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(*Dedicated to the memory of Professor Arthur Erdélyi*)

ON THE RATIO OF TWO GAMMA FUNCTIONS

By

YUDELL L. LUKE

Department of Mathematics, University of Missouri
Kansas City, Missouri 64110, U. S. A.

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We are concerned with evaluation of $\Gamma(z+a) / \Gamma(z+b)$ where a and b are fixed and z is large. In view of the property $\Gamma(y+1)=y\Gamma(y)$, we can always arrange matters so that the essential part of the calculation allows the pertinent variable to be sufficiently large in some sense. This is readily seen by writing

$$\frac{\Gamma(x+a)}{\Gamma(x+b)}=\frac{(x+b)_m}{(x+a)_m}\,\frac{\Gamma(z+a)}{\Gamma(z+b)},\ z=x+m, \tag{1}$$

where m is a positive integer to insure that z is sufficiently large. Further in the case of real variable and parameter we can take $a=0$ and $0<b<1$. See later remarks. Considerable information in this topic will be found in the volumbes by Luke [1, 2].

In [1951] Tricomi and Erdélyi [3] gave the asymptotic expansion

$$\frac{\Gamma(z+a)}{\Gamma(z+b)}=z^{a-b}\sum_{k=0}^{N-1}\frac{(-)^k B_k^{(c)}(a)\,(b-a)_k\,z^{-k}}{k\,!}$$

$$+z^{a-b}\,O(z^{-N}),c=a-b+1,$$

$$|z|\to\infty,\ |\arg(z+a)|\leqslant\pi-\epsilon,\ \epsilon>0, \tag{2}$$

where $B_k^{(c)}(a)$ is the generalized Bernoulli polynomial. This expansion is noted in the Luke references. In rather recent times, several workers have dealt essentially with the special case $a=0$, $b=\frac{1}{2}$ without noting the Tricomi-Erdélyi expansion.

Consider

$$h(n) = \frac{(2\pi)^{\frac{1}{2}}\,(2n+3)^{\frac{1}{2}}\,\Gamma(n+2)}{2^{n+2}\,\Gamma(n+{}^5/_2)} = \frac{\pi^{\frac{1}{2}}}{2^{n+1}}\,H(n). \tag{3}$$

Phillips and Sahney [4] give the first four terms in the asymptotic expansion of $H(n)$. Their result is but a special case of (2) with $z=n+{}^3/_2$, $a=\frac{1}{2}$, $b=1$ and with z^{-k} in the descending series portion of (2) expanded in descending powers of n. They also showed that the first two (three) terms of this series gives a lower (upper bound) for $H(n)$.

Bowman and Shenton [5] gave the first 16 coefficients in the asymptotic expansion for

$$g(m) = \frac{(m/2)^{\frac{1}{2}}\,\Gamma(m/2)}{\Gamma(m/2+\frac{1}{2})} \tag{4}$$

and the corresponding coefficients in the continued J-fraction. They note that these representations go back to Stieltjes, but they do not seem to recognize that their asymptotic expansion is a special case of (2) with $a=0$, $b=\frac{1}{2}$.

All of this is a prelude to the point that all the expansions noted above are inferior to a result of Fields [6] who proved that

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} = \omega^{a-b} \sum_{k=0}^{N} \frac{B_{2k}^{(2\rho)}(\rho)\,(1-2\rho)_{2k}\,\omega^{-2k}}{(2k)!}$$

$$+\,\omega^{a-b}\,O(\omega^{-2N}),\ \omega = z+a-\rho,$$

$$2\rho = 1+a-b,\ |z|\to\infty,\ |\arg(z+a)| \leqslant \pi-\varepsilon,\ \varepsilon>0. \tag{5}$$

Further, we will show that certain Padé fractions based on the series portion in (5) give rise to upper and lower bounds which are also quite powerful and certainly supperior to the inequalit previously noted. (In this connection, see the concluding paragraph of this paper.) It appears that (5) was also unknown to the above authors. It is also recorded in [1, 2] and in [1], some expressions for early Bernoulli polynomials $B_{2k}^{(2\rho)}(\rho)$ are posted. Further polynomials can be found using a recursion formula which is also given in [1, 2]. In a follow up paper Fields [7] gave an improved order estimate for the remainder. Note that (5) is an even series in ω^{-1} and it is this feature which makes (5) more powerful than (2). In the following, we give further remarks based on heuristic evidence to support this contention. If one is committed to the use of (2), and the variable and parameters are real, one can always adjust the variable so that in effect $a=0$ and $0<b<1$ whence $0<c<1$. In this event, the series in (2) is not alternating. Indeed, it can be shown that $B_k^{(c)}(0)$ vanishes for $k=2$, 4 and 6 for a value of c, $0<c<1$. On the other hand, in the use of (5), with real variable and parameters, simple adjustments can be made so that in essence, $0<\rho<\frac{1}{2}$. In this situation $B_{2k}^{(2\rho)}$ (∂) is positive (negative) when k is even (odd). Thus the series in (5) is alternating and so we get lower and upper bounds for N even odd respectively.

To illustrate the relative merits of (2) and (5), we use the model $a=\frac{1}{2}$, $b=1$ whence $\rho=\frac{1}{4}$.

