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(*Dedicated to the memory of Professor Arthur Erdélyi*)

A TRANSFORMATION FORMULA FOR A GENERAL HYPERGEOMETRIC FUNCTION OF THREE VARIABLES

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1. INTRODUCTION

The aim of this note is to establish a transformation formula involving the general hypergeometric function $F^{(3)}[x, y, z]$ of three variables introduced earlier by Srivastava [7]. Due to the very general nature of the transformation formula obtained here, a number of results (known as well as new) follows as particular cases.

The hypergeometric function $F^{(3)}[x, y, z]$ of three variables has been defined by Srivastava in the form (see [7], p. 428) :

(1) $$F^{(3)}\left[\begin{matrix}(a) :: (b) ; (b') ; (b'') : (c) ; (c') ; (c'') ; \\ (f) :: (g) ; (g') ; (g'') : (h) ; (h') ; (h'') ;\end{matrix} x, y, z\right]$$

$$= \sum_{m,n,p=0}^{\infty} \frac{((a))_{m+n+p} ((b))_{m+n} ((b'))_{n+p} ((b''))_{p+m} ((c))_m ((c'))_n}{((f))_{m+n+p} ((g))_{m+n} ((g'))_{n+p} ((g''))_{p+m} ((h))_m ((h'))_n} \cdot \frac{((c''))_p \, x^m y^n z^p}{((h''))_p \, m! \, n! \, p!},$$

which is valid under certain restrictions on the parameters, etc. Here (a) is taken to denote the sequence of A parameters $a_1, a_2, .., a_A$, that is,

unless otherwise stated, there are A of the a parameters, B of the b parameters, and so on. Thus $((a))_m$ is to be interpreted as $\prod_{j=1}^{A} (a_j)_m$; with similar interpretations for $((b))_m$, etc. Further $(a)_m = \Gamma(a+m) / \Gamma(a)$.
An empty product in (1) is to be treated as unity.

2. Main Result. We establish the following transformation formula :

$$(2) \qquad (1-x)^{-\lambda}(1-y)^{-\mu}(1-z)^{-\nu} F^{(3)}\left[\begin{matrix}(a)::(b);(b');(b''):(c);(c');(c'');\\ (f)::(g);(g');(g''):(h);(h');(h'');\end{matrix}\right.$$

$$\left.\left(\frac{-\alpha x}{(1-x)(1-y)(1-z)}\right)^{\delta}, \left(\frac{-\beta y}{(1-x)(1-y)(1-z)}\right)^{\delta}, \left(\frac{-\gamma z}{(1-x)(1-y)(1-z)}\right)^{\delta}\right]$$

$$= \sum_{R,S,T,=0}^{\infty} \frac{(\lambda)_R (\mu)_S (\nu)_T \, x^R \, y^S \, z^T}{R! \; S! \; T!}$$

$$. F^{(3)}\left[\begin{matrix}(a)::\Delta(\delta,\nu+T),(b);\Delta(\delta,\lambda+R),(b');\Delta(\delta,\mu+S),(b''):\\ \Delta(\delta,\lambda),\Delta(\delta,\mu),\Delta(\delta,\nu),(f)::(g);(g');(g''):\end{matrix}\right.$$

$$\left.\begin{matrix}\Delta(\delta,-R),(c);\Delta(\delta,-S),(c');\Delta(\delta,-T),(c'');\\ (h);(h');(h'');\end{matrix}\; \alpha^{\delta}, \beta^{\delta}, \gamma^{\delta}\right]$$

where δ is a positive integer, and $\Delta(m, a)$ stands for the m parameters a/m, $(a+1)/m$,..., $(a+m-1)/m$.

The formula (2) is valid under the conditions
$A+B+B''+C \leqslant F+G+G''+H$, $A+B+B'+C' \leqslant F+G+G'+H'$, $A+B'+B''+C'' \leqslant F+G'+G''+H''$, and $|x|<1, |y|<1, |z|<1$;
but if $A+B+B''+C=F+G+G''+H+1, A+B+B'+C'= F+G+G'+H'+1$, $A+B'+B''+C'' = F+G'+G''+H''+1$, then $|x|$, $|y|$, $|z|$, $|\alpha|$, $|\beta|$, and $|\gamma|$ are to be restricted appropriately, so that the series involved are either terminating of convergent.

