Jñānabha, Vol. 9/10, 1980

(*Dedicated to the memory of Professor Arthur Erdélyi*)

COMPARISON OF THE ORDER PROPERTY OF A GENERALIZED LAPLACE-STIELTJES INTEGRAL

by

SUSHIL KUMAR SINHA

Qtrs. No. H/45, Hinoo, Ranchi-834002, Bihar, India

(Received : November 20, 1979; Revised : April 28, 1980)

ABSTRACT

We establish theorems concerning the order condition of $A_\alpha(t)$ as $t \to \infty$, by investigating the order relations existing between different expressions involving $A_\alpha(t)$, the α-th Cesáro sum of the determining function $A(t)$ from the summability property of the integral

$$\int_0^\infty (2st)^\lambda \, W_{k,m}(2st) \, dA(t)$$

when the integral is summable $|C, \alpha|$. Here $W_{k,m}$ is the Whittaker function [1], and $A(t)$ is a function of bounded variation in every finite interval $0 \leqslant t \leqslant T$, and $s = \sigma + iT$ is a complex variable. We shall also give a formula for the abscissa of absolute summability of the integral whenever it is positive.

I. INTRODUCTION

Goyal [3] gave a generalization of the Laplace-Stieltjes integral

$$(1.1) \qquad f(s) = \int_0^\infty e^{-st} \, dA(t)$$

in the form :

$$(1.2) \qquad f(s) = \int_0^\infty (2st)^\lambda \, W_{k,m}(2st) \, dA(t)$$

where $W_{k,m}$ is Whittaker function [1], and $A(t)$ is a function of bounded variation in any finite interval $0 \leqslant t \leqslant T$.

When $\lambda = -1/4$, $k = 1/4$, and $m = \pm 1/4$, the transform (1.2) reduce to (1.1).

Bosanquet [2] gave the order condition of $A_\alpha(t)$ as $t \to \infty$, by investigating the order relations existing between different expressions involving $A_\alpha(t)$, when the integral (1.1) is sumable $| C, \alpha |$.

The purpose of the present paper is to obtain similar theorems for (1.2), and also to deduce a formula for the abscissa of absolute summability of integral (1 2) whenever it is positive.

2 NOTATIONS

We introduce the following notations :

(i) $\emptyset_{k,m}(2st) = (2st)^{\lambda} W_{k,m}(2st)$

(ii) $[\alpha]$ denotes the integral part of α. If $\alpha > 1$, we denote by h the greatest integer less than α ; if $0 < \alpha \leqslant 1$, then $h=0$, and if $\alpha=0$, then $h=-1$.

(iii) σ and T stand for the real and imaginary parts of the complex number s, $s = \sigma + iT$.

(iv) k and m are taken to be real.

(v) In (1.2), we shall take $k + \lambda \geqslant 0$, unless otherwise stated.

(vi) $A_\alpha(x)$ denotes the α-th Cesáro sum of $A(x)$.

3. Main Results

Teorem 3.1 If $\alpha \geqslant 0$, $\sigma_0 > 0$ and

$$x^{-\alpha} \emptyset_{k,m}(2s_0 x) A_\alpha(x) = o(1) \mid C, 0 \mid, (x \to \infty),$$

then $$\int_1^x | d[u^{-\alpha} A_\alpha(u)] | = o[2\sigma_0 x)^{-\lambda-k} e^{\sigma_0 x}].$$

Proof : We set $g(x) = x^{-\alpha} \emptyset_{k,m}(2s_0 x) A_\alpha(x)$

$$h(x) = x^{-\alpha} A_\alpha(x)$$

$$\text{and } g^{*}(x) = \int_{x}^{\infty} | \, dg\,(u) \, |.$$

Then $| g(x) | \leqslant \overset{*}{g}(x) = o\,(1)$, and hence by the use of a Theorem of Pollard ([4], 261) and by integration by part, we have

$$\int_{1}^{x} | \, dh\,(u) \, | = \int_{1}^{x} | \, d\,[u^{-\alpha} A_{\alpha}\,(u)\,] \, |$$

