alpha_j s)}$$

where an empty product is is interpreted as unity, L is a suitable contour and the integers M,N,P,Q satisfy $0 \leqslant M \leqslant Q$, $0 \leqslant N \leqslant P$. The coefficient a_j (j=1,..., P) and β_j (j=1, . , Q) are all positive.

In the sequel, we also require the *H*-function of two variables which was defined earlier [8, p. 117]. The notation used here for representing this function is a slight variant of the one given by Srivastava and Panda [13, p 266, Eq. (1.5)] :

(1.5) $$H^{o,n_1 : m_2, n_2 ; m_3, m_3}_{p_1,q_1 : p_2,q_2 ; p_3,q_3} \left[\begin{matrix} x \\ y \end{matrix} \middle| \begin{matrix} (a_j ; \alpha_j, A_j)_{1,p_1} : (c_j, \gamma_j)_{1,p_2} ; (e_j, E_j)_{1,p_3} \\ (b_j : \beta_j, B_j)_{1,q_1} : (d_j ; \delta_j)_{1,q_2} ; (f_j, F_j)_{1,q_3} \end{matrix} \right]$$

$$= \frac{-1}{4\pi^2} \int_{L_1} \int_{L_2} \theta_1(s)\, \theta_2(t)\, \emptyset(s,t)\, x^s y^t ds\, dt,$$

where

(1.6) $$\emptyset(s,t) = \frac{\prod_{j=1}^{n_1} \Gamma(1 - a_j + \alpha_j s + A_j t)}{\prod_{j=n_1+1}^{p_1} \Gamma(a_j - \alpha_j s - A_j t) \prod_{j=1}^{q_1} \Gamma(1 - b_j + \beta_j s + B_j t)}$$

and $\theta_1(s)$ and $\theta_2(t)$ are the Γ—quotients anologous to the quotient $\theta(s)$ in the equation (1.4) involving appropriately the parameters of the second and third sets, respectively. The contours L_1 and L_2 are suitably defined and the integers n_j, p_i, q_i (i=1,2,3), m_j (j=2,3) are such that $0 \leqslant n_i \leqslant p_i$, $q_1 \geqslant 0$, $0 \leqslant m_j \leqslant q_j$.

The coefficients A_j, B_j, E_j, F_j, α_j, β_j, γ_j, δ_j, are all positive. The conditions of existence of the functions defined in (1.3) and (1.5) are given in [5, p. 597, Eqns. (3.1) to (3.6)] and [8, p. 119, (i) to (vi)]. We assume that these conditions are satisfied by the different H-functions occurring in this paper.

2. Main Results

Theorem 1. Let

$$(2.1) \qquad F(x,t) = \sum_{n=0}^{\infty} C_n f_n(x)\, t^n,$$

where $f_n(x)$ *is a polynomial of degree* n *in* x.

Then

$$(2.2) \quad L\left[z^{\lambda-1} (1-z)^{\mu-1} H^{M,N}_{P,Q}\left[yz^{\sigma} \middle| \begin{matrix} (a_j, \alpha_j)_{1,P} \\ (b_j, \beta_j)_{1,Q} \end{matrix} \right] F(x, z^{\alpha} (1-z)^{\beta} t; p \right]$$

$$= p^{-\lambda} \sum_{n=0}^{\infty} C_n f_n(x) \frac{(t p^{-\alpha})^n}{\Gamma(1-\mu-\beta n)}$$

$$H^{0,1:1,1;M,N}_{1,0:1,1;P,Q}\left[\begin{matrix} -p^{-1} \\ yp^{-\sigma} \end{matrix} \middle| \begin{matrix} (1-\lambda-\alpha n;1,\sigma) : (\mu+\beta n, 1) ; (a_j, \alpha_j)_{1,P} \\ \text{———} \quad ; (0,1) \quad ; (b_j, \beta_j)_{1,Q} \end{matrix} \right],$$

provided that

$$Re\,(p) > 0,\ Re\,[\lambda+\sigma\,(b_i/\beta_i)] > 0\ (i=1,\dots,M),$$

$$\sigma > 0,\ \alpha,\ \beta,\ \mu \geqslant 0\ (\alpha,\ \beta \text{ being not zero simultaneously})$$