Proof : Substituting for $F^{(3)}$ in the left-hand side of (2) from (1) and making use of the elementry expansion :

$$(3)\quad \sum_{r=0}^{\infty} \frac{(n)_r\, x^r}{r\,!} = (1-x)^{-n}\,,\ |\,x\,| < 1,$$

we obtain

$$\sum_{m,n,\,p=o}^{\infty} \frac{((a))_{m+n+p}((b))_{m+n}\,((b'))_{n+p}\,((b''))_{p+m}\,((c))_m\,((c'))_n\,((c''))_p}{((f\,))_{m+n+p}\,((g))_{m+n}\,((g'))_{n+p}\,((g''))_{p+m}\,((h))_m\,((h'))_n\,((h''))_p}$$

$$.\ \frac{(-\alpha)^{m\delta}\,(-\beta)^{n\delta}\,(-\gamma)^{p\delta}}{m!\ n!\ p!} \sum_{r,s,t=0}^{\infty} \frac{(\lambda+m\delta+n\delta+p\delta)_r}{r\,!\ s\,!\ t\,!}$$

$$.\ (\mu+m\delta+u\delta+p\delta)_s\ (\nu+m\delta+n\delta+p\delta)_t\ x^{r+m\delta}\ y^{s+n\delta}\ z^{t+p\delta}.$$

Inverting the order of summation, which is justified due to absolute convergence of the series involved, substituting $r+m\delta = R,\ \ s+n\delta = S,\ \ t+p\delta=T,$ making use of the relationships [6, pp. 22, 32] :

$$(4)\qquad (a)_{mk} = m^{mk} \prod_{i=1}^{m} \left(\frac{a+i-1}{m}\right)_k ;\ (-n)_k = \frac{(-1)^k\, n\,!}{(n-k)\,!}\,,\ 0\leqslant k\leqslant n\ ;$$

and simplifying, we get the right-hand side of (2).

3. Special Cases—We give below some of the particular cases which are believed to be new.

(i) Putting $\delta=1$ in (2) and replacing x by x/λ, y by y/μ, z by z/ν, α bp $\lambda\alpha$, β by $\mu\beta$ and γ by $\nu\gamma$, we obtain

$$(1-x/\lambda)^{-\lambda}\ (1-y/\mu)^{-\mu}\ (1-z/\nu)^{-\nu}\ F^{(3)} \left[\begin{matrix} (a)::(b);\ (b');\ (b''):(c);\ (c');\ c''); \\ (f\,)::(g);\ (g');\ (g''):(h);\ (h');\ (h''); \end{matrix} \right.$$

$$\left. \frac{-\alpha x}{(1-x/\lambda)\,(1-y/\mu)\,(1-z/\nu)}\,,\ \frac{-\beta y}{(1-x/\lambda)\,(1-y/\mu)\,(1-z/\nu)}\,,\ \frac{-\gamma z}{(1-x/\lambda)(1-y/\mu)(1-z/\nu)} \right]$$

$$= \sum_{R,S,T=0}^{\infty} \frac{(\lambda)_R\,(\mu)_S\,(\nu)_T\,(x/\lambda)^R\,(y/\mu)^S\,(z/\nu)^T}{R!\ \ S!\ \ T!} \sum_{r,s,t=0}^{\infty} \frac{((a))_{r+s+t}}{((\lambda))_{r+s+t}}$$

$$\frac{(\nu+T)_{r+s}\,((b))_{r+s}\,(\lambda+R)_{s+t}\,((b'))_{s+t}\,((\mu+S)_{t+r}\,((b''))_{t+r}}{(\mu)_{r+s+t}\,(\nu)_{r+s+t}\,((f))_{r+s+t}\,((g))_{r+s}\,((g'))_{s+t}\,((g''))_{t+r}}$$

$$\frac{(-R)_r\,((c))_r\,(-S)_s\,((c'))_s\,(-T)_t\,((c''))_t\,((\lambda\alpha))^r\,((\mu\beta))^s\,(\nu\gamma)^t}{((h))_r\,((h'))_s\,((h''))_t\;r!\;s!\;t!}.$$