$$\leqslant \int_{1}^{x} | \, d\,[\,g\,(u) \,/\, \emptyset_{k,m}\,(2s_0 u)\,] \, |$$

$$\leqslant [\,1 \,/\, | \, \emptyset_{k,m}\,(2s_0 1) \, | \,] \; \overset{*}{g}\,(1)$$

$$+ 2 \int_{1}^{x} \overset{*}{g}\,(u) \, | \, (d/du) \, [1/\emptyset_{k,m}\,(2s_0 u)\,] \, | \, du.$$

$$= o\,[\,(2\,\sigma_0\,x)^{-\lambda-k} \; e^{\sigma_0 x}\,], \qquad (x \to \infty)$$

(By [5] ; Lemma 3.3)

Thus the theorem is completely established.

Theorem 3.2 If $\alpha \geqslant 0$ and $\sigma_0 > 0$ and if

(3.1) $$\int_{1}^{x} | \, dA_{\alpha}\,(u) \, | = o\,[\,x^{\alpha}\,(2\,\sigma_0\,x)^{-\lambda-k}\,e^{\sigma_0 x}\,], \; (x \to \infty),$$

then

(3.2) $$\int_{1}^{x} | \, d\,[u^{-\alpha}\,A_{\alpha}\,(u)] \, | = o\,[\,(2\,\sigma_0\,x)^{-\lambda-k}\,e^{\sigma_0 x}\,],$$

and conversely.

Proof : We first assume that (3.1) holds, and $\overset{*}{A}(x) = \int_{1}^{x} | \, dA_{\alpha}\,(u) \, |$;

then $| A_{\alpha}\,(x) | \leqslant | A_{\alpha}\,(1) | + \overset{*}{A}(x) = o\,[\,x^{\alpha}\,(2\,\sigma_0 x)^{-\lambda-k}\,e^{\sigma_0 x}\,]$

and hence by the use of a Theorem of Pollard ([4], 261) and by integration by part, we have

$$\int_{1}^{x} | \, d\,[\,u^{-\alpha}\,A_{\alpha}\,(u)\,] \, |$$

$$\leqslant x^{-\alpha} \overset{*}{A}(x) + \int_1^x \alpha\, u^{-\alpha-1} [\, \overset{*}{A}(u) + | A_\alpha (u) | \,] \, du$$

$$= o[(2\sigma_0 x)^{-\lambda-k} e^{\sigma_0 x}] + o[x^{-1} (2\sigma_0 x)^{-\lambda-k} e^{\sigma_0 x}]$$

$$= o[(2\sigma_0 x^{-\lambda-k} e^{\sigma_0 x}]$$

which gives (3.2).

Conversely, we assume that (3.2) holds, and

$$C_\alpha (x) = [A_\alpha (x) / x^\alpha];$$

then $\int_1^x | dC_\alpha (u) | = \int_1^x | d [u^{-\alpha} A_\alpha (u)] |$

$$= o[(2\sigma_0 x)^{-\lambda-k} e^{\sigma_0 x}].$$

We set $\overset{*}{C}(x) = \int_1^x | dC_\alpha (x) | = o[(2\sigma_0 x)^{-\lambda-k} e^{\sigma_0 x}]$

then $| C_\alpha (x) | = \left| \int_0^x dC_\alpha (x) \right| \leqslant C_\alpha (1) + \overset{*}{C}(x)$

$$= o[(2\sigma_0 x)^{-\lambda-k} e^{\sigma_0 x}], \; (x \to \infty).$$

Now $\int_0^x | dA_\alpha (x) | \leqslant \int_0^1 | dA_\alpha (u) | + \int_1^x | d [u^\alpha C_\alpha (u)] |$

$$\leqslant \int_0^l | dA_\alpha (u) | + x^\alpha \overset{*}{C}(x) + \alpha \int_1^x u^{\alpha-1} [\, \overset{*}{C}(u) + | C_\alpha (u) | \,] \, du$$

$$= o[x^\alpha (2\sigma_0 x)^{-\lambda-k} e^{\sigma_0 x}], \; (x \to \infty),$$

which gives (3.1).

Thus the theorem is completely established.

