Jñānābha, Vol. 48(2) (2018), 32-45

REAL ZEROS OF GENERALIZED MITTAG-LEFFLER FUNCTIONS OF TWO VARIABLES THROUGH INTEGRAL TRANSFORMS

By
Hemant Kumar
Department of Mathematics
D. A-V. Postgraduate College Kanpur-208001, Uttar Pradesh, India
Email:palhemant2007@rediffmail.com

(*Received : September* 15, 2018 ; *Revised: October* 18, 2018)

Abstract

In this work, we study the distribution of real zeros of two variables Mittag-Leffler functions and the integral functions involving some generalized Mittag-Leffler functions of two variables through integral transforms and use them in computation.
2010 Mathematics Subject Classifications: Primary; 33E12; Secondary; 33C65, 26A33, 44A20.
Keywords and phrases: Meromorphic functions, generalized Mittag- Leffler functions of two variables, real zeros, integral transforms.

1 Introduction

The gradual development of this work is divided into two subsections. In subsection 1.1, we present the study of distribution of real zeros of some integral functions found in the literature like gamma function, $\xi(z)$ function, one and two parameter Mittag-Leffler functions and also certain theorems to obtain the real zeros of integral functions. In subsection 1.2, the theorems concerning obtaining real zeros through integral transform methods are introduced.

1.1 Real zeros of integral functions

Titchmarsh [27, p.268] had written that a number of important functions have no complex zeros; for example, all the zeros of $\frac{1}{\Gamma(z)}, z \in \mathbb{C}$, are real. Here, $\frac{1}{\Gamma(z)}$ is a meromorphic function defined in following Weierstrass product form (see also Conway [4, p. 176])

$$\frac{1}{\Gamma(z)} = ze^{\gamma z}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}}, \text{where,}\gamma \text{ is chosen so that } \Gamma(1)=1. \tag{1.1}$$

Clearly, the function given in Eqn. (1.1) has zeros at $z = 0, -1, -2, -3, \ldots$.

On the other hand it is some time very difficult to decide whether the zeros are real or not; for example it was conjectured by Riemann, in 1859, that all the zeros of the function $\Xi(z)$ defined by (see in [27, p. 265])

$$\Xi(z) = \xi(\frac{1}{2}+iz), i = \sqrt{(-1)} \text{ and } \xi(s) = \frac{s}{2}(s-1)\pi^{-\frac{1}{2}s}\Gamma(\frac{1}{2}s)\zeta(s),$$

where, the Riemann zeta function $\zeta(s)$ is defined as

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}, \Re(s) > 1, \tag{1.2}$$

are real, but this has never been proved. Again, Titchmarsh [27, p. 269] suggested that following theorems are applicable to check that the function has real zeros:

Theorem 1.1. ***(The Laguerre's theorem)****.* (See also, Kumar and Pathan [13, p. 67]):

Suppose that the function $\Phi(w)$ *is having negative real non-zero zeros and that is given by*

$$\Phi(w) = ae^{kw}\prod_{n=1}^{\infty}(1+\frac{w}{\alpha_n})e^{-\frac{w}{\alpha_n}}, \tag{1.3}$$

where, a*,and* α_n *being all positive,* k *is any constant.*

Also, let $f(z)$ *be an integral function of the form*

$$f(z) = e^{bz+c}\prod_{n=1}^{\infty}\left(1+\frac{z}{z_n}\right), \tag{1.4}$$

b *and all* z_n *are positive.*

If

$$f(z) = \sum_{n=0}^{\infty} a_n z^n,\ a_n = \frac{f^{(n)}(0)}{n!}, \tag{1.5}$$

then,

$$g(z) = \sum_{n=0}^{\infty} a_n \Phi(n) z^n \tag{1.6}$$

is an integral function, all of whose zeros are real and negative.

