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ON THE FISHER DISTRIBUTION OF PALEOMAGNETIC DIRECTIONS ON A SPHERE

By

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Abstract

The Fisher distribution is one of the most important distributions in statistics to deal with the paleomagnetic data; (cf. Fisher, R. A. (1953). Dispersion on a sphere. *Proc. R. Soc. Lond. A*, **217** (1130), 295-305). In this paper, we have considered several distributional properties of the Fisher distribution. Based on these distributional properties, we have established some new characterizations of the Fisher distribution by truncated first moment, order statistics and upper record values.

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1 Introduction

Paleomagnetism is the study of the record of the Earth's magnetic field in rocks, sediment, or archeological materials; (cf. https://en.wikipedia.org/wiki/Paleomagnetism). "The conceptual model for Earth's magnetic field is that of a dipole (i.e. bar magnet) positioned at Earth's center and aligned with the rotational axis of the Earth (see Figure 1). This allows us to predict the direction of the magnetic field at any location on Earth's surface using the fundamental equations of a dipole field. This equation gives a direct relation between magnetic inclination and geographic latitude at the point of observation"; (cf. http://www.geo.mtu.edu/KeweenawGeoheritage/IRKeweenawRift/Paleomagnetism.html).

The British statistician Sir R. A. Fisher introduced a probability density function to study certain statistical problems in paleomagnetism, which, in literature, is now known as the Fisher distribution, see, for example, Fisher [11], Mardia and Jupp [22], among others. "However, as pointed out by Mardia and Jupp [22], Fisher distributions first appeared in a paper on statistical mechanics by Langevin [21]. Further, according to Mardia and Jupp [22], the maximum likelihood estimation in the Fisher distributions and a corresponding characterization were considered by Arnold [2]. Also, Kuhn and Grün [20] found Fisher distributions as approximate solution to a problem associated with paths and chains of random segments in three dimensions. Significant advances in statistical applications were made by Fisher [11], who used these distributions to investigate certain statistical problems in

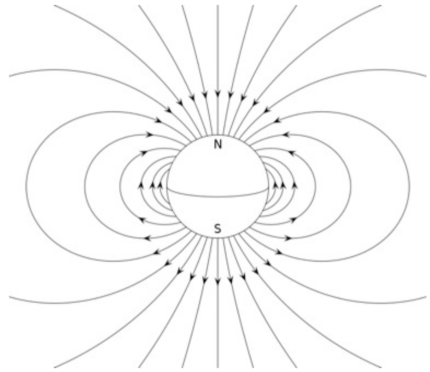

Figure 1.1: Earth's magnetic field (Source:http://www.geo.mtu.edu/KeweenawGeoheritage/IRKeweenawRift/Paleomagnetism.html)

paleomagnetism. Watson and Williams [26] studied the extension of the Fisher distribution to higher dimension." For further reading, the interested readers are also referred to some relevant literature, such as, Jammalamadaka and Sengupta [16], Jeffreys [17] and Jones and Pewsey [18], among others.

We say that a random variable X has the standard Fisher distribution, denoted as $X \sim FI(\kappa)$ (here we use FI in honor of Fisher) if its probability density function (*pdf*) is given by

$$f(\theta) = \frac{\kappa e^{\kappa \cos(\theta)} \sin(\theta)}{2 \sinh(\kappa)}, \quad 0 \leq \theta \leq \pi, \kappa \geq 0 \tag{1.1}$$

which defines the probability of finding a direction, where θ is the realization of X and denotes the angle from true mean direction (and is $= 0$ at true mean) and κ is the precision parameter and measures the concentration of the distribution about the true mean direction and is $= 0$ for a distribution of directions that is uniform over the sphere and approaches ∞ for directions concentrated at a point. Also, note that the distribution of directions is azimuthally symmetric about the true mean. It is easy to see that that the *pdf* in (1.1) integrates to 1. To describe the shapes of the Fisher distribution, $X \sim FI(\kappa)$, the plots of the *pdf* (1.1) for some values of the parameter κ are provided below in Figures 1.2 (a) and (b).