Theorem 3.3 If $\alpha \geq 0$, $\sigma_0 > 0$ and the integral (1.2) is summable $| C, \alpha |$ for $s = s_0$, then

$$\int_0^x | dA_\alpha (u) | = o[x^\alpha (2\sigma_0 x)^{-\lambda-k} e^{\sigma_0 x}], \; (x \to \infty)$$

Proof : Applying [7] ; Theorem 1, to our assumption, we have

$$x^{-\alpha} \emptyset_{k,m} (2s_0 x) A_\alpha (x) = o(1) \mid C, 0 \mid, (x \to \infty).$$

This implies, by virtue of Theorem 3.1, that

$$(3.3) \quad \int_1^x \mid d [u^{-\alpha} A_\alpha (u)] \mid = \int_1^x \mid d [u^{-\alpha} \{ u^\alpha / \emptyset_{k,m} (2s_0 u) \}] \mid$$

$$= o [(2 \sigma_0 x)^{-\lambda - k} e^{\sigma_0 x}] \quad (x \to \infty).$$

Hence, by applying the converse part of Theorem 3.2 to (3.3), we get

$$\int_0^x dA_\alpha (u) \mid = o [x^\alpha (2 \sigma_0 x^{-\lambda - k} e^{\sigma_0 x}], (x \to \infty).$$

Hence the theorem.

4. Abscissa of Absolute Summability

The following theorem gives the existence and expression for the abscissa of absolute summability of the integral (1.2) whenever it is positive.

Theorem 4.1 If $\alpha \geqslant 0, \bar{\sigma}_\alpha$ is the abscissa of absolute summability $\mid C, \alpha \mid$ of (1.2) and if $\bar{\sigma}_\alpha \geqslant 0$, then

$$\bar{\sigma}_\alpha = \overline{\lim_{x \to \infty}} \left[(1/x) \log \int_0^x \mid dA_\alpha (x) \mid \right].$$

Proof : Let

$$(4.1) \quad \overline{\lim_{x \to \infty}} \left[(1/x) \log \int_0^x \mid dA_\alpha (u) \mid \right] = \sigma_a$$

then we have $(1/x) \log \int_0^x \mid dA_\alpha (u) \mid < \sigma_a + \epsilon$, where ϵ is an arbitrary small positive number.

Hence, from [6] ; Theorem 2, the integral (1.2) is summable $\mid C, \alpha \mid$ for $\sigma > \sigma_a$.

Hence

$$(4.2) \quad \bar{\sigma}_\alpha \leqslant \sigma_a .$$

Now, if possible, let the integral (1.2) be summable $| C, \alpha |$ for $\sigma < \sigma_a - \epsilon$, $\epsilon > 0$, then by theorem 3.3, we have

$$(1/x) \log \int_0^x | dA_\alpha (x) | < \sigma_a - \epsilon,$$

which contradicts (4.1).

Hence, we have $\bar{\sigma}_\alpha = \sigma_a$.

This establishes the theorem.

Acknowledgements

I am thankful to Dr. G. L. Das for guidance in the preparation of this paper, and to Prof. H. M. Srivastava of the University of Victoria Canada, for his valuable suggestions for improvement of the paper.

REFERENCES

[1] E. T. Whittaker and G. N. Watson, *Modern Analysis*, Fourth edition, Cambridge University Press, 1962.

[2] L. S. Bosanquet, The summability of Laplace integrals, Proc. London Math. Soc. (3) **3** (1953), 267-304.

[3] R. P. Goyal, *A study of certain functional transformations*, Ph. D. Thesis, Agra University, India, 1966.

[4] S. Pollard, On the criteria for the convergence of a Fourier series, J. London Math. Soc. 2 (1927), 255-262.

[5] Sushil Kumar Sinha, Abelian theorem of the abscissa of summability of a generalized Laplace-Stieltjes integral, J. Inst. Sci (Nepal) **2** (1979), 67-82.

[6] Sushil Kumar Sinha, Summability and absolute summability property of a generalized Laplace-Stieltjes integral, Communicated for publication.

[7] Sushil Kumar Sinha, Absolute summability property of a generalized Laplace-Stieltjes integral, Communicated for publication.