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(Dedicated to the memory of Professor Arthur Erdélyi)

SOME REMARKS ON THE DERIVATIVE OF A POLYNOMIAL

By

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Let $P(z)$ be a polynomial of degree n and $P'(z)$ denote its derivative. Concerning $|P'(z)|$ the following results are known :

Thaorem A ([3], p. 58). *If* P(z) *is a polynomial of degree* n, *with* $|P(z)| \leqslant 1$ *on* $|z| \leqslant 1$ *and* P(z) *has no zeros in the disk* $|z| < k$, $k \geqslant 1$ *then for* $|z| \leqslant 1$

(1) $\qquad |P'(z)| \leqslant n/(1+k)$

and the equality in (1) *holds for* $P(z)=[(z+k)/(1+k)]^n$.

Theorem B ([2], p. 544). *If* P(z) *is a polynomial of degree* n, *with* $\max |P(z)| = 1$ *on* $|z| \leqslant 1$ *and* P(z) *has all its zeros in the disk* $|z| \leqslant k$, $k \geq 1$ *then*

(2) $\qquad \max\limits_{|z|=1} |P'(z)| \geqslant n/(1+k^n)$

and the equality in (2) *holds for* $P(z) = (z^n+k^n) / (1+k^n)$.

Theorem C ([4], p. 122). *If* P(z) *is a plynomial of degree* n, *then for* $p \geqslant 1$

(3) $$\left[\int_0^{2\pi} |P'(e^{i\theta})|^p \, d\theta\right]^{1/p} \leqslant C_p \, n \left[\int_0^{2\pi} |\operatorname{Re} P(e^{i\theta})|^p \, d\theta\right]^{1/p}$$

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where $C_p^p = \sqrt{\pi} \frac{\Gamma(\frac{1}{2}p+1)}{\Gamma(\frac{1}{2}p+\frac{1}{2})}$ *and the equality in* (3) *holds for* $P(z)=az^n$ *where* a *is an arbitrary constant.*

In this note, we present another proof (short and simple) of the above theorems. Actually, we note that these theorems are a consequence of the observations due to be Bruijn [1] which are inspired by a result known as :

Laguerre Theorem. *If* $P(z)$ *is a polynomial of degree* n *and* $P(z)\neq 0$ *in* $z \in C$, *where* C *is circular domain, then for all* $z \in C$, $\xi \in C$,

(4) $(\xi - z) P'(z) + nP(z) \neq 0.$

Remark : A circular domain is the image of the unit disk (open or closed) under a linear transformation.

In fact, with z_ν, $\nu = 1, 2, \ldots, n$ as the zeros of $P(z)$ in the right-hand side of (4) $\sum_{\nu=1}^{n} \frac{\xi - z_\nu}{z - z_\nu}$ cannot be zero as for fixed $\xi, z \in C$, the linear transformation $\frac{\xi - \lambda}{z - \lambda}$ maps z_ν, $\nu=1, 2, \ldots, n$ into a convex circular domain (the image of the complement of C) not containing zero.

With $P(z)$ replaced by $P(z)-\eta$, one can deduce from the Laguerre Theorem, the following

Lemma (*cf.* [1], p. 592). *Let* C *be a circular domain in the* z*-plane and* S *an arbitrary point set in the* w*-plane.* If $P(z)=w \in S$ *for* and $z \in C$, then we have for $z \in C$ *and* $\xi \in C$.

(5) $\frac{\xi}{n} P'(z) + P(z) - z \frac{P'(z)}{n} \in S.$

Alternate Proof of Theorem A Since $P(z)$ has no zeros in $|z| > k$, $k \geqslant 1$, from Laguerre Theorem for $|z| \leqslant 1$, $0 < |\xi| < k$ and a suitable choice of argument ξ in (4), we have

(6) $\left| \xi \frac{P'(z)}{n} \right| < \left| z \frac{P'(z)}{n} P(z) \right|$

or

(7) $\left| \xi \dfrac{P'(z)}{n} \right| > \left| z \dfrac{P'(z)}{n} - P(z) \right|$

But a sufficiently small value of $|\xi|$ contradicts (7), so (6) is true and letting $|\xi| \to k$, one has

(8) $k \left| \dfrac{P'(z)}{n} \right| \leqslant \left| z \dfrac{P'(z)}{n} - P(z) \right|$

for $|z| \leqslant 1$.

Now let C be the unit disk $|z| \leqslant 1$ and S be the disk $|w| \leqslant 1$. From the hypothesis, the image of the disk $|z| \leqslant 1$ under the maping $P(z)$ is contained in $|w| \leqslant 1$. As ξ varies in the unit disk, (5) implies that a disk of radius $\left| \dfrac{P'(z)}{n} \right|$ and centre $z \dfrac{P'(z)}{n} - P(z)$ (which also belong to S, take $\xi = 0$) must be contained in the disk $|w| \leqslant 1$. Since from (8) the centre is at a distance greater than $k \left| \dfrac{P'(z)}{n} \right|$ from the origin, we have $k \left| \dfrac{P'(z)}{n} \right| + \left| \dfrac{P'(z)}{n} \right| \leqslant 1$. This completes our proof of Theorem A.

