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(Dedicated to the memory of Professor Arthur Erdélyi)

REDUCTION FORMULAE FOR SOME TRIPLE HYPERGEOMETRIC FUNCTIONS

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The functions considered in this paper belong to the class of triple Gaussian hypergeometric functions, i. e., functions that reduce to ${}_2F_1$ when ever exactly one variable is non-zero. Reduction formulae expressing some of these functions in terms of double Gaussian hypergeometric functions (the fourteen functions introduced by Appell and by Horn; *cf.*, *e.g.*, [1, § 5.7.1]), or even simpler functions, will be given. The parameters of the triple series will satisfy *two* conditions and the variables will be $(x, y, -y)$.

To state the results conveniently we shall use a modification of the earlier notations (*cf.* [4] and [5]). For a triple Gaussian hypergeometric series with the general term $C(m, n, p)\, x^m y^n z^p$, imagine the pattern

$$\begin{matrix} m, m\ ; n, n\ ; p, p \\ -m; \ -n; \ -p \end{matrix}$$

and the symbol

$$F\left[\begin{matrix} ,\ ;\ ,\ ;\ , \\ ;\ \ ; \end{matrix}\ \middle|\ x\ ; y\ ; z\right]$$

with nine blank spaces for the parameters. Each parameter is now written in the space (*s*) corresponding to its Pochhammer symbol subscript. An example will clarify the principle :

$$(1) \qquad F\left[\begin{matrix} a, a\,;\, b, d\,;\, c, d \\ e\,;\quad a\,;\quad a \end{matrix}\middle| x\,;y;\,z \right] =$$

$$= \sum_{m,n,p} \frac{(a)_{2m-n-p}\,(b)_n\,(c)_p\,(d)_{n+p}\,x^m\,y^n\,z^p}{(e)_m\,m\,!\,n\,!\,p\,!}.$$

If two parameters are equal, one will be enclosed in square brackets. All summations are from 0 to ∞.

The reduction formulae will be obtained from the result*

$$(2) \quad \sum_{m,n,p} \frac{A\,(m,n+p)\,(b)_n\,(b)_p\,x^m\,y^n\,(-y)^p}{m\,!\,n\,!\,p\,!} =$$

$$= \sum_{m,k} \frac{A\,(m,\,2k)\,(b)_K\,x^m\,y^{2k}}{m\,!\,k\,!},$$

which we applied to some similar problems in a previous paper [3] ; absolute convergence is, of course, assumed. Proofs will merely be outlined; the elementary rules for the Pochhammer symbol are, of course, applied.

The formula (2), which actually implies one parameter condition, is applicable to a few triple Gaussian series (the one considered in (1) is among these). In each case a higher-order double hypergeometric series S is obtained. S may reduce to a double Gaussian hypergeometric series after introduction of a suitable parameter condition; in this way the following formulae (as well as variants having the denominator parameters $2b$, $b+\frac{1}{2}$ replaced by $2b-1$, $b-\frac{1}{2}$) are found :

$$(3) \quad F\left[\begin{matrix} c, d\,;\, a, b\,;\, a, [b] \\ a\,;\quad 2b\,;\quad 2b \end{matrix}\middle| x\,;y\,;-y \right] = H_7\,(a, c, d, b+\tfrac{1}{2}\,;\,\tfrac{1}{4}y^2, x),$$

$$(4) \quad F\left[\begin{matrix} a, a\,;\, a, b\,;\, a, [b] \\ d\,;\quad 2b;\quad 2b \end{matrix}\middle| x\,;y\,;-y \right] = F_4\,(\tfrac{1}{2}a,\, \tfrac{1}{2}+\tfrac{1}{2}a,\, d,\, b+\tfrac{1}{2};\, 4x, y^2),$$

$$(5) \quad F\left[\begin{matrix} a, c\,;\, a, b\,;\, a, [b] \\ d\,;\quad 2b\,;\quad 2b \end{matrix}\middle| x\,;y\,;-y \right] = H_4\,(a, c, b+\tfrac{1}{2}, d\,;\,\tfrac{1}{4}y^2, x).$$

* Notice that (2) is an obvious triple-series analogue of an identity due to Srivastava [6, p. 297 (17)] who indeed gave several other classes of reduction formulas involving double series with essentially arbitrary terms.