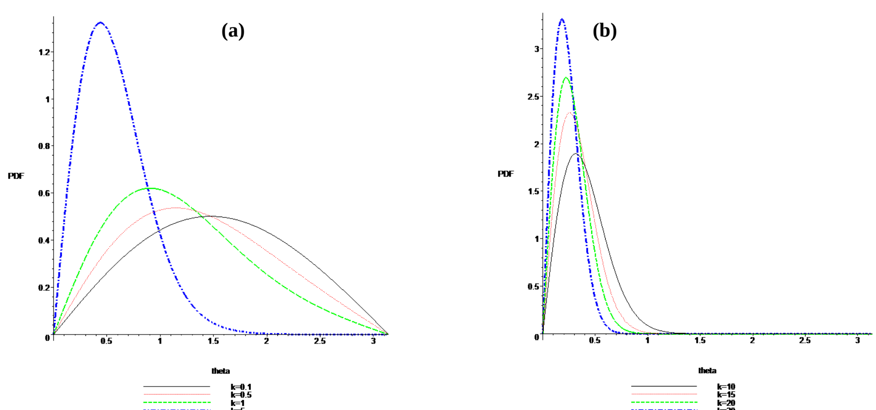

Figure 1.2: (a and b): Plots of the *pdf* of the standard Fisher distribution, $X \sim FI(\kappa)$.

The effects of the parameter can easily be seen from the Figures 1.2 (a and b). Similar plots can be drawn for others values of the parameters. It is obvious from these figures that, as κ increases, the probability mass becomes more concentrated about $\theta = 0$.

The organization of this paper is given as follows: Some distributional properties of the Fisher distribution are given in Section 2. Section 3 discusses the characterizations of the Fisher distribution. Finally, some concluding remarks are given in section 4.

2 Distributional Properties

In this section, we will discuss some of the distributional properties of the Fisher distribution, $X \sim FI(\kappa)$.

2.1 Cumulative Distribution Function

The cumulative distribution function (*cdf*) of $X \sim FI(\kappa)$ is given by

$$F(\theta) = \frac{e^{\kappa} - e^{\kappa\cos(\theta)}}{2\sinh(\kappa)}, \quad 0 \leq \theta \leq \pi, \kappa \geq 0. \tag{2.1}$$

Since

$$e^{\kappa\cos\theta} = (I_0(\kappa) + 2\sum_{j=1}^{\infty} I_j(\kappa)\cos(j\theta)),$$

where $I_0(\kappa)$ is the modified Bessel function of the first kind and order 0 and $I_j(\kappa)$ is the modified Bessel function of the first kind and order j given by

$$I_j(\kappa) = (\frac{\kappa}{2})^j \sum\nolimits_{m=0}^{\infty} (\frac{\kappa}{2})^{2m} \frac{1}{(m!)\Gamma(j+m+1)},$$

where $\Gamma(.)$ denotes the gamma function, see, for example, Gradshteyn and Ryzhik [15], we can also write the *pdf* (1.2), that is, $f(\theta)$ and the *cdf* (2.1), that is, $F(\theta)$, respectively as follows:

$$f(\theta) = \frac{\kappa\sin(\theta)}{2\sinh(\kappa)}(I_0(\kappa) + 2\sum_{j=1}^{\infty} I_j(\kappa)\cos(j\theta)), \quad 0 \leq \theta \leq \pi, \kappa \geq 0, \tag{2.2}$$

and

$$F(\theta) = \frac{e^{\kappa} - (I_0(\kappa) + 2\sum_{j=1}^{\infty} I_j(\kappa)\cos(j\theta))}{2\sinh(\kappa)}, \quad 0 \leq \theta \leq \pi, \kappa \geq 0. \tag{2.3}$$

The plots of the *cdf* (2.1) for some values of the parameter κ are provided below in Figures 2.1 (a and b).

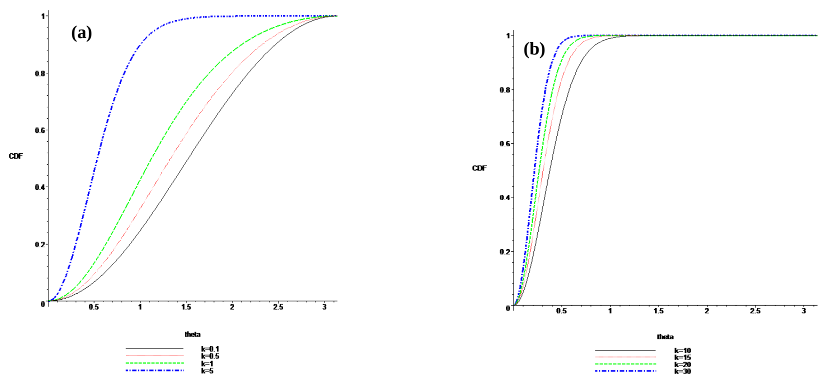

Figure 2.1: (a and b): Plots of the *cdf* standard Fisher distribution, $X \sim FI(\kappa)$.