It is remarked that the integral functions or entire functions are those meromorphic functions which have only zeros, but not poles. For example, Titchmarsh [27, p. 285] had pointed out that the generalized hypergeometric function defined by the formula

$${}_pF_q(\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q;z) = \sum_{n=0}^{\infty}\frac{(\alpha_1)_n\ldots(\alpha_p)_n}{(\beta_1)_n\ldots(\beta_q)_n}\frac{z^n}{n!},$$
$$(\alpha)_n = \alpha(\alpha+1)\ldots(\alpha+n-1), (\alpha)_0 = 1,$$

is an integral function, if $q \geq p$, which has no poles.

Making an application of above theorem of Laguerre (Theorem 1.1), Peresyolkova [19] has derived that the function defined by

$$\varphi_{\rho_1,\rho_2}(z,\mu_1,\mu_2) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\mu_1+\frac{n}{\rho_1})\Gamma(\mu_2+\frac{n}{\rho_2})} \tag{1.7}$$

is an integral function whose all zeros are real and negative as the Mittag-Leffler function given by

$$E_{\rho_1}(z,\mu_1) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\mu_1+\frac{n}{\rho_1})} \tag{1.8}$$

has real and negative non-zero zeros when $\rho_1 \leq \frac{1}{2}$, and also that may be represented in infinite product form as

$$E_{\rho_1}(z,\mu_1) = \frac{1}{\Gamma(\mu_1)}\prod_{n=1}^{\infty}(1+\frac{z}{z_n}), \tag{1.9}$$

where all z_n are positive and that is an entire function having its non -zero real and negative zeros.

On the other hand, Craven and Csordas [5] have defined Laguerre multiplier sequences and then prove that for $\beta > 2$ and $\gamma > 0$, a positive integer $m_0 \in m_0(\beta, \gamma)$ and $m \geq m_0$, the meromorphic function defined by

$$E^{(m)}_{\beta,\gamma}(z) = \sum_{n=0}^{\infty} \frac{\Gamma(m+n+1)}{\Gamma(\beta(m+n)+\gamma)} \frac{z^n}{n!} \tag{1.10}$$

is a meromorphic function of Laguerre-Polya class and has only real zeros. In this connection, the Po'lya-Benz and Po'lya-Schur type theorems are very useful to study the real zeros of polynomials (see Aleman et al. [2]).

Again, from Eqn. (1.8) we have the relation $E_{\frac{1}{\beta}}(z, 1) = E_\beta(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\beta n+1)}$. The complex zeros of this Mittag-Leffler function $E_\beta(z)$ were studied by Wiman [28] and found that the asymptotic curve along which the zeros are located for $0 < \beta < 2$ and showed that they fall on the negative real axis for $\beta \geq 2$.

Hilfer and Seybold [10] also studied the location of zeros of generalized Mittag Leffler function $E_\beta(z)$ as function of β, for the case $1 < \beta < 2$ and noted that with increasing β more and more pairs of zeros collapse onto the negative real axis.

Hanneken et al. [9, pp. 15-26] enumerated the complex zeros of the Mittag-Leffler function $E_\beta(z)$ when $0 < \beta < 1$, the finite real zeros of the $E_\beta(z)$ for the case $1 < \beta < 2$ and that infinite real zeros when $\beta \geq 2$.

Further, for a generalized Mittag-Leffler function, defined by the formula $E_{\beta,\gamma}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\beta n+\gamma)}$, Popov [20] and Popov and Sedletski [21] have worked on its distribution of real zeros and found that all zeros of this function $E_{\beta,\gamma}(z)$ are real negative and simple when $\beta > 2$ and $0 < \gamma \leq (2\beta - 1)$. Also, in the work of [20], it has shown that all zeros of the function $E_{4,9}(z)$ are real, negative and simple.

Recently, Kumar and Pathan [13] have developed another method of sequence of integrals on making an appeal to the Theorem 1.1 and the techniques of Kumar, Pathan and Chandel [15] and Kumar and Srivastava [16] to obtain the distribution of real non-zero zeros of generalized Mittag-Leffler function of Shukla and Prajapati [22].

