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(*Dadicated to the memory of Professer Arthur Erdélyi*)

A TYPE OF GENERALISED OSCILLATORY REGRESSION

By

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SUMMARY

Bessel-Clifford function of quardratic argument are employed in a statistical time-series regression model to provide a type of least-squares fit which generalises the use of standard Fourier analysis. An example from econometrics is given and a further generalisation is indicated.

1. INTRODUCTION

If $Y_1(t)$ and $Y_2(t)$ constitute a fundamental system of solutions of a linear ordinary differential -equation of the second order, then we may model a statistical time-series by means of the expression

$$y(t) = A_1 Y_1(\lambda t) + A_2 Y_2(\lambda t) + \in(t), \qquad \text{...(1.1)}$$

where A_1, A_2 and λ are constants and where $\in(t)$ represents the correspon-ding residuals wh ch are to be minimised by considering the sum of their squares as usual. The model (1.1) is envisaged as a generalisation of the simple Fourier case, where Y_1 and Y_2 are then the circular func ions Cos and Sin, respectively. {See Bloomfield [1], p. 11, for example. } It is assu-med that the mean and any secular component of the time-series have been removed, and that the functions Y_1 and Y_2 are oscillatory over the range of the independent variable under consideration.

In this paper, we let the functions $Y_1(t)$ and $Y_2(t)$ take the form

$$\mathrm{Ec}(b, t) = {}_0F_1\left(-;\ \frac{1+b}{2};-\frac{t^2}{4}\right) \qquad \text{...(1.2)}$$

and $$Es\,(b, t) = t^{1-b}\ {}_0F_1\left(-\,;\ \frac{3-b}{2};\ -\frac{t^2}{4}\right), \qquad \text{...(1.3)}$$

where the parameter b is less than unity. We note that Ec and Es reduce, respectively, to Cos and Sin when we let b become zero.

The two functions defined by (1.2) and (1.3) are linearly independent solutions of the Bessel-Clifford equation provided that b is not equal to a negative odd integer because $Ec\,(b, t)$ is not defined under those circumstances. {See Hayek [4].}

The function ${}_0F_1\,(-\,;\,b\,;\,x)$ is a case of the generalised hypergeometric function of one variable defined by the relation

$${}_AF_B\left(\begin{matrix} a_1, \ldots, a_A\,; \\ b_1, \ldots, b_B\,; \end{matrix}\ x\right) = \sum_{r=0}^{\infty} \frac{(a_1)_r \cdots (a_A)_r}{(b_1)_r \cdots (b_B)_r} \frac{x^r}{r\,!}, \qquad \text{...(1.4)}$$

$$(a)_r = a\,(a+1)\,(a+2)\ldots(a+r-1),\ (a)_0 = 1. \qquad \text{...(1.5)}$$

{ See Slater [6], p. 40. }

It will be seen that the functions (1.2) and (1.3) are related to the Bessel function of the first kind, but are preferred in the present context because the corresponding fundamental system of solutions of the Bessel equation $J_\nu\,(t)$ and $J_{-\nu}(t)$, where ν is not an integer or zero, is such that one member vanishes at the origin and the other becomes infinite there, depending upon the sign of ν.

The use of these generalisations of the circular functions seems to be most useful in dealing with relatively small segments of a longer time-series, and the least-squares fit can often dramatically improved by suitably adjusting the parameter b.

