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(*Dedicated to the memory of Professor Arthur Erdélyi*)

THE DISTRIBUTION OF A LINEAR COMBINATION AND THE RATIO OF PRODUCTS OF RANDOM VARIABLES ASSOCIATED WITH THE MULTIVARIABLE H-FUNCTION

By

S. P. Goyal and R. K. Agrawal

*Department of Mathematics, B. V. College of Arts and Science, Banasthali Vidyapith-***304022***, Rajasthan, India*

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ABSTRACT

The present paper deals with the probability density function of a linear combination and the ratio of products of random variables having as their probability density function in terms of the *H*-function of several complex variables, defined by H. M. Srivastava and R. Panda. The results obtained in this paper are quite general and useful in nature. The results established recently by A. M. Mathai and R. K. Saxena, R. K. Saxena and S. P. Dash, and several others, follow as particular cases of our findings.

1. INTRODUCTION

In a large variety of statistical problems such as total time of service required in a medical check-up, problem of inter-live-birth interval, etc., the distribution of a linear combination of random variables plays an important role. Again, the linear combination and the ratio of independent random variables (when density function belongs to the same family) are useful in the theory of sampling distribution. There is a vast litrature on the distribution of linear combination and the ratio of products of random variables, when individual variables are assumed to have particular type of probability density models. For example, Mathai and Saxena [2], Srivastava and Singhal [7], and Saxena and Dash [4] have discussed these problems, where each component variable is assumed to have a density associated with Gauss's ${}_2F_1$—function or Fox's H-function. In this paper, we have studied the aforementioned problems for the probability density function in terms of the multivariable H-function defined by Srivastava and Panda [5, p. 271, Eq. (4.1) *et seq.*]. The probability density function considered here contains (as particular cases) a large variety of such functions introduced in the literature from time to time. Thus our findings will unify and extend the results on linear combination and the ratio of products of random variables studied by several research workers. Indeed, as long as one can find the practical situations, when the introduction of a more general function is justifiable, the generalization can be put to practical use. The technique employed here to derive the results is that of the Laplace transform and its inverse.

The multivariable H-function occurring in this paper is a special case of the general multivariate H-function introduced and studied earlier by Srivastava and Panda ([5] and [6]). The parameters of this function will be displayed in the following contracted notation, which slightly differs from that of Srivastava and Panda [6, p. 130, Eq. (1.3)]:

$$H[x_1,\dots,x_r] = H^{0,n:1,n_1;\dots;1,n_r}_{p,q:p_1,q_1;\dots;p_r,q_r}\left[\begin{matrix} x_1 \\ \vdots \\ x_r \end{matrix}\middle|\begin{matrix} (a_j;\alpha_j',\dots,\alpha_j^{(r)})_{1,p} : (c_j',\epsilon_j')_{1,p_1};\dots;(c_j^{(r)},\epsilon_j^{(r)})_{1,p_r} \\ (b_j;\beta_j',\dots,\beta_j^{(r)})_{1,q} : (0,1),(d_j',\delta_j')_{2,q_1};\dots;(0,1),(d_j^{(r)},\delta_j^{(r)})_{2,q_r} \end{matrix}\right]$$

$$= (2\pi\omega)^{-r}\int_{L_1}\dots\int_{L_r}\phi(s_1,\dots,s_r)\prod_{i=1}^{r}\{\theta_i(s_i)\,\Gamma(-s_i)\,(x_i)^{s_i}\,ds_i\}, \qquad \dots(1.1)$$

where $\omega=\sqrt{-1}$, and

$$\phi(s_1,\dots,s_r) = \prod_{j=1}^{n}\Gamma\left(1-a_j+\sum_{i=1}^{r}\alpha_j^{(i)}s_i\right)$$

$$\cdot \left[\prod_{j=n+1}^{p} \Gamma \left(a_j - \sum_{i=1}^{r} \alpha_j^{(i)} s_i\right). \prod_{j=1}^{q} \Gamma \left(1-b_j+\sum_{i=1}^{r} \beta_j^{(i)} s_i\right) \right]^{-1}, \qquad \ldots(1.2)$$

$$\theta_i (s_i) = \prod_{j=1}^{n_i} \Gamma \left(1-c_j^{(i)} + \epsilon_j^{(i)} s_i\right) \left[\prod_{j=n_i+1}^{p_i} \Gamma \left(c_j^{(i)} - \epsilon_j^{(i)} s_i\right) \prod_{j=2}^{q_i} \Gamma \left(1 - d_j^{(i)} + \delta_j^{(i)} s_i\right) \right]^{-1} \qquad \ldots(1.3)$$

$$(i = 1, \ldots, r)$$

i in the superscript (i) stands for the number of primes, e. g, $b^{(1)} = b'$, $b^{(2)} = b''$, and so on ; $(a_j ; \alpha_j', \ldots, \alpha_j^{(r)})_{1,p}$ would abbreviate $(a_1; \alpha_1', \ldots, \alpha_1^{(r)})$, $\ldots, (a_p ; \alpha_p', \ldots, \alpha_p^{(r)})$ and $(c_j, \epsilon_j)_{n+1,p}$ the $(p-n)$ parmeter squence $(c_{n+1}, \epsilon_{n+1}), \ldots, (c_p, \epsilon_p)$ for integers n and p such that $0 \leqslant n \leqslant p$, and so on.