2.2 Hazard Rate Function

Using the equations (1.1) and (2.1), the hazard rate (or failure rate) $h(\theta)$ of Fisher distribution, $X \sim FI(\kappa)$ is given by

$$h(\theta) = \frac{f(\theta)}{1 - F(\theta)} = \frac{\kappa e^{\kappa \cos(\theta)} \sin(\theta)}{2\sinh(\kappa) - [e^{\kappa} - e^{\kappa \cos(\theta)}]}, \quad 0 \leq \theta \leq \pi, \kappa \geq 0. \tag{2.4}$$

From (2.4), it is easily seen that its hazard rate, $h(\theta)$, has the following property: $h'(\theta) \geq 0$, that is, $f'(\theta)[1 - F(\theta)] + [f(\theta)]^2 \geq 0$, for all θ in $0 \leq \theta \leq \pi$ and for all $, \kappa \geq 0$, where $f'(\theta) = \frac{1}{2} \frac{\kappa e^{\kappa \cos(\theta)}}{\sinh(\kappa)} [\cos(\theta) - \kappa \sin(\theta)]$ and $1 - F(\theta) = 1 - (\frac{e^{\kappa} - e^{\kappa \cos(\theta)}}{2\sinh(\kappa)})$. Thus, $X \sim FI(\kappa)$ has an increasing failure rate (IFR). To describe the shapes of the hazard rate (or failure rate) of the Fisher distribution, $X \sim FI(\kappa)$, the plots of the hazard rate(2.4) for some values of the parameter κ are provided below in Figures 2.2 (a and b). The effects of the parameter can easily be seen from these graphs. It is obvious from the Figure 2.2 (a) that the hazard rate, $h(\theta)$ of $X \sim FI(\kappa)$, is concave up, that is, bathtub shaped, when $0 < \kappa < 1$. Also, we observe from the Figure 2.2 (b) that the hazard rate, $h(\theta)$ of $X \sim FI(\kappa)$, is concave down, that is, upside down bathtub shaped, when $\kappa \geq 10$. Similar plots can be drawn for others values of the parameters.

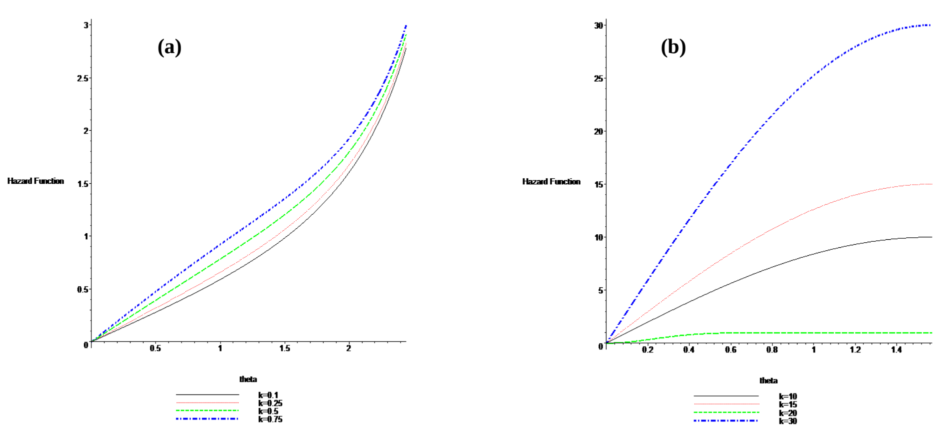

Figure 2.2: (a and b): Plots of the Hazard Function standard Fisher distribution, $X \sim FI(\kappa)$

2.3 Moment

Using the *pdf* in (2.2), we have the following expression for the first moment of the Fisher distribution, $X \sim FI(\kappa)$:

$$E(X) = \int_0^{\pi} \theta[\frac{\kappa \sin(\theta)}{2\sinh(\kappa)} + \frac{1}{I_0(\kappa)} \frac{\kappa}{\sinh(\kappa)} \sum\nolimits_{j=1}^{\infty} I_j(\kappa) \sin(\theta)\cos(j\theta)]d\theta,$$

which, after integration and simplification, easily gives

$$E(X) = \frac{\pi\kappa}{2\sinh(\kappa)} + \frac{1}{I_0(\kappa)} \frac{\kappa}{\sinh(\kappa)} \sum\nolimits_{j=1}^{\infty} I_j(\kappa) \frac{2j\sin(\pi j) - \pi(j^2-1)\cos(\pi j)}{(j^2-1)^2},$$

where $I_0(\kappa)$ is the modified Bessel function of the first kind and order 0 and $I_j(\kappa)$ is the modified Bessel function of the first kind and order j.