In our work, we use the techniques due to ([13], [15] and [16]) and consider the conditions due to the work of Popov [20] and Popov and Sedletski [21], and prove that:

Theorem 1.2. *For $z \in \mathbb{C}, \Re(\beta) > 2, 0 < \Re(\gamma) \leq 2\beta - 1, \Re(\delta) > 0, q \in (0,1) \cup \mathbb{N}, q \leq \beta$ and all z_n be positive. The generalized Mittag-Leffler function defined due to Shukla and Prajapati [22] given by*

$$E^{\delta,q}_{\beta,\gamma}(z) = \sum_{n=0}^{\infty} \frac{(\delta)_{qn}}{\Gamma(\beta n+\gamma)n!} z^n, \tag{1.11}$$

here,

$$(\delta)_{qn} = \frac{\Gamma(\delta+qn)}{\Gamma(\delta)} \text{ and when } q \in \mathbb{N}, (\delta)_{qn} = \prod_{r=1}^{q} (\frac{\delta+r-1}{q})_n, \tag{1.12}$$

may be represented by Weierstrass product form (see Kumar and Pathan [13])

$$E^{\delta,q}_{\beta,\gamma}(z) = \frac{1}{\Gamma(\gamma)} \exp[\frac{(\delta)_q \Gamma(\gamma)}{\Gamma(\gamma+\beta)} z] \prod_{n=1}^{\infty} ((1+\frac{z}{z_n}) \exp[-\frac{z}{z_n}]) \tag{1.13}$$

Also that the function $E^{\delta,q}_{\beta,\gamma}(z)$ defined in Eqns. (1.11)-(1.13) is an entire function and it has simple real non-zero and negative zeros.

It is remarkable that on making specialization of the parameters of the function given in Eqns. (1.11)-(1.13), we get the results identical due to the Popov [20] and Popov and Sedletski [21].

1.2 Real zeros due to integral transforms

Csordas and Varga [6] have presented the theorem of Po'lya such that:

Theorem 1.3. *If $K(t)$ and $|K(t)|$ are integrable over $\mathbb{R}$ and $K(t) = O(\exp(-|t|^{2+\alpha})), \alpha > 0$, $t \to \infty$, and $K : \mathbb{R} \to \mathbb{R}$, is real analytic on an interval about origin such that*

$$K(t) = \sum_{n=0}^{\infty} c_n t^n, t \in (-r, r), r > 0, c_n \in \mathbb{R}, n = 0, 1, 2, \ldots. \tag{1.14}$$

Then

$$H(z) = \int_0^{\infty} t^{z-1} K(t) dt \tag{1.15}$$

is a meromorphic function and if $H(z)$ has only real negative zeros, then the entire function

$$F_q(z) = \int_{-\infty}^{\infty} K(t^{2q}) e^{izt} dt \ (q = 1, 2, 3, \ldots) \tag{1.16}$$

has only real zeros.

Here, in our investigations, we study the distribution of real zeros of generalized Mittag-Leffler functions of two variables through above said integral transforms methods and make its computation.

2 Generalized Mittag-Leffler functions of two variables

In this section, we present following generalized Mittag-Leffler functions of two variables:

The two variable generalized Mittag-Leffler functions due to Garg et al. [8] are defined in the form

$$\begin{aligned} E_1(x, y) &= E_1\left(\begin{matrix} \delta_1, \alpha_1; \delta_2, \beta_1 \\ \gamma_1, \alpha_2, \beta_2; \gamma_2, \alpha_3; \gamma_3, \beta_3 \end{matrix} \middle| \begin{matrix} x \\ y \end{matrix}\right) \\ &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\delta_1)_{\alpha_1 m} (\delta_2)_{\beta_1 n}}{\Gamma(\gamma_1 + \alpha_2 m + \beta_2 n)} \frac{x^m}{\Gamma(\gamma_2 + \alpha_3 m)} \frac{y^n}{\Gamma(\gamma_3 + \beta_3 n)} \end{aligned} \tag{2.1}$$

provided that $\gamma_1, \gamma_2, \gamma_3, \delta_1, \delta_2, x, y \in \mathbb{C}, \min\{\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3\} > 0$.