From (2),

$$U(z)=\frac{\Gamma(z+\frac{1}{2})}{\Gamma(z+1)}=z^{-\frac{1}{2}}V_N(z)+O(z^{-N-\frac{1}{2}}),$$

$$V_N(z)=\sum_{k=0}^{N-1} a_k z^{-k}, \tag{6}$$

where for example $a_0=1$, $a_1=-1/8$, $a_2=1/128$, and $a_3=5/1024$. From (5)

$$U(z)=\frac{\Gamma(z+\frac{1}{2})}{\Gamma(z+1)}=2y^{-\frac{1}{2}}W_N(z)+O(z^{-2N-\frac{1}{2}}),$$

$$W_N(z)=\sum_{k=0}^{N-1}(-)^k\, b_k\, y^{-2k},\ y=4z+1, \tag{7}$$

where for example $b_0=1$, $b_1=1/4$, $b_2=21/32$, $b_3=671/128$ and $b_4=180323/2048$. The coefficients b_5, b_6, and b_7 are given in [1, 2]. In the following tables, we show computations for $U(z)$ using the above approximations for $z=3$ and $z=4$.

$z = 3, U(z) = .55389\ 18284$

N	$z^{-1/2} V_N(z)$	Error	$2y^{-1/2} W_N(z)$	Error
1	.57735 02692	−.235 (−1)	.55470 01962	−.808 (−3)
2	.55329 40080	.598 (−3)	.55387 96338	.122 (−4)
3	.55379 51801	.976 (−4)	.55389 23792	−·551 (−6)
4	.55389 95909	−.777 (−5)	.55389 17768	.516 (−7)

$z = 4, U(z) = \cdot 48465\ 53499$

N	$z^{-1/2} V_N(z)$	Error	$2y^{-1/2} W_N(z)$	Error
1	.5	−.154 (−1)	·48507 12501	−.416 (−3)
2	.48437 5	.280 (−3)	.48465 16383	.371 (−5)
3	.48461 91406	.362 (−4)	.48465 54496	−.997 (−7)
4	.48465 72876	−.194 (−5)	.48465 53443	.56 (−8)

The superiority of (7) is manifest. Note that the errors in the approximants from (7) alternate in sign as N sncreases and so we get the bounds as previously noted.

We now consider $W_N(z)$. Let $U_{n,n}$ be the main diagonal Padé approximant of order n. Let $U_{n,n+1}$ be the first subdiagonal Padé approximant. In the former case, the numerator and denominator are polynomials in $1/y^2$ of degree n, while in the latter case, the numerator and denominator are polynomials in $1/y^2$ of degree n and $n+1$ respectively. We find it convenient to write the equivalent forms

$$U_{0,0}(z) = 1,\ U_{0,1}(z) = \frac{4y^2}{4y^2+1},\ U_{1,1}(z) = \frac{8y^2+19}{8y^2+21},$$

$$U_{1,2}(z) = \frac{608y^4+5048y^2}{608y^4+5200y^2+901},$$

$$U_{2,2}(z) = \frac{57664y^4+1202712y^2+1719379}{57664y^4+1217128y^2+1985819}. \qquad (8)$$

For $z>0$, we have the inequalities

$$U_{n,n+1}(z) < (y^{1/2}/2)\, U(z) < U_{n,n}(z). \qquad (9)$$

The following tables illustrate use of the Padé approximants for $z = 1\,(1)\,4$. Calculations for $2y^{-1/2}\,U_{2,3}\,(z)$ are not provided.

$z = 1,\ U(z) = .88622\ 69255$

n	$2y^{-1/2}\,U_{n,n}\,(z)$	Error	$2y^{-1/2}\,U_{n,n+1}\,(z)$	Error
0	.89442 71910	–.820 (–2)	.88557 14762	.656 (–3)
1	.88633 28273	–.106 (–3)	.88619 72165	.297 (–4)
2	.88623 68357	–.791 (–5)	———	———

$z = 2,\ U(z) = .66467\ 01941$

n	$2y^{-1/2}\,U_{n,n}(z)$	Error	$2y^{-1/2}\,U_{n,n+1}\,(z)$	Error
0	.66666 66667	–.200 (–2)	.66461 53847	.547 (–3)
1	.66467 36423	–.345 (–5)	.66466 97750	.417 (–6)
2	.66467 02639	–.698 (–7)	———	———

$z = 3,\ U(z) = .55389\ 18284$

n	$2y^{-1/2}\,U_{n,n}\,(z)$	Error	$2y^{-1/2}\,U_{n,n+1}\,(z)$	Error
0	.55470 01962	–.808 (–3)	.55388 08459	.110 (–4)
1	.55389 21843	–.156 (–6)	.55389 18052	.232 (–7)
2	.55389 18306	–.22 (–8)	———	———

$z = 4,\ U(z) = .48465\ 53499$

n	$2y^{-1/2}\,U_{n,n}\,(z)$	Error	$2y^{-1/2}\,U_{n,n+1}(z)$	Error
0	.48507 12501	–.416 (–3)	.48465 20010	.335 (–5)
1	.48465 54153	–.654 (–7)	.48465 53472	.27 (–8)
2	.48465 53500	–.1 (–9)	———	———

Neglecting the remainder in (7), we can rearrange this expression to read

$$\pi \sim \frac{4}{y}\left\{\frac{2^{2z}\,\Gamma^2\,(z+1)}{\Gamma\,(2z+1)}\right\}^2\ \{W_N(z)\}^2 \qquad (10)$$

If z is a positive integer, we can evaluate the right hand side and so have an approximation for π for each N. Equation (6) can be treated in a similar fashion. Numerics showing the superiority of (7) will be found in [1, 2].

In a previous paper (8), I noted the works of numerous authors concerning two sided inequalities for $\Gamma(z+1)/\Gamma(z+\frac{1}{2})$. Further inequalities for the latter have been given by Slavic [9] and Shafer [10]. For the same number of terms, the inequalities noted in (8, 9) are much sharper than any others known to me,

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