2.4 Shannon Entropy

An entropy provides an excellent tool to quantify the amount of information (or uncertainty) contained in a random observation regarding its parent distribution (population). A large value of entropy implies the greater uncertainty in the data. As proposed by Shannon [25], if X is a continuous random variable with *pdf* $f_X(x)$, defined over an interval Ω then Shannon's entropy of X, denoted by $H(f)$, is defined as

$$H(f) = E[-\ln\{f_X(x)\}] = -\int_{\Omega} f_X(x) \ln\{f_X(x)\}dx. \tag{2.5}$$

Now, using the *pdf* (1.1) of the Fisher distribution, $X \sim FI(\kappa)$, in Eq. (2.5) and integrating with respect to θ and simplifying, we obtain an explicit expression of Shannon entropy as follows:

$$\begin{aligned} H(f) &= -\int_0^{\pi} \frac{\kappa e^{\kappa\cos(\theta)} \sin(\theta)}{2\sinh(\kappa)} \ln[\frac{\kappa e^{\kappa\cos(\theta)} \sin(\theta)}{2\sinh(\kappa)}]d\theta \\ &= -\ln(\frac{\kappa}{2\sinh(\kappa)}) - [\kappa + \frac{(\kappa+1)e^{-\kappa}}{2\sinh(\kappa)}] - \int_0^{\pi} \frac{\kappa e^{\kappa\cos(\theta)} \sin(\theta)\ln[\sin(\theta)]}{2\sinh(\kappa)}d\theta. \end{aligned} \tag{2.6}$$

Now, by taking $\kappa\cos(\theta) = u$ in the integral in Eq. (2.6), recalling the definition of the definite integral of an even function and also the definition of the logarithmic series, we have

$$\begin{aligned} &\int_0^{\pi} \frac{\kappa e^{\kappa\cos(\theta)} \sin(\theta)\ln[\sin(\theta)]}{2\sinh(\kappa)}d\theta = \int_0^{\kappa} \frac{e^u \ln(1-\frac{u^2}{\kappa^2})}{2\sinh(\kappa)}du \\ &= \sum\nolimits_{l=1}^{\infty} (-1)^{2l+1} \frac{1}{2\sinh(\kappa)} (\frac{1}{\kappa^2})^l \int_0^{\kappa} e^u u^{2l} du \\ &= \sum\nolimits_{l=1}^{\infty} (-1)^{2l+1} \frac{1}{2\sinh(\kappa)} (\frac{1}{\kappa^2})^l B(1, 2l+1)\kappa^{2l} {}_1F_1(2l;\ 1+2l; \kappa), \end{aligned} \tag{2.7}$$

which follows from Eq. 3.383.1, Page 318, of Gradshteyn and Ryzhik [15] and where $B(.,.)$ denotes the beta function and ${}_1F_1(.;.;.)$ denotes the degenerate hypergeometric function; see, for example, Gradshteyn and Ryzhik [15], among others. Thus, in view of (2.7), the expression (2.6) of Shannon entropy of $X \sim FI(\kappa)$ is given by

$$\begin{aligned} H(f) = &-\ln(\frac{\kappa}{2\sinh(\kappa)}) - [\kappa + \frac{(\kappa+1)e^{-\kappa}}{2\sinh(\kappa)}] \\ &- \sum\nolimits_{l=1}^{\infty} (-1)^{2l+1} \frac{1}{2\sinh(\kappa)} (\frac{1}{\kappa^2})^l B(1, 2l+1)\kappa^{2l} {}_1F_1(2l;\ 1+2l; \kappa). \end{aligned} \tag{2.8}$$

2.5 Percentile Points

Here we compute the percentage points of the standard Fisher distribution, $X \sim FI(\kappa)$, with the *pdf* (1.1) and the *cdf* (2.1). For any $0 < p < 1$, the 100 p^{th} percentile, (also called the quantile of order p), of $X \sim FI(\kappa)$, with *pdf* $f_X(\theta)$, is a number θ_p such that the area under $f_X(\theta)$ to the left of θ_p is p. That is, θ_p is any root of the equation given by

$$F(\theta_p) = \int_0^{\theta_p} f_X(u)du = p.$$

The percentage points θ_p associated with the *cdf* (2.1) of $X \sim FI(\kappa)$ are computed for some selected values of the parameters by using Maple software. These are provided in the Table 2.1 below:

Table 2.1: Percentage Points of the standard Fisher distribution, $X \sim FI(\kappa)$