$$\begin{aligned} E_2(x, y) &= E_2\left(\begin{matrix} \delta_1, \alpha_1, \beta_1; \delta_2, \alpha_2 \\ \gamma_1, \alpha_3, \beta_2; \gamma_2, \alpha_4; \gamma_3, \beta_3 \end{matrix} \middle| \begin{matrix} x \\ y \end{matrix}\right) \\ &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\delta_1)_{\alpha_1 m + \beta_1 n} (\delta_2)_{\alpha_2 m}}{\Gamma(\gamma_1 + \alpha_3 m + \beta_2 n)} \frac{x^m}{\Gamma(\gamma_2 + \alpha_4 m)} \frac{y^n}{\Gamma(\gamma_3 + \beta_3 n)} \end{aligned} \tag{2.2}$$

provided that $\gamma_1, \gamma_2, \gamma_3, \delta_1, \delta_2, x, y \in \mathbb{C}, \min\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3\} > 0$.

Recently, Kumar and Pathan [14] have found a two variable generalized Mittag-Leffler function in the study of statistical characteristics, operational formulae of special functions and anomalous diffusion in the form

$$E^{(\delta)}_{\alpha,\beta}(x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\delta)_{m+n} x^m y^n}{m! n! \Gamma(\alpha m + \beta n + 1)}, \forall x, y, \delta \in \mathbb{C}, \alpha > 0, \beta > 0, \Re(\delta) > 0. \tag{2.3}$$

Then, Kumar and Pathan [13] has obtained the distribution of non-zero zeros of generalized Mittag-Leffler function of two variables through its infinite product formula (1.13) and with the help of Eqn. (1.10) and they [13] defined other generalized Mittag-Leffler function of two variables in the form

$$E^{(\delta)}_{\alpha,\beta;\gamma}(x,y)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\delta)_{m+n}x^m y^n}{m!n!\Gamma(\alpha m+\beta n+\gamma)} \tag{2.4}$$

provided that $x,y,\delta,\gamma\in\mathbb{C},\alpha>0,\beta>0,\ \Re(\gamma)>0,\Re(\delta)>0$.

Very recently, Pathan and Kumar [18] have obtained an infinite product formula for the distribution of non-zero zeros of multi-index generalized Mittag-Leffler function.

Kumar [12] has generalized the two variable Mittag-Leffler function given in Eqn. (2.4) in the form

$$E^{(\delta;\alpha_1,\beta_1)}_{\alpha_2,\beta_2;\gamma_1:\alpha_3;\gamma_2:\beta_3;\gamma_3}(x,y)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\delta)_{\alpha_1 m+\beta_1 n}x^m y^n}{\Gamma(\alpha_2 m+\beta_2 n+\gamma_1)\Gamma(\alpha_3 m+\gamma_2)\Gamma(\beta_3 n+\gamma_3)}, \tag{2.5}$$

provided that $x,y,\delta,\gamma_i\in\mathbb{C},\alpha_i>0,\beta_i>0,\ \Re(\gamma_i)>0,\Re(\delta)>0,i=1,2,3$, and studied its convergence conditions and some of its relations to various special functions.

3 Real zeros of Mittag-Leffler functions and integral functions involving generalized Mittag-Leffler functions of two variables through integral transforms

In this section, to obtain the distribution of real zeros of Mittag-Leffler functions of two variables and the integral functions involving two variables Mittag-Leffler functions, we present some theorems.