and $F(x,t)$ *is such that the left-hand side of (2.2) exists.*

Theorem 2. *Let* $F(x,t)$ *be as given by (2.1).*

Then

$$(2.3) \quad I^{q}_{y}\left[z^{\lambda-1} (1-z)^{\mu-1} H^{M,N}_{P,Q}\left[az^{\sigma} \middle| \begin{matrix} (a_j, \alpha_j)_{1,P} \\ (b_j, \beta_j)_{1,Q} \end{matrix} \right] F(x,z^{\alpha} (1-z)^{\beta} t) \right]$$

$$= y^{q+\lambda-1} \sum_{n=0}^{\infty} C_n f_n(x) \frac{(ty^{\alpha})^n}{\Gamma(1-\mu-\beta n)}$$

$$H^{0,1:1,1;M,N}_{1,1:1,1;P,Q}\left[\begin{matrix} -y \\ ay^{\sigma} \end{matrix}\middle| \begin{matrix} (1-\lambda-\alpha n;1,\sigma):(\mu+\beta n,1);(a_j,\alpha_j)_{1,P} \\ (1-\lambda-q-\alpha n;1,\sigma):(0,1)\quad;(b_j,\beta_j)_{1,Q} \end{matrix}\right]$$

provided that $Re(q) > 0$, $Re[\lambda+\sigma(b_i/\beta_i)] > 0$ $(i=1,..M)$, $\sigma > 0$, $\alpha,\beta,\mu \geqslant 0)$ (*α and β being not zero simultaneously*) *and* F(x,t) *is such that left-hand side of* (2.3) *exists.*

Proofs : To prove (2.2), we substitute for F(x,t) from (2.1) into the left-hand side of (2.2), use [5, p. 600, Eqn. (4.7)], take the Laplace transform term-by-term, and apply [8, p. 122, Eqn. (2.2)]. To prove (2.3), we use the convolution property of the Laplace transforms [3, p. 131, Eqn. (20)], take the inverse Laplace transform of the right-hand side of (2.2), term-by-term with the help of [8, p. 122, Eqn. (2.2)], and invoke (1.2).

3, Applications

(*a*) *Special case of Theorem 1* :

If in Theorem 1, we take $\alpha=1$, $\beta=0$, $C_n=(n!)^{-1}$,

$$f_n(x) = {}_{u+1}F_v\left[\begin{matrix} -n,(a'_u) \\ (b'_v) \end{matrix} ; x\right],$$

so that from [12, p. 233, Eqn. (13)], we have

$$F(x,t) = e^{t}\, {}_uF_v\left[\begin{matrix} (a'_u) \\ (b'_v) \end{matrix} ; -xt\right] \tag{3.1}$$

then (2.2), with these substitutions, gives

$$L\left[z^{\lambda-1}(1-z)^{\mu-1} H^{M,N}_{P,Q}\left[yz^{\sigma}\middle| \begin{matrix} (a_j,\alpha_j)_{1,P} \\ (b_j,\beta_j)_{1,Q} \end{matrix}\right] e^{tx} \right. \tag{3.2}$$

$$\left. {}_uF_v\left[\begin{matrix} (a'_u) \\ (b'_v) \end{matrix} ; -xtz\right] ; p \right]$$

$$\frac{p^{-\lambda}}{\Gamma(1-\mu)} \sum_{n=0}^{\infty} {}_{u+1}F_{v}\left[\begin{matrix}-n, & (a'_u) \\ & (b_v')\end{matrix}; x\right] \frac{(tp^{-1})^n}{n!} \times$$

$$H_{1,0:1,1;P,Q}^{0,1:1,1;M,N}\left[\begin{matrix}-p^{-1} \\ yp^{-\sigma}\end{matrix}\middle| \begin{matrix}(1-\lambda-n;1,\sigma):(\mu,1);(a_j,\alpha_j)_{1,p} \\ :(0,1);(b_j,\beta_j)_{1,Q}\end{matrix}\right].$$

On applying [5, p. 600, Eqn. (4.6)] and [6, p. 124, Eqn. (3.6)], putting $\mu=1$, using [8, p. 122, Eqn. (2.2)], replacing t by pt, and y by yp^{σ}, we are led to special cases of [13, p. 267, Eqn. (2.1)] and also of [11, p. 302, Eqn. (2.2)].

