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(*Dedicated to the memory of Professor Arthur Erdélyi*)

A NOTE ON THE NUMBER OF DERANGEMENTS

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Numerical evidences led Prof. G. Kreweras* to conjecture that

$$(1) \quad \sum_{k=0}^{n} (-1)^{k+1} \frac{d\,(n+k+2,\, k+1)}{n+k+2} = \frac{B_{n+1}}{n+1}\,,\; n=0, 1, \ldots,$$

where $d\,(n, k)$ is the number of derangements of N, $|N| = n$, with k orbits, or permutations with k cycles of length $\geqslant 2$. These numbers have the generating function [1, p. 256]

$$(2) \qquad e^{-ut}\,(1-t)^{-u} = \sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} d\,(n, k)\, u^{k}\, \frac{t^{n}}{n!}.$$

The B_k are the Bernoulli numbers. Let $B_k^{(n)}(x)$ be the Bernoulli polynomials of order n, [2, p. 124ff]. Special cases of these polynomials appear in a variety of contexts. We have

$$(3) \qquad B_k \equiv B_k^{(1)}(0) = (-1)^k\, B_k^{(1)}(1),\; k = 0, 1, \ldots,$$

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and, in a moment, we shall use the expansion

$$\left[\frac{t}{ln(1+t)}\right]^n = \sum_{k=0}^{\infty} B_K^{(k-n+1)}(1)\, \frac{t^k}{k!}.$$

It is easy to see that $d(0,0)=1$, and

$d(n,0) = 0$ for $n = 1, 2, \ldots$; hence the right-hand side of (2) can be written

$$1+\sum_{n=0}^{\infty}\sum_{k=0}^{n} d(n+k+2, k+1)\, u^{k+1}\, \frac{t^{n+k+2}}{(n+k+2)!}$$

and replacing u by u/t, we have

$$(4)\quad \sum_{k=0}^{n} \frac{d(n+k+2, k+1)}{(n+k+2)!} u^{k+1} = \frac{1}{(n+1)!}\, D_t^{n+1}\left[e^{-u}(1-t)^{-u,t}\right]\Big|_{t=0},$$

$$n = 0, 1, 2, \ldots,$$

using Taylor's theorem.

In order to establish (1), consider the operational formula obtained by replacing u by $D_x \equiv d/dx$ in the last relation. Operating each member of the resulting equation on x^{-n-1}, we have

$$\sum_{k=0}^{n} \frac{d(n+k+2, k+1)}{(n+k+2)!} \cdot (-1)^{k+1}\, \frac{(n+k+1)!}{n!}\, x^{-n-k-2}$$

$$= \frac{1}{(n+1)!}\, D_t^{n+1}\left[e^{-\left(1+\frac{ln(1-t)}{t}\right)D_x} x^{-n-1}\right]\Bigg|_{t=0}$$

$$= \frac{1}{(n+1)!}\, D_t^{n+1}\left[\left(x-1-\frac{ln(1-t)}{t}\right)^{-n-1}\right]\Big|_{t=0}.$$

Thus, for $x=1$,

$$\sum_{k=0}^{n} (-1)^{k+1} \frac{d(n+k+2, k+1)}{n+k+2}$$

$$= \frac{1}{n+1} D_t^{n+1} \left[\frac{-t}{ln(1-t)}\right]^{n+1}\Big|_{t=0}$$

$$= \frac{1}{n+1} \sum_{k=n+1}^{\infty} (-1)^k B_k^{(k-n)}(1) \frac{t^{k-n-1}}{(k-n-1)!}\Big|_{t=0}$$

$$= \frac{(-1)^{n+1}}{n+1} B_{n+1}^{(1)}(1) = \frac{B_{n+1}}{n+1}.$$

This completes our derivation of (1).

Next, differentiating (2) with respect to t, we have

$$\sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} d(n+2, k+1) \frac{u^k t^n}{(n+1)!}$$

$$= e^{-ut} (1-t)^{-u-1}$$

$$= (1-t)^{-1} \sum_{r=0}^{\infty} \sum_{k=0}^{[r/2]} d(r, k) \frac{u^k t^r}{r!}$$

$$= \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \sum_{k=0}^{[r/2]} d(r, k) \frac{u^k t^{n+r}}{r!}$$

$$= \sum_{n=0}^{\infty} \sum_{r=0}^{n} \sum_{k=0}^{[r/2]} d(r, k) \frac{u^k t^n}{r!}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} \sum_{r=2k}^{n} d(r, k) \frac{u^K t^n}{r!}.$$

Hence

$$(5) \qquad d(n+2, k+1) = (n+1)! \sum_{r=2k}^{n} \frac{d(r, k)}{r!}, \; n = 0, 1, \ldots .$$

Iterating this recurrence, we find the explicit representation :

$$(6)\quad d\,(n+2,\,k+1)=(n+1)\,!\sum_{t_1=0}^{n-2k}\ \sum_{t_2=0}^{t_1}\cdots\sum_{t_k=0}^{t_{k-1}}\frac{1}{(t_1+2k)(t_2+2k-2)\dots(t_k+2)},$$

for $n = 2k,\, 2k+1, \dots$.

A similar representation can be obtained for the Bernoulli nnmber by merely substituting the last expression in (1).

REFERENCES

[1] L. Comtet, *Advanced Combinatorics*, D. Reidel Publishing Co., Boston 1974.

[2] L. M. Milne—Thomson, *The Calculus of Finite Differences*, Macmillan, London 1933.