Parameter *k*	**75%**	**80%**	**85%**	**90%**	**95%**	**99%**
0.1	2.05022	2.17331	2.30945	2.46708	2.66794	2.93089
0.2	2.00444	2.13044	2.27065	2.43402	2.64358	2.91964
0.5	1.86007	1.99238	2.14285	2.32248	2.55921	2.87979
1.0	1.61635	1.74851	1.90487	2.10130	2.37862	2.78781
1.5	1.40138	1.52266	1.67003	1.86317	2.15714	2.65419
2.0	1.23068	1.33807	1.46981	1.64587	1.92683	2.47416
2.5	1.10012	1.19516	1.31177	1.46810	1.72147	2.26223
3.0	0.99992	1.08512	1.18934	1.32861	1.55403	2.05094
3.5	0.92141	0.99893	1.09343	1.21914	1.42125	1.86614
4.0	0.85836	0.92982	1.01668	1.13172	1.31537	1.71441
4.5	0.80652	0.87310	0.95382	1.06036	1.22940	1.59147
5.0	0.76299	0.82556	0.90125	1.00088	1.15816	1.49084
5.5	0.72582	0.78500	0.85650	0.95038	1.09799	1.40707
6.0	0.69359	0.74989	0.81781	0.90683	1.04635	1.33607
6.5	0.66531	0.71911	0.78394	0.86878	1.00140	1.27495
7.0	0.64023	0.69183	0.75396	0.83516	0.96180	1.22161
7.5	0.61779	0.66744	0.72718	0.80517	0.92658	1.17452
8.0	0.59756	0.64546	0.70307	0.77820	0.89498	1.13255
8.5	0.57919	0.62553	0.68121	0.75378	0.86642	1.09481
9.0	0.56242	0.60733	0.66128	0.73152	0.84043	1.06064
9.5	0.54703	0.59064	0.64299	0.71113	0.81666	1.02950
10.0	0.53283	0.57525	0.62615	0.69236	0.79480	1.00097

3 Characterizations

Since characterization can be used to confirm whether the given continuous probability distribution satisfies the underlying requirements,many researchers have investigated the characterizations of probability distributions by different methods, see, for example, Galambos and Kotz [12], Glänzel [13], Glänzel et al. [14], Kotz and Shanbhag [19] and Nagaraja [23], among others. For recent developments on the characterizations of probability distributions by truncated moment method, the interested readers are referred to Ahsanullah [3], Ahsanullah and Shakil [4], Ahsanullah et al. [5, 6 and 7].It appears from the literature that not much attention has been paid to the characterizations of the Fisher distribution which can be usefully employed. However, as Mardia and Jupp [22] point out, "Arnold [2] gave a maximum likelihood characterization of the Fisher distribution, which was later proved by Breitenberger [9] by a simpler method. The general result was proved by Bingham and Mardia [8]." In this section, we present the characterizations of the Fisher distribution, $X \sim FI(\kappa)$ by truncated moments. For this, we will need some assumption and lemmas as provided below.

Assumption 3.1. *Suppose the random variable X is absolutely continuous with cumulative distribution function (cdf) $F(x)$ and probability density function (pdf) $f(x)$. We assume that $\gamma = \{x|F(x) > 0\}$and $\delta = \inf\{x|F(x) < 1\}$. We further assume that $E(X)$ exists.*

Lemma 3.1. *Under the assumption 3.1, if $E(X|X \leq x) = g(x)\tau(x)$, where $\tau(x) = \frac{f(x)}{F(x)}$ and $g(x)$ is a continuous differentiable function of x with the condition that $\int_\gamma^x \frac{u-g'(u)}{g(u)}du$ is finite for all x, $\gamma < x < \delta$, then $f(x) = ce^{\int_\gamma^x \frac{u-g'(u)}{g(u)}du} = ce^{\int_\gamma^x \frac{f'(u)}{f(u)}du}$, where $\frac{u-g'(u)}{g(u)} = \frac{f'(u)}{f(u)}$ and c is determined by the condition $\int_\gamma^\delta f(x)dx = 1$.*

Proof. For proof, see Ahsanullah and Shakil [4]. □

Lemma 3.2. *Under the assumption 3.1, if $E(X|X \geq x) = g(x)r(x)$, where $r(x) = \frac{f(x)}{1-F(x)}$ and $g(x)$ is a continuous differentiable function of x with the condition that $\int_x^\delta \frac{u+g'(u)}{g(u)}du$ is finite for all x, $\gamma < x < \delta$, then $f(x) = ce^{-\int_x^\delta \frac{u+g'(u)}{g(u)}du} = ce^{\int_x^\delta \frac{f'(u)}{f(u)}du}$, where $-\frac{u+g'(u)}{g(u)} = \frac{f'(u)}{f(u)}$ and c is determined by the condition $\int_\gamma^\delta f(x)dx = 1$.*

Proof. For proof, see Ahsanullah and Shakil [4]. □

3.1 Characterization by Truncated Moments

Following theorems contain the characterizations of the Fisher distribution by truncated moments. Without loss of generality, we will consider the e *pdf* $f(\theta)$ of the standard Fisher distribution, $X \sim FI(\kappa)$, as given in 2.2 above.