(b) Special cases of Theorem 2 :

1. On letting $y\to 1$, using [1, p. 215, Eqn. (1)], putting $\alpha=\mu=1$, $\beta=0$ in Theorem 2, it reduces to [9, p. 187, Theorem (2.1)].

2. On setting $\alpha=0$, $\beta=1$, in Theorem 2 and choosing $f_n(x)$ and C_n as in (a) above, expressing the ${}_uF_v$ function and the exponential function as power series, and putting $\sigma=1$, M=N=P=Q=1, $a_1=1-v$, $\alpha_1=1$, $b_1=0$, $\beta_1=1$, therein, it gives on replacing a by a/y.

$$(3.3)\quad \sum_{n=0}^{\infty} {}_{u+1}F_{v}\left[\begin{matrix}-n, & (a'_u) \\ & (b'_v)\end{matrix}; x\right] F_1(\lambda, 1-\mu-n, \nu; \lambda+q; y, -a)\frac{t^n}{n!}$$

$$=e^{t} \sum_{n,r=0}^{\infty} \frac{(a'_u)_n(-xt)^n(-yt)^r(\lambda)_r}{(b'_v)_n(\lambda+q)_r\, r!\, n!} F_1(\lambda+r, 1-\mu-n, \nu; \lambda+r+q; y, -a)$$

If we put $\lambda=\beta$, $\lambda+q=\delta$, $u=\nu=1$, $a'_1=\alpha$, $b'_1=\gamma$, $\mu=1$. $v=0$ and let $a\to -y$, then (3.3) yields [10, *p*. 473, Eqn. (20)].

3. Lastly, if in Theorem 2, we take $f_n(x)=L_n^{(\nu-n)}(x)$, $C_n=1$, so that from [2, p. 189, Eqn. (19)], $F(x,t)=(1-t)^{\nu}e^{-xt}$, and put $\alpha=1$, $\beta=0$, then on proceeding as before, we get

$$(3.4)\quad \sum_{n=0}^{\infty} \frac{(-xt)^n}{n!\,\Gamma(-\nu)} H_{1,1:1,1;P,Q}^{0,1:1,1;M,N}\left[\begin{matrix}t \\ a\end{matrix}\middle| \begin{matrix}(1-\lambda-n;1,\sigma):(1+\nu,1);(a_j,\alpha_j)_{1,p} \\ (1-\lambda-n-q;1,\sigma):(0,1);(b_j,\beta_j)_{1,Q}\end{matrix}\right]$$

$$= \sum_{n=0}^{\infty} L_n^{(\nu-n)}(x)\, H^{M,\, N+1}_{P+1,\, Q+1}\left[a \,\middle|\, \begin{matrix} (1-\lambda-n,\, \sigma),\, (a_j,\, \alpha_j)_{1,P} \\ (b_j,\, \beta_j)_{1,Q} \end{matrix} \right] t^n.$$

For $\sigma=1$, $M=N=P=Q=1$, $a_1=1-\gamma$, $\alpha_1=1$, $\lambda=\beta$, $\lambda+q=\delta$.

$b_1=0$, $\beta_1=1$, (3.4) reduces to

$$(3.5) \quad \sum_{n=0}^{\infty} \frac{(\beta)_n}{(\delta)_n} F_1(\beta+n, -\nu, \gamma; \delta+n; t, -a) \frac{(-xt)^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{(\beta)_n}{(\delta)_n} L_n^{(v-n)}(x)\, {}_2F_1(\beta+n, \gamma; \delta+n; -a)\, t^n.$$

if we let $a \to 0$, (3.5) corresponds to [7, *p.* 57, Eqn (4.11)].

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