2. A Simple Least-Squares Model—When applying the method of least-squares to the model.

$$y\,(t) = Ac\ Ec\,(b, \lambda\,t) + As\ Es\,(b, \lambda\,t) + \epsilon(t), \qquad \text{.. (2.1)}$$

we minimise the expression

$$T = \sum_{r=0}^{n} \{\,y\,(t_r) - Ac\ Ec\,(b, \lambda\,t_r) - As\ Es\,(b, \lambda\,t_r)\,\}^2 \qquad \text{...(2.2)}$$

where $n+1$ is the number of elements of the time-series, which are taken to be equally spaced. After putting the partial derivatives $\frac{\partial T}{\partial Ac}$ and $\frac{\partial T}{\partial As}$ equal to zero we may solve the resulting normal equations which yield

$$Ac = \frac{\sum_{r=0}^{n} y(t_r)\, Ec\,(b,\lambda t_r) \sum_{r=0}^{n} \{Es\,(b,\lambda t_r)\}^2 - \sum_{r=0}^{n} y(t_r) Es(b,\lambda t_r) \sum_{r=0}^{n} Es(b,\lambda t_r) Ec(b,\lambda t_r)}{\sum_{r=0}^{n} \{Ec\,(b,\lambda\, t_r)\}^2 \sum_{r=0}^{n} \{Es\,(b,\lambda t_r)\}^2 - \left\{\sum_{r=0}^{n} Es\,(b,\lambda\, t_r)\; Ec\,(b,\lambda\, t_r)\right\}^2} \qquad \ldots(2.3)$$

and

$$As = \frac{\sum_{r=0}^{n} y(t_r)\, Es\,(b,\lambda t_r) \sum_{r=0}^{n} \{Ec\,(b,\lambda t_r)\}^2 - \sum_{r=0}^{n} y(t_r) Ec'b,\lambda t_r) \sum_{r=0}^{n} Es(b,\lambda t_r) Ec(b,\lambda t_r)}{\sum_{r=0}^{n} \{Ec\,(b,\lambda\, t_r)\} \sum_{r=0}^{n} \{Es\,(b,\lambda t_r)\}^2 - \left\{\sum_{r=0}^{n} Es\,(b,\lambda\, t_r)\; Ec\,(b,\lambda\, t_r)\right\}} \qquad \ldots(2.4)$$

The values of λ and b which minimise the function T must then be obtained by numerical computation. When $b = 0$, T will generally be approximately minimised when $\sqrt{A_C^2 + A_S^2}$ is maximised, a fact which is exploited in classical periodogram analysis. For general values of b, however, this does not hold, and the exact expression for T must be used.

3. Practical Computation —The series representations (1.2) and (1.3) converge quite rapidly when the argument is not too large. In fact, these functions may be computed to an accuracy of two decimal places for $t \leqslant 14$, and a greater degree of accuracy would seldom if ever be required in practice. For larger values of t an asymptotic expansion could be employed, but since circular functions feature prominently in this expansion, no additional advantage will accrue for $t > 14$, approximately. This is the main reason why the generalised analysis discussed here is best applicable to short segments of time-series where each segment does not include more than about two cycles of the principal oscillatory component.

The author has found, in practice, that each segment should be in length somewhat less than one cycle and more than one-half of a cycle of the principal component. From a numerical point of view the behaviour of the fnnction T is more sensitive to changes in λ than to changes in b, so that we may find the approximate value of λ using circular functions directly which is a much quicker procedure that using series expansions. This gives the optimum minimisation of T for $b=0$. The computation is then repeated for progressively varying values of b within the range of λ already determined.

As an example, we use the time-series of relative prices of the export of steel bars from the U. S. A. for the period 1957–1965 given in the following table :—

1957	1958	1959	1960	1961	1962	1963	1964	1965
1.06	1.02	1.15	1.12	1.10	1.13	1.23	1.24	0.99

{ See Tarr [7].}

After removal of the mean and multiplying by 100 for convenience, we obtain the time-series.

$$-6.7, -0.7, 2.3, -0.7, -2.7, 0.3, 10.3, 11.3, -13.7.$$

The parameter λ is computed for $b=0$ by the straight forward use of circular functions and b is then varied until the sum of the squares, T, is minimised. The results are set out below :

T	λ	b
185.95	11.9	0
151.04	12.0	—0.5
96.03	12.0	—2 5
79.16	11.9	—3.5
61.94	10.8	—4.5
53.79	10.7	—5.5
50.82	10.7	—6.5
51.21	10.7	—7.5
50.65	10.70	—6.8

The values of the constants Ac and As corresponding to the last entry for optimised T are -0.18 and -9.61×10^{-5}, respectively.