Proceeding to limits as $\lambda\to\infty$, $\mu\to\infty$, $\nu\to\infty$, we obtain

$$(5)\quad e^{x+y+z}\,F^{(3)}\left[\begin{matrix}(a)::(b);(b');(b''):(c);(c');(c'');\\(f)::(g);(g');(g''):(h);(h');(h'');\end{matrix}\;-\alpha x,-\beta y,-\gamma z\right]$$

$$=\sum_{R,S,T=0}^{\infty}\frac{x^R\,y^S\,z^T}{R!\,S!\,T!}\,F^{(3)}\left[\begin{matrix}(a)::(b);(b')(b''):-R,(c);-S,(c');-T,(c'');\\(f)::(g);(g');(g''):(h);(h');(h'');\end{matrix}\;\alpha,\beta,\gamma\right]$$

(ii) If we take $\delta=2$, replace α^2 by $\lambda^2\alpha$, β^2 by $\mu^2\beta$, γ^2 by $\nu^2\gamma$, x by x/λ, y by y/μ, z by z/ν and proceed as in (i) above, we get the transformation formula

$$(6)\quad e^{x+y+z}\,F^{(3)}\left[\begin{matrix}(a)::(b);(b');(b''):(c);(c');(c'');\\(f)::(g);(g');(g''):(h);(h');(h'');\end{matrix}\;\alpha x^2,\beta y^2,\gamma z^2\right]$$

$$=\sum_{R,S,T=0}^{\infty}\frac{x^R\,y^S\,z^T}{R!\,S!\,T!}\,F^{(3)}\left[\begin{matrix}(a)::(b);(b');(b''):\Delta(2,-R),(c);\Delta(2,-S),(c');\\(f)::(g);(g');(g''):(h);(h');(h''):\end{matrix}\right.$$

$$\left.\begin{matrix}\Delta(2,-T),(c'');\\ \text{———————};\end{matrix}\;4\alpha,4\beta,4\gamma\right].$$

(iii) On taking $A=B=B'=B''=F=G=G'=G''=0$ in (2), $F^{(3)}$ breaks into the product of three hypergeometric functions and it takes the form

$$(7)\quad (1-x)^{-\lambda}(1-y)^{-\mu}(1-z)^{-\nu}\;{}_CF_H\left[\begin{matrix}(c);\\(h);\end{matrix}\left(\frac{-\alpha x}{(1-x)(1-y)(1-z)}\right)^{\delta}\right]$$

$$\cdot\;{}_{C'}F_{H'}\left[\begin{matrix}(c');\\(h');\end{matrix}\left(\frac{-\beta y}{(1-x)(1-y)(1-z)}\right)^{\delta}\right]\;{}_{C''}F_{H''}\left[\begin{matrix}(c'');\\(h'');\end{matrix}\left(\frac{-\gamma z}{(1-x)(1-y)(1-z)}\right)^{\delta}\right]$$

$$=\sum_{R,S,T=0}^{\infty}\frac{(\lambda)_R\,(\mu)_S\,(\nu)_T\,x^R\,y^S\,z^T}{R!\,S!\,T!}\,F^{(3)}\left[\begin{matrix}-::\Delta(\delta,\nu+T);\Delta(\delta,\lambda+R;\\ \Delta(\delta,\lambda),\Delta(\delta,\mu),\Delta(\delta,\nu)::\end{matrix}\right.$$

$$\triangle\ (\delta,\mu+S) : \triangle(\delta, -R), (c); \triangle\ (\delta, -S), (c'); \triangle(\delta, -T), (c''); \quad \alpha^{\delta}, \beta^{\delta}, \gamma^{\delta} \Big].$$
$$\text{———} : (h) ; (h') ; (h'') ;$$

On specializing the parameters in $F^{(3)}$, it yields the fourteen hypergeometric functions $F_1, F_2, \ldots, F_{14}$ of three variables, defined explicitly by Lauricella [4, p. 114], and the three additional functions H_A, H_B and H_C, defined by Srivastava (*cf.*, *e.g.*, [8], pp. 99-100); it also reduces to Appell functions and other functions of similar nature. Thus the transformation discussed here becomes a key result. In particular, the results of Mahajan [5], Brafman [1], Chaundy [3], and Jain and Dave [4], can be obtained by the proper choices of the parameters and variables involved.

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