The conditions of convergence for the mnltiple contour integral (1.1), and other details for the H-function of several variables can be found in the papers by Srivastava and Panda ([5] and [6]).

2. Useful Results

The following results will be required in the course of our analysis :

Result 1

$$H[x_1, \ldots, x_r] = \sum_{v_1=0}^{\infty} \cdots \sum_{v_r=0}^{\infty} \emptyset (v_1, \ldots, v_r) \prod_{i=1}^{r} \left\{ \frac{\theta_i (v_i) (-x_i)^{v_i}}{v_i !} \right\} \qquad \ldots(2.1)$$

where $\emptyset (v_1, \ldots, v_r)$ is defined by (1.2) and $\theta_i (v_i)$ is defined by (1.3).

The above result follows easily from a series expansion given by Saxena [3, p. 225, Eq. 4.1)].

Result 2

$$\int_0^\infty \cdots \int_0^\infty \prod_{i=1}^{r} \left\{ (x_i)^{\sigma_i-1} \exp(-k_i x_i) \right\} H\left[y_1 x_1^{\lambda_1}, \ldots, y_r x_r^{\lambda_r} \right] dx_1 \ldots dx_r$$

$$= \prod_{i=1}^{r} \left\{ (k_i)^{-\sigma_i} \right\} H^{0, n : 1, n_1+1; \ldots; 1, n_r+1}_{p, q : p_1+1, q_1; \ldots; p_r+1, q_r} \left[\begin{array}{c} y_1 k_1^{-\lambda_1} \\ \vdots \\ y_r k_r^{-\lambda_r} \end{array} \right|$$

$$(a_j ; \alpha_j', \ldots, \alpha_j^{(r)})_{1,p} : (1-\sigma_1, \lambda_1), (c_j', \epsilon_j')_{1,p_1} ; \ldots ; (1-\sigma_r, \lambda_r),$$
$$(b_j ; \beta_j', \ldots, \beta_j^{(r)})_{1,q} : (0, 1), (d_j', \delta_j')_{2,q_1} ; \ldots ; (0, 1),$$

$$\left. \begin{array}{l} (c_j^{(r)}, \epsilon_j^{(r)})_{1,p_r} \\ (d_j^{(r)}, \delta_j^{(r)})_{2,q_r} \end{array} \right] = J(k_1, \ldots, k_r ; \sigma_1, \ldots, \sigma_r) \qquad \ldots(2.2)$$

where

$$\min_{1 \leq i \leq r} \{Re(k_i), Re(\sigma_i)\} > 0, \lambda_i > 0, U_i > 0, |\arg y_i| < (1/2) U_i \pi, \qquad \ldots(2.3)$$

and

$$U_i = -\sum_{j=n+1}^{p} \alpha_j^{(i)} - \sum_{j=1}^{q} \beta_j^{(i)} + \sum_{j=1}^{n_i} \epsilon_j^{(i)} - \sum_{j=n_i+1}^{p_i} \epsilon_j^{(i)} - \sum_{j=2}^{q_i} \delta_j^{(i)} + 1 \qquad \ldots(2.4)$$

The evaluation of the above result is quite straightforward, and therefore we omit the details.

Result 3. Let the probability density function

$$f_{x_1, \ldots, x_r}(k_1, \ldots, k_r ; \sigma_1, \ldots, \sigma_r) = \{ J(k_1, \ldots, k_r ; \sigma_1, \ldots, \sigma_r) \}^{-1}$$

$$\prod_{i=1}^{r} \left\{ (x_i)^{\sigma_i-1} \exp(-k_i x_i) \right\} H\left[y_1 x_1^{\lambda_1}, \ldots, y_r x_r^{\lambda_r} \right], \qquad \ldots(2.5)$$

provided that the inequalities given with (2.3) are satisfied, and

$f_{x_1, \ldots x_r}(k_1, \ldots, k_r ; \sigma_1, \ldots, \sigma_r) = 0$, elsewhere.