Theorem 3.1.1. *Suppose that X is absolutely continuous bounded random variable with cdf $F(x)$ such that $F(0) = 0$ and $F(\pi) = 1$, then $E(X|X \leq \theta) = g(\theta)\tau(\theta)$, where $\tau(\theta) = \frac{f(\theta)}{F(\theta)}$ and $g(\theta)$ is a continuous differentiable function of θ, $0 < \theta < \pi$, given by*

$$g(\theta) = \frac{e^{-k\cos\theta}}{\sin(\theta)}\{I_0(\kappa)(\sin(\theta) - \theta\cos(\theta))$$
$$+ \sum_{j=1}^{\infty} I_j(\kappa)[\frac{\sin((j+1)\theta) - (j+1)\theta\cos((j+1)\theta)}{(j+1)^2} - \frac{\sin((j-1)\theta) - (j-1)\theta\cos((j-1)\theta)}{(j-1)^2}]\},$$

if and only if

$$f(\theta) = \frac{\kappa e^{\kappa\cos(\theta)}\sin(\theta)}{2\sinh(\kappa)}, \quad 0 \leq \theta \leq \pi, \kappa \geq 0.$$

Proof. If

$$f(\theta) = \frac{\kappa e^{\kappa \cos(\theta)} \sin(\theta)}{2\sinh(\kappa)}, \quad 0 \leq \theta \leq \pi, \kappa \geq 0,$$

that is,

$$f(\theta) = \frac{\kappa \sin(\theta)}{2\sinh(\kappa)}(I_0(\kappa) + 2\sum_{j=1}^{\infty} I_j(\kappa)\cos(j\theta)), \quad 0 \leq \theta \leq \pi, \kappa \geq 0,$$

then it is easily seen, by direct integration and simplification, that

$$g(\theta) = \frac{\int_0^\theta uf(u)du}{f(\theta)} = \frac{\int_0^\theta [I_0(\kappa)u\sin(u) + 2\sum_{j=1}^{\infty} I_j(\kappa)u\sin(u)\cos(ju)]du}{e^{k\cos\theta}\sin(\theta)}$$

$$= \frac{e^{-k\cos\theta}}{\sin(\theta)}\{I_0(\kappa)(\sin(\theta) - \theta\cos(\theta))$$

$$+ \sum_{j=1}^{\infty} I_j(\kappa)[\frac{\sin((j+1)\theta) - (j+1)\theta\cos((j+1)\theta)}{(j+1)^2} - \frac{\sin((j-1)\theta) - (j-1)\theta\cos((j-1)\theta)}{(j-1)^2}]\}$$

Suppose that

$$g(\theta) = \frac{e^{-k\cos\theta}}{\sin(\theta)}\{I_0(\kappa)(\sin(\theta) - \theta\cos(\theta))$$

$$+ \sum_{j=1}^{\infty} I_j(\kappa)[\frac{\sin((j+1)\theta) - (j+1)\theta\cos((j+1)\theta)}{(j+1)^2} - \frac{\sin((j-1)\theta) - (j-1)\theta\cos((j-1)\theta)}{(j-1)^2}]\}.$$

Then, differentiating both sides of the above equations with respect to θ and simplifying, it is easily seen that

$$g'(\theta) = \theta - [\cot(\theta) - k\sin(\theta)]g(\theta), \text{since } e^{\kappa\cos\theta} = (I_0(\kappa) + 2\sum_{j=1}^{\infty} I_j(\kappa)\cos(j\theta)).$$

Thus $\dfrac{\theta - g'(\theta)}{g(\theta)} = \cot(\theta) - k\sin(\theta)$,

from which, on using Lemma 3.1, we have

$$\frac{f'(\theta)}{f(\theta)} = \frac{\theta - g'(\theta)}{g(\theta)} = \cot(\theta) - k\sin(\theta),$$

or, $$d[\ln(f(\theta))] = \cot(\theta) - k\sin(\theta).$$

On integrating the above equation with respect to θ, we obtain $f(\theta) = c\sin(\theta)e^{k\cos\theta}$, where c is a constant to be determined.

On integrating the above equation with respect to θ from 0 to π and using the condition

$$\int_0^\pi f(\theta)d\theta = 1, \text{we easily obtain } c = \frac{\kappa}{2\sinh(\kappa)}, \text{and thus}$$

$$f(\theta) = \frac{\kappa e^{\kappa\cos(\theta)}\sin(\theta)}{2\sinh(\kappa)}, \quad 0 \leq \theta \leq \pi, \kappa \geq 0,$$

which is the *pdf* of the standard Fisher distribution, $X \sim FI(\kappa)$. This completes the proof of the Theorem 3.1.1. □