One interpretation of the period of the principal local oscillation of a time-series whose segments are analysed by the above method may be taken to be twice the interval between the first two positive zeros of the function

$$F(t) = Ac\, Ec\,(b, \lambda\, t) + As\, Es\,(b, \lambda\, t), \quad \text{...(3.1)}$$

multiplied by a factor equal to the length of time occupied by the segment of the time-series under discussion. An indication of the amplitude of this component may be furnished by taking the mean of the moduli of the turning values of F (t) in the interval $0 \leqslant t \leqslant 1$. The local period and amplitude of the above series were found to be 5.98 years and 7 units respectively. All these calculations were carried out by means of a micro-computer.

4. A Further Generalisation. Conclusion. A basic analogue of the system $Ec\,(b, t)$ and $Es\,(b, t)$ is given by

$$Ec\,(q\,;\, b, t) = \sum_{r=0}^{\infty} \frac{q^{\frac{1}{2}r(r-1)}(-t^2/4)^r}{[\frac{1}{2}+\frac{1}{2}b]_r\,[r]\,!} \quad \text{.. (4.1)}$$

and

$$Es\,(q\,;\, b, t) = \sum_{r=0}^{\infty} \frac{q^{\frac{1}{2}r(r-1)}(-t^2\, q^{\frac{1}{2}-\frac{1}{2}b}/4)^r}{[3/2-\frac{1}{2}b]_r\,[r]\,!}, \quad \text{...(4.2)}$$

where q is theoretically any uumber, real or complex, called the base. Also

$$[a]_r = [a]\,[a+1]\,[a+2]\,\ldots\,[a+r-1], \qquad [a]_0 = 1,$$

$$[r]\,! = [1]\,[2]\,[3]\,\ldots \qquad \text{and } [a] = (1-q^a)/(1-q).$$

{ See Jackson [5], for example.}

In the present application, the base is taken to be real and restricted to the range $0 < q \leqslant 1$. When $q \rightarrow 1$, $Ec\,(q\,;\, b, t)$ and $Es\,(q\,;\, b, t)$ tend to $Ec\,(b, t)$ and $Es\,(b, t)$, respectively. The functions (4.1) and (4.2) are for general values of b linearly independent solutions of a basic analogue of the Bessel-Clifford equation. {See Exton [2] and [3]. }

The basic functions $Ec\,(q\,;\, b, t)$ and $Es\,(q\,;\, b, t)$ may be used as a basis for the model (1.1), and it is envisaged that segments of time series will arise in which the corresponding least-squares fit will be further improved by varying q as an additional parameter after first finding the optimum values of λ and b.

The method outlined in this paper constitutes a generalisation of complex demodulation of statistical time-series ; see Bloomfield [1], Chapter 6. An estimate of the spectrum of each segment of the time series under discussion may be built-up by repeated application of the above analysis.

REFERENCES

[1] P. Bloomfield, *Fourier Analysis of Time-Series*, Wiley, New York, 1976.

[2] H. Exton On a basic analogue of the generalised Laguèrre equation, Funkcial Ekvac., **20** (1977), 1–8.

[3] H. Exton, A basic analogue of the Bessel-Clifford equation, Jñānabha **8** (1978),

[4] N. Hayek, Estudio de la ecuacion differencial $xy''+(a+1)y'+y=0$ y de sus applicaciones, Collect. Math. **18** (1967), 57-174.

[5] F. H. Jackson, On q-functions and a certain difference operator, Trans. Roy. Soc. Edinburg **46** (1908), 253-281.

[6] L. J. Slater, *Generalised Hypergeometric Functions*, Cambridge University Press, 1966.

[7] D. G. Tarr, Cyclical dumping : the case of steel production, J. International. Econ. **9** (1979), 57-63.