Since there exists at least one set of parameters for which the function $f_{x_1,\dots,x_r}(k_1,\dots,k_r;\sigma_1,\dots,\sigma_r)$ in (2.5) is non-negative, it is assumed that the parameters are such that

$f_{x_1,\dots,x_r}(k_1,\dots,k_r;\sigma_1,\dots,\sigma_r) \geqslant 0$ for $0 < x_i < \infty$ $(i=1,\dots,r)$.

It can be remarked here that, on putting $n = p = q = 0$, the function given by (2.5) breaks up into the probability density function involving several Fox's H-functions, which happens to be a particular case of the probability density function considered by Saxena and Dash [4]. Incidentally, the value of $C(\tau, \sigma, \xi, d)$ involved in the function taken by Saxena and Dash ([4], p. 59, Result 3) contains some misprints.

3. Distribution of a Linear Combination of Several Random Variables

The theorem given below gives us the distribution of a linear combination of several independent variables associated with the probability density function defined by (2.5).

Theorem I. *Let* X_i $(i=1,\dots,r)$ *be the* r *independent random variables, where* X_i *has the probability density function defined by* (2.5). *Then the probability function* h(u) *of*

$U=\sum_{i=1}^{r} z_i X_i$ *is as follows :*

$$h(u)=\sum_{v_1=0}^{\infty}\cdots\sum_{v_r=0}^{\infty}\phi(v_1,\dots,v_r)\prod_{i=1}^{r}\left\{\frac{\theta_i(v_i)\,\Gamma(R_i)\,(-y_i)^{v_i}\,(z_i)^{-R_i}}{v_i\,!}\right\}$$

$$\cdot\frac{u^{\sum_{i=1}^{r}(R_i)-1}\;\phi_2\left(R_1,\dots,R_r;\ \sum_{i=1}^{r}R_i;\ -\frac{k_1}{z_1}u,\dots,-\frac{k_r}{z_r}u\right)}{\Gamma\left(\sum_{i=1}^{r}R_i\right)\ J(k_1,\dots,k_r;\sigma_1,\dots,\sigma_r)},$$

where

$$R_i = \sigma_i + \lambda_i v_i \qquad \dots(3.2)$$

$$\phi_2(c_1,\ldots,c_r;d;z_1',\ldots,z_r') = \sum_{m_1=0}^{\infty}\cdots\sum_{m_r=0}^{\infty}\frac{(c_1)_{m_1}\cdots(c_r)_{m_r}(z_1')^{m_1}\cdots(z_r')^{m_r}}{(d)_{m_1+\ldots+m_r}\; m_1!\ldots m_r!}, \qquad \ldots(3.3)$$

and $\phi(v_1,\ldots,v_r)$ *and* $\theta_i(v_i)$ *are defined by* (1.2) *and* (1.3), *respectively. The density function (3.1) is valid under the following conditions :*

$\lambda_i > 0$, $\min_{1\leq i\leq r}\{\mathrm{Re}(k_i), \mathrm{Re}(\sigma_i)\} > 0$, $|\arg y_i| < (1/2)\,U_i\,\pi$, $(i=1,\ldots,r)$,

(U_i is given by (2.4) and the series on the right-hand side of (3.1) *converges absolutely.*

Proof : Let $\bar{\phi}(s,u)$ denote the Laplace transform of U. Then

$$\bar{\phi}(s,U) = E[\exp(-sU)] = E\left[\exp\left(-s\sum_{i=1}^{r} z_i X_i\right)\right]$$

$$= \frac{\int_0^\infty\cdots\int_0^\infty \prod_{i=1}^{r}\left[\exp\{-(k_i+sz_i)x_i\}(x_i)^{\sigma_i-1}\right] H\left[y_1 x_1^{\lambda_1},\ldots,y_r x_r^{\lambda_r}\right] dx_1\ldots dx_r}{J(k_1,\ldots,k_r;\sigma_1,\ldots,\sigma_r)}$$

$$= \frac{J(k_1+s z_1,\ldots,k_r+s z_r;\sigma_1,\ldots,\sigma_r)}{J(k_1,\ldots,k_r;\sigma_1,\ldots,\sigma_r)}, \qquad (3.4)$$

where E stands for "Mathematical expectation".