In other cases, the double series is written in the form

$$(6)\qquad S=\sum_k B(k)\, y^{2k} F(k,x),$$

where F is a ${}_2F_1$ with variable depending upon x and parameters involving k. It may reduce to power functions after introduction of one parameter condition on account of [1, § 2.8 (5), (6) . This procedure leads to the following reduction formulae :

$$(7)\quad F\left[\begin{matrix} a, a ; a, b ; a, [b] \\ \tfrac{1}{2} ; \quad d ; \quad d \end{matrix}\middle| x ; y ; -y \right] =$$

$$= \tfrac{1}{2}(1+2\sqrt{x})^{-a}\, {}_3F_2(\tfrac{1}{2}a, \tfrac{1}{2}+\tfrac{1}{2}a, b ; \tfrac{1}{2}d, \tfrac{1}{2}+\tfrac{1}{2}d ; y^2 (1+2\sqrt{x})^{-2})$$
$$+\tfrac{1}{2}(1-2\sqrt{x})^{-a}\, {}_3F_2(\tfrac{1}{2}a, \tfrac{1}{2}+\tfrac{1}{2}a, b ; \tfrac{1}{2}d, \tfrac{1}{2}+\tfrac{1}{2}d ; y^2 (1-2\sqrt{x})^{-2}),$$

$$(8)\quad F\left[\begin{matrix} a, a ; d, b ; d, [b] \\ \tfrac{1}{2}; \quad a ; \quad a \end{matrix}\middle| x ; y ; -y \right] =$$

$$= \tfrac{1}{2}(1+2\sqrt{x})^{-a}\, {}_3F_2(\tfrac{1}{2}d, \tfrac{1}{2}+\tfrac{1}{2}d, b ; \tfrac{1}{2}-\tfrac{1}{2}a, 1-\tfrac{1}{2}a ; y^2(1+2\sqrt{x})^2)$$
$$+\tfrac{1}{2}(1-2\sqrt{x})^{-a}\, {}_3F_2(\tfrac{1}{2}d, \tfrac{1}{2}+\tfrac{1}{2}d, b ; \tfrac{1}{2}-\tfrac{1}{2}a, 1-\tfrac{1}{2}a ; y^2(1-2\sqrt{x})^2),$$

$$(9)\quad F\left[\begin{matrix} a, a ; 1-a, b ; 1-a, [b] \\ 1-a ; \quad a ; \quad a \end{matrix}\middle| x ; y ; -y \right] =$$

$$= (\tfrac{1}{2}+\tfrac{1}{2}\sqrt{(1+4x)}\,)^{1-a} (1-y^2 (\tfrac{1}{2}+\tfrac{1}{2}\sqrt{(1+4x)}\,)^2)^{-b}/\sqrt{(1+4x)},$$

$$(10)\quad F\left[\begin{matrix} a, a; -a, b ; -a, [b] \\ -a ; \quad a ; \quad a \end{matrix}\middle| x ; y ; -y \right] =$$

$$= (\tfrac{1}{2}+\tfrac{1}{2}\sqrt{(1+4x)}\,)^{-a}\, {}_2F_1 (-\tfrac{1}{2}a, b ; 1-\tfrac{1}{2}a; y^2(\tfrac{1}{2}+\tfrac{1}{2}\sqrt{(1+4x)})^2),$$

$$(11)\quad F\left[\begin{matrix} a, a ; a, b ; a, [b] \\ 1+a; 1+a; 1+a \end{matrix}\middle| x; y; -y \right] =$$

$$= (\tfrac{1}{2}+\tfrac{1}{2}\sqrt{(1-4x)}\,)^{-a}\, {}_2F_1 (\tfrac{1}{2}a, b ; 1+\tfrac{1}{2}a; 4y^2 (1+\sqrt{(1-4x)}\,)^{-2}).$$

Finally, to prove the formula

$$(12)\quad F\left[\begin{matrix} a, c ; d, 1-\tfrac{1}{2}a; d, [1-\tfrac{1}{2}a] \\ d ; \quad a ; \quad a \end{matrix}\middle| x; y ; -y \right] =$$
$$= (1+x)^{-c}\, H_7(d, 1-a-d, c, \tfrac{1}{2}-\tfrac{1}{2}a; \tfrac{1}{4} y^2, -x/1+x),$$

and the variant obtained by interchanging the parameters $1-\frac{1}{2}a$ and $\frac{1}{2}-\frac{1}{2}a$, we perform a linear transformation of $F(k,\varkappa)$ in (6) before introducing the second parameter condition.

Some of the functions considered have occurred in the literature. Thus, we have $D^{1,3}_{(3)}$ in (3), F_G in (5), ${}^{(1)}H^{(3)}_3$ in (11), and G_B in (12). (Definitions of these functions can be found in [2, Ch. 3].)

REFERENCES

[1] A Erdélyi et al., Higher Transcendental Functions, Vol. I McGraw-Hill, New York, 1953.

[2] H. Exton, Muliiple Hypergeometric Functions and Applications, Ellis Horwood, Chichester, 1976.

[3] P. W. Karlsson, Reduction of certain hypergeometric functions of three variables, Glasnik Mat. Ser. III **8** (1973), 199-204.

[4] S. Saran, Hypergeometric functions of three variables, Ganita **5** (1954), 77-91; corr. ibid. **7** (1956), 65.

[5] H. M. Srivastava, Some integrals representing triple hypergeometric functions, Rend. Circ. Mat. Palermo Ser. II **16** (1967), 99-115.

[6] H. M. Srivastava, On the reducibility of Appell's function F_4, Canad. Math. Bull. **16** (1973), 295-298.