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(Dedicated to the memory of Professor Arthur Erdélyi)

ON MULTIDIMENSIONAL INTEGRAL TRANSFORMS

By

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ABSTRACT

In this paper, two general theorems involving multidimensional integral transforms have been established. The first expresses an interesting relationship between images and originals of related functions in multiple integral transforms, while the second reveals the interconnections between images of related functions in the respective transforms. These theorems are sufficiently general in nature and shed light on the basic structure of integral transforms involved. They unify and extend a number of scattered theorems in the literature involving single and double integral transforms. Again by taking specific transforms in our theorems we can obtain a large number of results which may prove to be useful in solving certain boundary value problems.

1. INTRODUCTION AND DEFINITIONS

Let $f(x_1, \ldots, x_r)$ belong to a prescribed class of real or complex—valued functions or r real variables $x_1, \ldots, x_r$ defined over the region $R : 0 \leqslant x_i < \infty, i = 1, \ldots r$. Then (in line with the definition of linear integral transforms), the multidimensional integral transform $T\{f(x_1,\ldots,x_r) ; p_1, \ldots, p_r\}$ of the function $f(x_1, \ldots, x_r)$ is defined and represented as follows :

$$\phi(p_1, \ldots, p_r) = T\{f(x_1, \ldots, x_r); p_1, \ldots, p_r\}$$

$$= \int_0^{\infty} \cdots \int_0^{\infty} k(x_1, \ldots, x_r; p_1, \ldots, p_r)\, f(x_1, \ldots, x_r)\, dx_1 \ldots dx_r, \qquad \ldots (1.1)$$

where the functions $f(x_1, \ldots, x_r)$ and $k(x_1, \ldots, x_r; p_1, \ldots, p_r)$ and the parameters $p_1, \ldots, p_r$ are always so chosen that the integral (1.1) is absolutely convergent, i. e.

(i) The product $(x_1)^{c_1} \ldots (x_r)^{c_r} f(x_1, \ldots, x_r)$ is integrable (in the sense of Lebesgue) over every finite region

$R(R_1, \ldots, R_r): \leqslant x_i \leqslant R_i,\ R_i > 0,\ i = 1, \ldots, r$

where $k(x_1, \ldots, x_r; p_1, \ldots, p_r)$

$= O(x_1^{c_1} \ldots x_r^{c_r}),\ \max\{x_1, \ldots, x_r\} \to 0$

and

(ii) the limit of the finite form of the multiple integral in (1.1) with

$\int_0^{\infty} \cdots \int_0^{\infty}$ replaced by $\int_0^{R_1} \cdots \int_0^{R_r}$,

exists at the point $(p_1, \ldots, p_r)$ when $R_1, \ldots, R_r \to \infty$.

For a specific transform, $k(x_1, \ldots, x_r; p_1, \ldots p_r)$ is a definite function of $x_1, \ldots, x_r, p_1, \ldots, p_r$ and is known as the kernel of the transform. Also $\phi(p_1, \ldots, p_r)$ is called the image of the function $f(x_1, \ldots, x_r)$ in the transform defined by (1.1), and $f(x_1, \ldots, x_r)$ the original. For a systematic study of two general classes of multidimensional integral transformations, in which the kernels involve the H-functions of several variables, the reader is referred to a series of recent papers by Srivastava and Panda [7] who also cite a number of special instances of their transforms in the literature (cf. [7], Part I, p. 119; see also pp. 122-124).

For an integral transform of the type given by (1.1), it is easy to verify that if

$$\phi_1(p_1, \ldots, p_r) = T\{f_1(x_1, \ldots, x_r); p_1, \ldots, p_r\} \qquad \ldots (1.2)$$

and

$$\phi_2(p_1, \ldots, p_r) = T\{f_2(x_1, \ldots, x_r); p_1, \ldots p_r\} \qquad \ldots (1.3)$$

then, under appropriate conditions of convergence of the integrals involved, the following formula analogous to the Parseval-Goldstein type of formula for the Laplace transform in one and more variables holds :

$$\int_0^\infty \cdots \int_0^\infty f_2(x_1, \ldots, x_r)\, \emptyset_1(x_1, \ldots, x_r)\, dx_1 \ldots dx_r$$

$$= \int_0^\infty \cdots \int_0^\infty f_1(x_1, \ldots, x_r)\, \emptyset_2(x_1, \ldots, x_r)\, dx_1 \ldots dx_r. \qquad \ldots(1.4)$$

The above result will be referred to as the generalized Parseval-Goldstein formula and will be required in the sequel. For two interesting special forms of (1.4), see Srivastava and Panda [7, Part I, p. 129, Theorem 3].