Expanding the multivariable H-function involved in numerator of (3.4) with the help of (2.1), we get, after a little simplification, that

$$\bar{\phi}(s,U) = \{J(k_1,\ldots,k_r;\sigma_1,\ldots,\sigma_r)\}^{-1}\sum_{v_1=0}^{\infty}\cdots\sum_{v_r=0}^{\infty}\phi(v_1,\ldots,v_r)$$

$$\prod_{i=1}^{r}\left\{\frac{\theta_i(v_i)\,\Gamma(R_i)\,(-y_i)^{v_i}\,(k_i+s z_i)^{-R_i}}{(v_i)!}\right\} \qquad \ldots(3.5)$$

Using a known result [1, p. 238, Eq. (9)] for finding out the inverse Laplace transform of the terms $\prod_{i=1}^{r}(k_i+sz_i)^{-R_i}$, involving s in (3.5), we arrive at the right-hand side of $h(u)$ given by (3.1),

4. Distribution of the Ratio of Products of several Random variables

Let $W = \frac{X_1 \ldots X_h}{X_{h+1} \ldots X_r}$ $(1 \leqslant h \leqslant r)$, where X_i $(i=1,\ldots, r)$ be the r independent variables and the probability density function for X_i is given by (2.5). The Theorem given below gives the probability density function $g(w)$ of W.

Theorem 2. *Let*

$$V = \log_e W = \sum_{i=1}^{h} \log_e X_i - \sum_{i=h+1}^{r} \log_e X_i .$$

Then the density function g (w) *is given by*

$$g(w) = \{ J(k_1, \ldots, k_h ; \sigma_1, \ldots \sigma_h) \, J(k_{h+1}, \ldots, k_r ; \sigma_{h+1}, \ldots, \sigma_r) \}^{-1}$$

$$\sum_{v_1=0}^{\infty} \cdots \sum_{v_r=0}^{\infty} \phi(v_1, \ldots, v_h) \, \phi(v_{h+1}, \ldots, v_r) \prod_{i=1}^{r} \left\{ \frac{\theta_i (v_i (-y_i)^{v_i} (k_i)^{-R_i}}{v_i !} \right\}$$

$$G^{h, r-h}_{r-h, h} \left(\frac{k_1 \ldots k_h}{k_{h+1} \ldots k_r} W \middle| \begin{matrix} (1-R_j)_{h+1, r} \\ (R_j)_{1, h} \end{matrix} \right). \qquad \ldots (4.1)$$

where $1 \leqslant h \leqslant r$ *and* R_i *is defined by* (3.2). *Incidently, the conditions of existenee for* g (w) *are same as those for* h (u) *given by Theorem 1.*

Proof : Suppose $\bar{\phi}(s, V)$ is the Laplace transform of V. Then

$$\bar{\phi}(s, V) = E[\exp(-sV)] = E\left[\prod_{i=1}^{h} (X_i)^{-s} \prod_{i=h+1}^{r} (X_i)^{s} \right]$$

$$= \left[\int_0^{\infty} \cdots \int_0^{\infty} f_{x_1, \ldots, x_h}(k_1, \ldots, k_h ; \sigma_1, \ldots, \sigma_h) \left\{ \prod_{i=1}^{h} (x_i)^{-s} \right\} dx_1 \ldots dx_h \right]$$

$$\cdot \left[\int_0^{\infty} \cdots \int_0^{\infty} f_{x_{h+1}, \ldots, x_r}(k_{h+1}, \ldots, k_r ; \sigma_1, \ldots \sigma_r) \left\{ \prod_{i=h+1}^{r} (x_i)^{s} \right\} dx_{h+1} \ldots dx_r \right]$$

$$= \frac{J(k_1,\ldots,k_h;\sigma_1-s,\ldots,\sigma_h-s)\,J(k_{h+1},\ldots,k_r;\sigma_{h+1}+s,\ldots,\sigma_r+s)}{J(k_1,\ldots,k_h;\sigma_1,\ldots,\sigma_h)\;J(k_{h+1},\ldots,k_r;\sigma_{h+1},\ldots,\sigma_r)} \qquad \ldots(4.2)$$

Expand the multivariate H-function involved in the numerator of (4.2) with the help of (2.1) and collect the terms involving s, which are

$$\prod_{i=1}^{h} (k_i)^s\,\Gamma(R)-s_i\cdot\prod_{i=h+1}^{r} (k_i^{-s})\,\Gamma(R_i+s)\,. \qquad \ldots(4.3)$$

Now taking the inverse Laplace transform of the expression (4.3) and substituting the value thus obtained in the expanded form of (4.2), we arrive at (4.1).

5. Special Cases

At the outset we should remark that the multivariable H-function defined by (1.1) includes a large variety of elementary special functions involving one or more variables as its particular cases. Thus for probability density function given by (2.5) is quite general in nature and from it all the known statistical distributions, such as generalized beta & gamma distributions, generalized F-distribution, student's t-distribution, normal distribution, exponential distribution, etc, can be derived as specialized or limiting cases of our distribution (2.5). Indeed for all these distributions, the probability density function for U and W can be obtained from (3.1) and (4.1) by suitably specializing the various parameters involved. We, however, prefer to omit the details.

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