Theorem 3.1.2. *Suppose that X is absolutely continuous bounded random variable with cdf $F(x)$ such that $F(0) = 0$ and $F(\pi) = 1$, then $E(X|X \geq \theta) = h(\theta)r(\theta)$, where $r(\theta) = \frac{f(\theta)}{1-F(\theta)}$ and $h(\theta)$ is a continuous differentiable function of θ, $0 < \theta < \pi$, given by*

$$
\begin{aligned}
h(\theta) &= \frac{\int_\theta^\pi uf(u)du}{f(\theta)} = \frac{\int_\theta^\pi [I_0(\kappa)u\sin(u) + 2\sum_{j=1}^\infty I_j(\kappa)u\sin(u)\cos(ju)]du}{e^{k\cos\theta}\sin(\theta)} \\
&= \frac{e^{-k\cos\theta}}{\sin(\theta)}\{I_0(\kappa)(\pi - \sin(\theta) + \theta\cos(\theta)) \\
&+ \sum_{j=1}^\infty I_j(\kappa)[\frac{(j+1)\theta\cos((j+1)\theta) + (j+1)\pi\cos(\pi j) - \sin((j+1)\theta) - \sin(\pi j)}{(j+1)^2} \\
&- \frac{(j-1)\theta\cos((j-1)\theta) + (j-1)\pi\cos(\pi j) - \sin((j-1)\theta) - \sin(\pi j)}{(j-1)^2}]\}
\end{aligned}
$$

if and only if

$$f(\theta) = \frac{\kappa e^{\kappa\cos(\theta)}\sin(\theta)}{2\sinh(\kappa)}, \quad 0 \leq \theta \leq \pi, \kappa \geq 0.$$

Proof. The proof is similar to the Theorem 3.1.1 and easily follows from Lemma 3.2. □

3.2 Characterizations by Order Statistics

Here, we will provide the characterizations based on order statistics, for which we first recall the following well-known results. Let$X_1, X_2, ..., X_n$ be n independent copies of the random variable X having absolutely continuous distribution function $F(x)$ and *pdf* $f(x)$. Suppose that $X_{1,n} \leq X_{2,n} \leq ... \leq X_{n,n}$ are the corresponding order statistics. It is known that $X_{j,n}|X_{k,n} = x$, for $1 \leq k < j \leq n$, is distributed as the $(j-k)^{\text{th}}$ order statistics from $(n-k)$ independent observations from the random variable V having the pdf $f_V(v|x)$ where $f_V(v|x) = \frac{f(v)}{1-F(x)}$, $0 \leq v < x$, see, for example, Arnold et al. [1], chapter 2, among others. Further, $X_{i.,n}|X_{k,n} = x, 1 \leq i < k \leq n$, is distributed as i^{th} order statistics from k independent observations from the random variable W having the pdf $f_W(w|x)$ where $f_W(w|x) = \frac{f(w)}{F(x)}, w < x$. Let

$$S_{k-1} = \tfrac{1}{k-1}(X_{1,n} + X_{2,n} + ... + X_{k-1,n}),$$

and

$$T_{k,n} = \tfrac{1}{n-k}(X_{k+1,n} + X_{k+2,n} + ... + X_{n.n}).$$

In the following two theorems, we will provide the characterization of the standard Fisher distribution, $X \sim FI(\kappa)$, based on order statistics.

Theorem 3.2.1. *Suppose the random variable X satisfies the assumption 1 with $\gamma = 0$ and $\delta = \pi$, then*

$$E(S_{k-1}|X_{k,n} = \theta) = g(\theta)\tau(\theta), \text{where } \tau(\theta) = \frac{f(\theta)}{F(\theta)}$$

and

$$g(\theta) = \frac{e^{-k\cos\theta}}{\sin(\theta)}\{I_0(\kappa)(\sin(\theta) - \theta\cos(\theta))$$

$$+\sum_{j=1}^{\infty} I_j(\kappa)[\frac{\sin((j+1)\theta)-(j+1)\theta\cos((j+1)\theta)}{(j+1)^2}-\frac{\sin((j-1)\theta)-(j-1)\theta\cos((j-1)\theta)}{(j-1)^2}]\},$$

if and only if

$$f(\theta)=\frac{\kappa e^{\kappa\cos(\theta)}\sin(\theta)}{2\sinh(\kappa)},\quad 0\le\theta\le\pi,\kappa\ge 0.$$