Alternative Proof of Theorem B. Since $P(z) \neq 0$ for $|z| > k$, one has

(9) $\xi \dfrac{P'(z)}{n} \neq z \dfrac{P'(z)}{n} - P(z)$

for all $|\xi|$ and $|z| > k$. For these ξ and z, the inequality like (6) is violated for large $|\xi|$, thus

(10) $k \left| \dfrac{P'(ke^{i\theta})}{n} \right| \geqslant \left| \dfrac{ke^{i\theta} P'(ke^{i\theta})}{n} - P(ke^{i\theta}) \right|.$

Since $P'(z)$ is a polynomial of degree $n-1$ add $k \geqslant 1$, making use of (10), we have

(11) $$k^n \max \left| \frac{P'(e^{i\theta})}{n} \right| \geqslant k \max \left| \frac{P'(ke^{i\theta})}{n} \right|$$

$$\geqslant \max \left| \frac{ke^{i\theta} P'(Re^{i\theta})}{n} - P(ke^{i\theta}) \right|$$

$$\geqslant \max \left| \frac{e^{i\theta} P'(e^{i\theta})}{n} - P(e^{i\theta}) \right|$$

$$\geqslant |P(e^{i\theta})| - \left|\frac{P'(e^{i\theta})}{n}\right|$$

which implies that

$$\max_{|z|=1} |P'(z)| \geqslant n/(1+k^n).$$

This completes our proof of Theorem B.

Alternative Proof of Theorem C. From a result of de Bruijn [1] we know that

$$(12) \qquad (e^{i(\eta+\theta)} - e^{i\theta}) \frac{P'(e^{i\theta})}{n} + P(e^{i\theta}) = \sum_w \lambda(n, w) P(we^{i\theta})$$

where w runs through the nth roots of $e^{i(n-1)\theta+\eta}$, $\lambda \geqslant 0$ and $\sum_w \lambda(n,w)=1$.

Taking real parts of both sides in (12) and then integrating the pth power of the absolute value, one gets

$$(13) \qquad \int_0^{2\pi} \left| Re\{P(e^{i\theta}) - e^{i\theta}\frac{P'(e^{i\theta})}{n}\} + Re\, e^{i(\eta+\theta)} \frac{P'(e^{i\theta})}{n}\right|^p d\theta$$

$$\leqslant \int_0^{2\pi} \left| Re\, P(e^{i\theta})\right|^p d\theta$$

where η is arbitrary. Put $A(\theta) = P(e^{i\theta}) - e^{i\theta}\frac{P'(e^{i\theta})}{n}$ and $B(\theta) = e^{i\theta}\frac{P'(e^{i\theta})}{n}$. Note that one can vary η in a sector of angle π so that both $Re\, A(\theta)$ and $Re\, e^{i\eta} B(\theta)$ have the same sign. In fact, if $Re\, A(\theta) \geq 0$, let $-\pi/2 - \arg B(\theta) \leqslant \eta \leqslant \pi/2 - \arg B(\theta)$ and in case $Re\, A(\theta) \leqslant 0$, choose $\pi/2 - \arg B(\theta) \leqslant \eta \leqslant 3\pi/2 - \arg B(\theta)$. This implies that for such a choice of η in (13), we have

$$(14) \qquad \int_0^{2\pi} \left| Re\, e^{i\eta} e^{i\theta} P'(e^{i\theta})\right|^p d\theta \leqslant n^p \int_0^{2\pi} \left| Re\, P(e^{i\theta})\right|^p d\theta.$$

From where

$$(15)\qquad \int_0^{2\pi} \left|P'(e^{i\theta})\right|^p \left|1+e^{i(2\eta-\alpha(\theta))}\right|^p d\theta \leqslant \int_0^{2\pi} \left|Re\, P(e^{i\theta})\right|^p d\theta;$$

$\alpha(\theta) = \arg e^{i\theta}\; P'(e^{i\theta})$. By integrating (15) with respect to η over the corresponding interval of length π, making a change of variable and then interchanging the order of integrations, one obtains

$$(16)\quad \int_0^{2\pi} \left|P'(e^{i\theta})\right|^p d\theta \,.\int_0^{2\pi} \left|\frac{1+e^{i\eta}}{2}\right|^p d\eta \leqslant 2\pi\, n^p \int_0^{2\pi} \left|Re\, P(e^{i\theta})\right|^p d\theta.$$

Consequently,

$$\int_0^{2\pi} \left|P'(e^{i\theta})\right|^p d\theta \leqslant \sqrt{\pi}\, \frac{\Gamma(\frac{1}{2}p+1)}{\Gamma(\frac{1}{2}p+\frac{1}{2})}\, n^p \int_0^{2\pi} \left|Re\, P(e^{i\theta})\right|^p d\theta.$$

This completes our proof of Theorem C.

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