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(Dedicated to the memory of Professor Arthur Erdélyi)

ON A NEW INTEGRAL TRANSFORM. I

By

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ABSTRACT

In the present paper, we introduce a new integral transform, involving Bessel functions as kernel. Inversion formula is established and some properties are given. The transform can be used to solve certain class of mixed boundary value problems.

1. Definition and Inversion Formula We consider the Bessel's differential equation [3]

(1) $$x^2 y'' + xy' + (\lambda^2 x^2 - \nu^2) y = 0$$

with the conditions

(2) $$y(a) = 0\,, \quad y(b) + h\,y'(b) = 0.$$

The general solution of (1) can be written as

(3) $$y(x) = C_1 J_\nu(\lambda x) + C_2\, y_\nu(\lambda x)$$

where C_1, C_2 are arbitrary constants, and $J_\nu(x)$ and $y_\nu(x)$ are the Bessel's functions of first and second kind respectively.

We want to obtain solutions of (1) which satisfies the conditions (2). Hence, we have

(4) $$C_1 J_\nu(\lambda a) + C_2\, y_\nu(\lambda a) = 0$$

(5) $$C_1 \left[J_\nu(\lambda b) + h \lambda J'_\nu(\lambda b) \right] + C_2 \left[y_\nu(\lambda b) + h \lambda y'_\nu(\lambda b) \right] = 0.$$

From (5) and (6), we can deduce

(6) $$\frac{C_1}{C_2} = -\frac{y_\nu(\lambda a)}{J_\nu(\lambda a)} = -\frac{[y_\nu(\lambda b) + h \lambda y'_\nu(\lambda b)]}{[y_\nu(\lambda b) + h \lambda J'_\nu(\lambda b)]}.$$

Thus, the function given by (3) is the solution of the equation (1), with conditions (2), if λ is a root of the transcendental equation

(7) $$J_\nu(\lambda a) [y_\nu(\lambda b) + h\lambda y'_\nu(\lambda b)] - y_\nu(\lambda a) [J_\nu(\lambda b) + h \lambda J'_\nu(\lambda b)] = 0.$$

Introducing the following notations,

$$A_\nu(\lambda x) = J_\nu(\lambda x) + h \lambda J'_\nu(\lambda x);$$

$$B_\nu(\lambda x) = y_\nu(\lambda x) + h \lambda y'_\nu(\lambda x)$$

(7) can be written as

(8) $$J_\nu(\lambda a) B_\nu(\lambda b) - y_\nu(\lambda a) A_\nu(\lambda b) = 0.$$

Let λ_i ($i = 1, 2, 3, \ldots$) be the positive roots of the equation (8). Then, from (4) and (5), we have

$$y_i(x) = \frac{C_1}{y_\nu(\lambda_i a)} [J_\nu(\lambda_i x) y_\nu(\lambda_i a) - y_\nu(\lambda_i x) J_\nu(\lambda_i a)]$$

(9) $$y_i(x) = \frac{C_1}{B_\nu(\lambda_i b)} [J_\nu(\lambda_i x) B_\nu(\lambda_i b) - y_\nu(\lambda_i x) A_\nu(\lambda_i b)].$$

Then the following functions are the solutions of the equation (1), with conditions (2) :

(10) $$C_\nu(\lambda_i x) = [y_\nu(\lambda_i a) + B_\nu(\lambda_j b)] J_\nu(\lambda_i x) - [J_\nu(\lambda_i a) + A_\nu(\lambda_i b)] y_\nu(\lambda_i x).$$

Now according the theory of Sturm-Liouville [1], the functions of the system (10) are orthogonal on the interval $[a, b]$ with weight function x, that is

(11) $$\int_a^b x\, C_\nu(\lambda_i x)\, C_\nu(\lambda_j x)\, dx = 0, \quad i \neq j$$

(12) $$\int_a^b x\, C^2_\nu(\lambda_i x)\, dx = \| C_\nu(\lambda_i x) \|^2.$$

If a function $f(x)$ and its first derivative are piecewise continuous on the interval $[a, b]$, then the relation

$$T[f(x), a, b, \nu; \lambda_i] = \bar{f}_\nu(\lambda_i) = \int_a^b x f(x)\, C_\nu(\lambda_i x)\, dx$$

defines an integral transform, where λ_i are the positive roots of the equation (8). To obtain the inversion formula, let

(13) $$f(x) = \sum_{i=1}^{\infty} a_i\, C_\nu(\lambda_i x).$$

Multiplying both sides by $x\, C_\nu(\lambda_j x)$, (j ; *fixed*) integrating with respect to x between a and b, we get

(14) $$a_i = \frac{\int_a^b x f(x)\, C_\nu(\lambda_i x)\, dx}{\| C_\nu(\lambda_i x) \|^2} = \frac{\bar{f}_\nu(\lambda_i)}{\| C_\nu(\lambda_i x) \|^2} \cdot \quad i = 1, 2, 3, \ldots$$

Hence

(15) $$f(x) = \sum_{i=1}^{\infty} \frac{\bar{f}_\nu(\lambda_i)}{\| C_\nu(\lambda_i x) \|^2}\, C_\nu(\lambda_i x)$$

where the summation is taken over all the positive roots of the equation (8).