Proof. It is known, see David and Nagaraja [10], that

$$E(S_{k-1}|X_{k,n}=\theta)=E(X|X\le\theta).$$

Thus the result follows from Theorem 3.1.1. □

Theorem 3.2.2. *Suppose the random variable X satisfies the assumption* 1 *with $\gamma=0$ and $\delta=\pi$, then*

$$E(T_{k,n}|X_{k,n}=\theta)=h(\theta)r(\theta),\text{where } r(\theta)=\frac{f(\theta)}{1-F(\theta)} \text{ and}$$

$$\begin{aligned}h(\theta)&=\frac{\int_\theta^\pi uf(u)du}{f(\theta)}=\frac{\int_\theta^\pi[I_0(\kappa)u\sin(u)+2\sum_{j=1}^\infty I_j(\kappa)u\sin(u)\cos(ju)]du}{e^{k\cos\theta}\sin(\theta)}\\&=\frac{e^{-k\cos\theta}}{\sin(\theta)}\{I_0(\kappa)(\pi-\sin(\theta)+\theta\cos(\theta))\\&+\sum_{j=1}^{\infty}I_j(\kappa)[\frac{(j+1)\theta\cos((j+1)\theta)+(j+1)\pi\cos(\pi j)-\sin((j+1)\theta)-\sin(\pi j)}{(j+1)^2}\\&-\frac{(j-1)\theta\cos((j-1)\theta)+(j-1)\pi\cos(\pi j)-\sin((j-1)\theta)-\sin(\pi j)}{(j-1)^2}]\}\end{aligned}$$

if and only if $$f(\theta)=\frac{\kappa e^{\kappa\cos(\theta)}\sin(\theta)}{2\sinh(\kappa)},\quad 0\le\theta\le\pi,\kappa\ge 0.$$

Proof. It is known, see David and Nagaraja [10], that

$$E(T_{k,n}|X_{k,n}=\theta)=E(X|X\ge\theta).$$

Thus the result follows from Theorem 3.1.2. □

3.3 Characterization by Upper Record Values

Here, we will provide the characterizations based on upper record values, for which we first recall the following definitions. Suppose that $X_1, X_2, ...$ is a sequence of independent and identically distributed absolutely continuous random variables with distribution function $F(x)$ and *pdf* $f(x)$. Let $Y_n=\max(X_1,X_2,...,X_n)$ for $n\ge 1$. We say that X_j is an upper record value of $\{X_n, n\ge 1\}$ if $Y_j>Y_{j-1}, j>1$. The indices at which the upper records occur are given by the record times $\{U(n)>\min(j|j>U(n+1), X_j>X_{U(n-1)}, n>1)\}$ and $U(1)=1$. We will denote the n^{th} upper record value as $X(n)=X_{U(n)}$. In the following theorem, we will provide the standard Fisher distribution, $X\sim FI(\kappa)$, based on upper record values.

Theorem 3.3.1. *Suppose the random variable X satisfies the assumption 1 with $\gamma = 0$ and $\delta = \pi$, then $E(X(n+1)|X(n) = \theta) = h(\theta)r(\theta)$, where $r(\theta) = \frac{f(\theta)}{1-F(\theta)}$ and*

$$h(\theta) = \frac{\int_\theta^\pi uf(u)du}{f(\theta)} = \frac{\int_\theta^\pi [I_0(\kappa)u\sin(u) + 2\sum_{j=1}^\infty I_j(\kappa)u\sin(u)\cos(ju)]du}{e^{k\cos\theta}\sin(\theta)}$$
$$= \frac{e^{-k\cos\theta}}{\sin(\theta)}\{I_0(\kappa)(\pi - \sin(\theta) + \theta\cos(\theta))$$
$$+ \sum_{j=1}^\infty I_j(\kappa)[\frac{(j+1)\theta\cos((j+1)\theta) + (j+1)\pi\cos(\pi j) - \sin((j+1)\theta) - \sin(\pi j)}{(j+1)^2}$$
$$- \frac{(j-1)\theta\cos((j-1)\theta) + (j-1)\pi\cos(\pi j) - \sin((j-1)\theta) - \sin(\pi j)}{(j-1)^2}]\}$$

if and only if $$f(\theta) = \frac{\kappa e^{\kappa\cos(\theta)}\sin(\theta)}{2\sinh(\kappa)}, \quad 0 \le \theta \le \pi, \kappa \ge 0.$$

Proof. It is known, see Nevzorov [24], that

$$E(X(n+1)|X(n) = \theta) = E(X|X \ge \theta).$$

Thus the result follows from Theorem 3.1.2. □

4 Concluding Remarks

Characterization of a probability distribution plays an essential role in probability and statistics and other applied sciences. It can be used to endorse whether the given probability distribution satisfies the underlying requirements before a particular probability distribution model is applied to fit the real world data. The Fisher distribution is one of the most important distributions in statistics to deal with the paleomagnetic data. In this paper, we have considered its several distributional properties. Based on these distributional properties, we have established some new characterizations of the Fisher distribution by truncated first moment, order statistics and upper record values. We believe that the findings of the paper will be useful for researchers in the fields of probability, statistics and other applied sciences.

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