2. We first establish a theorem which exhibits an interesting relationship between images and originals of related functions in the transforms T_1 and T_2 defined below

$$T_1\{f(x_1, \ldots, x_r); p_1, \ldots, p_r\}$$

$$= \int_0^\infty \cdots \int_0^\infty f(x_1, \ldots, x_r)\, k_1(p_1 x_1, \ldots, p_r x_r)\, dx_1 \ldots dx_r, \qquad \ldots(2.1)$$

$$T_2\{f(x_1, \ldots, x_r); p_1, \ldots, p_r\}$$

$$= \int_0^\infty \cdots \int_0^\infty f(x_1, \ldots, x_r)\, k_2(p_1 x_1, \ldots, p_r x_r)\, dx_1 \ldots dx_r, \qquad \ldots(2.2)$$

provided that the multiple integrals involved in (2.1) and (2.2) convergeo absolutely. Our result may be stated as

Theorem I. *If*

$$h_1(p_1, \ldots, p_r) = T_1\{h_2(x_1, \ldots, x_r)\, g(x_1, \ldots, x_r); p_1, \ldots, p_r\}, \qquad \ldots(2.3)$$

and

$$h_2(p_1^{\sigma_1}, \ldots, p_r^{\sigma_r}) = T_2\{f(x_1, \ldots, x_r); p_1, \ldots, p_r\}, \qquad \ldots(2.4)$$

then

$$h_1(p_1, \ldots, p_r) = \{\sigma_1 \ldots \sigma_r\} \int_0^\infty \cdots \int_0^\infty f(x_1, \ldots, x_r)$$

$$\cdot\, \emptyset(x_1, \ldots, x_r, p_1, \ldots, p_r)\, dx_1 \ldots dx_r, \qquad \ldots(2.5)$$

where

$$\emptyset(p_1,\ldots,p_r,\alpha_1,\ldots,\alpha_r)=T_2\{x_1^{\sigma_1-1}\ldots x_r^{\sigma_r-1}\,g(x_1^{\sigma_1},\ldots,x_r^{\sigma_r})$$

$$.k_1(\alpha_1x_1^{\sigma_1},\ldots,\alpha_r x_r^{\sigma_r});p_1,\ldots,p_r\}, \qquad \ldots(2.6)$$

the transforms T_1 *and* T_2 *are defined by means of equations* (2.1) *and* (2.2), $\sigma_1,\ldots\sigma_r$ *are non-zero real numbers of the same sign, each* $\alpha_1,\ldots,\alpha_r$ *is independent of* $p_1,\ldots,p_r$ *and all the multiple integrals involved in* (2.3) *to* (2.6) *are assumed to be absolutely convergent.*

Proof. Applying the generalized Parseval-Goldstein formula given by (1.4) to the operational pairs (2.4) and (2.6), we get

$$\int_0^\infty\ldots\int_0^\infty\{x_1^{\sigma_1-1}\ldots x_r^{\sigma_r-1}\}\,g(x_1^{\sigma_1},\ldots,x_r^{\sigma_r})\,h_2(x_1^{\sigma_1},\ldots,x_r^{\sigma_r})$$

$$.k_1(\alpha_1x_1^{\sigma_1},\ldots,\alpha_r x_r^{\sigma_r})\,dx_1\ldots dx_r$$

$$=\int_0^\infty\ldots\int_0^\infty\emptyset(x_1,\ldots,x_r,\alpha_1,\ldots,\alpha_r)\,f(x_1,\ldots,x_r)\,dx_1\ldots dx_r, \qquad \ldots(2.7)$$

Now replacing $\alpha_1,\ldots,\alpha_r$ by $p_1,\ldots,p_r$ in (2.7), changing the variables of integration slightly on its left-hand side, and interpreting the result thus obtained in terms of (2.3), we easily arrive at Theorem 1.

The presence of the arbitrary function $g(x_1,\ldots,x_r)$, and the general nature of kernel involved in the above theorem, enable it to yield general, deeper and useful results. Thus a large number of results, which express relationships between images and originals of related functions in any two integral transforms of the type given by (2.1) and (2.2), found recently by several authors and scattered in the literature, are all unified and extended by this theorem.