Using some well known properties of the Bessel functions [2, p. 634, 968, 969] we can easily derive

(16) $$\| C_\nu(\lambda_i x) \|^2 = \tfrac{1}{2} M^2(\lambda_i, a, b)\,\{ b^2 P(\lambda_i; b; \nu) - a^2 P(\lambda_i; a; \nu)\}$$
$$- M(\lambda_i; a, b)\, N(\lambda_i, a, b)\,\{ b^2 Q(\lambda_i, b, \nu) - a^2 Q(\lambda_i, a, \nu)\} +$$
$$+ \tfrac{1}{2} N^2(\lambda_i, a, b)\,\{ b^2 R(\lambda_i, b, \nu) - a^2 R(\lambda_i, a, \nu)\}$$

where

$$M(\lambda_i, a, b) = y_\nu(\lambda_i a) + B_\nu(\lambda_i b)$$

$$N(\lambda_i, a, b) = J_\nu(\lambda_i a) + A_\nu(\lambda_i b)$$

$$P(\lambda_i, \mu; \nu) = J^2_\nu(\lambda_i \mu) - J_{\nu-1}(\lambda_i \mu)\, J_{\nu+1}(\lambda_i \mu)$$

$$Q(\lambda_i, \mu; \nu) = J'_\nu(\lambda_i \mu)\, y_{\nu-1}(\lambda_i \mu) - \frac{1}{\lambda_i \mu} J_{\nu-1}(\lambda_i \mu)\, y_\nu(\lambda_i \mu)$$

$$-J'_{\nu-1}(\lambda_i \mu)\, y_\nu(\lambda_i \mu)$$

(17) $$R(\lambda_i, \mu; \nu) = y_\nu^2(\lambda_i \mu) - y_{\nu-1}(\lambda_i \mu)\, y_{\nu+1}(\lambda_i \mu)$$

$(\mu = a, b)$.

2. Some Properties of the Transform. The following propcrties can be easily varified from the definition of the transform

(18) $$T[\alpha f(x) + \beta g(x), a, b, \nu; \lambda_i] = \alpha T[f(x), a, b, \nu; \lambda_i] + \beta T[g(x), a, b, \nu; \lambda_i]$$

(19) $$T\Big[f(ax), a, b, \nu; \lambda_i\Big] = \frac{1}{a^2} T\Big[f(x), a\alpha, b\alpha, \nu; \frac{\lambda_i}{\alpha}\Big].$$

Transform of $g(x) = \dfrac{d^2 f}{dx^2} + \dfrac{1}{x}\dfrac{df}{dx} - \dfrac{\nu^2}{x^2} f$

Let

(20) $$I = \int_a^b x \left(f''(x) + \frac{1}{x} f'(x) \right) C_\nu(\lambda_i x)\, dx$$

$$= \int_a^b x f''(x)\, C_\nu(\lambda_i x)\, dx + \int_a^b f'(x)\, C_\nu(\lambda_i x)\, dx$$

$$= \Big\{ x C_\nu(\lambda_i x) f'(x) \Big\}_a^b - \lambda_i \int_a^b x f'(x)\, C'_\nu(\lambda_i x)\, dx$$

$$= \Big\{ x \Big[C_\nu(\lambda_i x) f'(x) - \lambda_i C'_\nu(\lambda_i x) f(x) \Big] \Big\}_a^b$$

$$+ \int_a^b x^{-1} \Big[\lambda_i^2 x^2 C''_\nu(\lambda_i x) + (\lambda_i x) C'_\nu(\lambda_i x) \Big] f(x)\, dx.$$

As the functions $C_\nu(\lambda_i x)$ satisfies the Bessel's differential equation we have

(21) $$\bar{g}_\nu(\lambda_i) = \int_a^b x \left(f'' + \frac{1}{x} f' - \frac{\nu^2}{x^2} f \right) C_\nu(\lambda_i x)\, dx$$

$$= \left\{ x \left[C_\nu (\lambda_i x) f'(x) - \lambda_i C'_\nu (\lambda_i x) f(x) \right] \right\}_a^b - \lambda_i^2 \bar{f}_\nu (\lambda_i),$$

which, on simplification gives

$$\bar{g}_\nu (\lambda_i) = \frac{b}{h} C_\nu (\lambda_i b) \left[f + h \frac{df}{dx} \right]_{x=b} + \lambda_i a C'_\nu (\lambda_i a) f(a) - \lambda^2 \bar{f}_\nu (\lambda_i)$$

(22)

Transform of x^ν. From the definition we have

$$T[x^\nu, a, b, \nu; \lambda_i] = \int_a^b x^{\nu+1} C_\nu (\lambda_i x)\, dx.$$

Using the result [2, p. 634]

$$\int x^{\rho+1} Z_\rho (x)\, dx = x^{\rho+1} Z_{\rho+1} (x)$$

where $Z_\rho (x)$ is one of the Bessel functions, and after a little simplification, we obtain

$$\text{(23)} \qquad T[x^\nu, a, b, \nu; \lambda_i] = \frac{b^{\nu+1}}{\lambda_i^2} \left(\frac{\nu}{b} + \frac{1}{h} \right) C_\nu (\lambda_i b) + \frac{a^{\nu+1}}{\lambda_i} C'_\nu (\lambda_i a)$$

Transform of a constant. We can easily derive

$$\text{(24)} \qquad T[a, a, b, 0; \lambda_i] = \frac{a}{\lambda_i^2} \left[\frac{b}{h} C_0 (\lambda_i b) + a \lambda_i C'_0 (\lambda_i a) \right.$$

3. Applications. The transform introduced in the previous sections can be used to solve a certain class of mixed boundary value problems. For example, the transform can be applied in problems of conduction of heat in hollow cylinders, when one lateral face is kept at a prescribed temperature, while the other radiates heat in the surrounding medium. A systematic use of this transform along with some other suitable transform can be used to solve problems of hollow cylinders (finite, semi-infinite or infinite).

In the second part of this paper the authors propose to solve some problems of conduction of heat in concentric hollow cylinders.

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