3. Special Cases of Theorem I

If in the above theorem, we put $\sigma_1=\sigma_2=\ldots=\sigma_r=1$ and take both the transforms T_1 and T_2 as Laplace transforms of r variables, $\emptyset(p_1,\ldots,p_r,\alpha_1,\ldots,\alpha_r)$ occurring in (2.6) can be given a slightly altered form In such a case we have

$$\emptyset(p_1,\ldots,p_r,\alpha_1,\ldots,\alpha_r)=L\{g(x_1,\ldots,x_r)\,e^{-\alpha_1x_1-\ldots-\alpha_rx_r};p_1,\ldots,p_r\}$$

$$=\theta(p_1+\alpha_1,\ldots,p_r+\alpha_r), \qquad \ldots(3.1)$$

where

$$\theta(p_1,\ldots,p_r) = L\{g(x_1,\ldots,x_r);p_1,\ldots,p_r\}$$

$$=\int_0^\infty \ldots \int_0^\infty e^{-p_1x_1-\ldots-p_rx_r}\, g(x_1,\ldots,x_r)\,dx_1\ldots dx_r\,, \quad \ldots(3.2)$$

and the theorem takes the following interesting form :

Corollary I. *If*

$$h_1(p_1,\ldots,p_r) = L\{g(x_1,\ldots,x_r)\,h_2(x_1,\ldots,x_r);p_1,\ldots,p_r\}, \quad \ldots(3.3)$$

and

$$h_2(p_1,\ldots,p_r) = L\{f(x_1,\ldots,x_r);p_1,\ldots,p_r\}, \quad \ldots(3.4)$$

then

$$h_1(p_1,\ldots,p_r) = \int_0^\infty \ldots \int_0^\infty f(x_1,\ldots,x_r)\,\theta(x_1+p_1,\ldots,x_r+p_r)\,dx_1\ldots dx_r \quad \ldots(3.5)$$

where

$$\theta(p_1,\ldots,p_r) = L\{g(x_1,\ldots,x_r);p_1,\ldots,p_r\}, \quad \ldots(3.6)$$

$\mathrm{Re}(p_i) > 0$ $(i=1,\ldots,r)$, *and the various multiple integrals involved in (3.3) to (3.6) are absolutely convergent.*

The special case of Theorem 1, when $r=2$, is of interest in ilself Thus on suitable choices of the corresponding transforms T_1, T_2, and the function $g(x, y)$, we easily get the known results obtained earlier by Bose [2, p. 176], Goyal [3. p. 139], and Jain [6, p. 314]. The analogue of Theorem 1 for integral transforms involving one variable, i. e. when $r=1$, is also quite interesting. It generalizes a theorem of Agrawal [1, p. 538] and a very large number of other results scattered in the literature, as we pointed out in an earlier paper [4].

4. Now we establish another theorem which reveals the inter-connections between images and related functions in the transforms defined by (2.1) and (2.2). We first state

Theorem 2. *If*

$$h_1(p_1,\ldots,p_r) = T_1\{x_1^{(c_1/\sigma_1)-1}\ldots x_r^{(c_r/\sigma_r)-1} f(x_1,\ldots,x_r);p_1,\ldots,p_r\} \quad \ldots(4.1)$$

and

$$h_2(p_1,\dots,p_r) = T_2\ \{ f(x_1^{-\sigma_1},\dots,x_r^{-\sigma_r})\ ;\ p_1,\dots,p_r \}, \qquad \dots(4.2)$$

then

$$h_1(p_1,\ ,p_r) = \{\sigma_1 \dots \sigma_r\} \int_0^\infty \dots \int_0^\infty h_2(x_1,\dots,x_r)$$

$$\cdot\ \emptyset(x_1,\dots,x_r, p_1,\dots,p_r)\ dx_1 \dots dx_r, \qquad \dots(4.3)$$

where

$$\{ p_1^{-c_1-1} \dots p_r^{-c_r-1} \}\ k_1(a_1 p_1^{-\sigma_1},\dots, a_r p_r^{-\sigma_r})$$

$$= T_2\ \{\emptyset(x_1,\dots,x_r, a_1,\dots,a_r)\ ;\ p_1,\dots,p_r\}, \qquad \dots(4.4)$$

$\sigma_1,\dots,\sigma_r$ are non-zero real numbers of the same sign, each of $a_1,\dots,a_r$ is independent of $p_1,\dots,p_r$ and all the multiple integrals involved in equations (4.1) to (4.4) are assumed to converge absolutely.

Proof. Theorem 2 easily follows on applying (1.4) to the pairs given by (4.2) and (4.4) and proceeding on lines similar to those indicated in the proof of Theorem 1.

Theorem 2 also sufficiently general in nature. Its analogues for one-and two-dimensional integral transforms were obtained by the author in his earlier papers ([4], [5]). These analogues are of interest in themselves and unify and extend a vast number of results as pointed therein.

In conclusion, we remark that a number of theorems involving specific multidimensional integral transforms introduced from time to time by several authors can be obtained from Theorems 1 and 2 ; however, we do not record them here for want of space.

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