Motivated by above work, Theorem 1.3 and the work of Cardon [3], in our investigations, we present following theorems:

Theorem 3.1. *If $K(t)$ and $|K(t)|$ are integrable over $\mathbb{R}$ and $K(t)=O(\exp(-|t|^{2+\alpha})),\alpha>0,t\to\infty$, and $K:\mathbb{R}\to\mathbb{R}$, is real analytic on an interval about origin such that*

$$K(t)=\sum_{n=0}^{\infty}c_n t^n,t\in(-r,r),r>0,c_n\in\mathbb{R},n=0,1,2,\ldots.$$

Also, if we let

$$G(t)=e^{pt}K(t),p>0. \tag{3.1}$$

Then

$$H(p,z)=\int_0^{\infty}e^{-pt}t^{z-1}G(t)dt \tag{3.2}$$

is a meromorphic function and if $H(p,z)$ has only real negative zeros, then the entire function

$$F_q(p,z)=\int_{-\infty}^{\infty}e^{-pt^{2q}}G(t^{2q})e^{izt}dt \quad (p>0,q=1,2,3,\ldots) \tag{3.3}$$

has only real zeros and hence then inverses also have only real zeros for all $q=1,2,3,...$

Proof. On using Theorem 1.3 we easily satisfy the Eqns. (3.1) and (3.2). Again Eqn. (3.2) shows that $H(p,z)$ is Laplace transform of the function $t^{z-1}G(t)$. Therefore the inverse Laplace transform of $H(p,z)$ is an entire function and it has negative real zeros as when $H(p,z)$ has negative real zeros. On applying the result for Dirac delta function such that

$\frac{1}{2\pi i}\lim_{\beta\to\infty}\int_{\gamma-i\beta}^{\gamma+i\beta} e^{(t-\theta)p}dp = \delta(t-\theta)$, (see Kanwal [11, p. 68]), and Eqn. (3.2), this implies that the inverse Laplace transform of $H(p,z)$ i.e.

$$\begin{aligned}
L^{-1}\{H(p,z)\} &= \frac{1}{2\pi i}\lim_{\beta\to\infty}\int_{\gamma-i\beta}^{\gamma+i\beta} e^{tp}H(p,z)dp \\
&= \frac{1}{2\pi i}\lim_{\beta\to\infty}\int_{\gamma-i\beta}^{\gamma+i\beta} e^{tp}\int_0^\infty e^{-p\theta}\theta^{z-1}G(\theta)d\theta dp \\
&= \int_0^\infty \theta^{z-1}G(\theta)\frac{1}{2\pi i}\lim_{\beta\to\infty}\int_{\gamma-i\beta}^{\gamma+i\beta} e^{(t-\theta)p}dpd\theta \\
&= \int_0^\infty \theta^{z-1}G(\theta)\delta(t-\theta)d\theta = \begin{cases} 0, \text{when}, t=0, z>1, \\ 0, \text{when}, \theta \neq t. \end{cases} \qquad (3.4)
\end{aligned}$$

In the similar manner, the inverse Laplace transform of $F_q(p,z)$ is an entire function and it has real zeros. That is

$$L^{-1}\{F_q(p,z)\} = \int_{-\infty}^{\infty} G(\theta^{2q})e^{iz\theta}\delta(t-\theta^{2q})d\theta, \qquad (3.5)$$

has real zeros for $q = 1,2,3,...$ □

On the other hand, we also present:

Theorem 3.2. *If F be of order $e^{\alpha t}, t>0$ and for some $\alpha > 0$ that is F be of $O(e^{\alpha t})$ for some α and if the image $f(p) = \int_0^\infty e^{-pt}F(t)dt, \mathfrak{R}(p) > \alpha$, is an analytic and entire function of the type given in Eqn. (1.4), of Theorem 1.1, and has negative real non-zero zeros and then, for any $q \geq \alpha$ the function $G(t) = e^{-qt}F(t), t>0, G(t) = 0, t<0$ be an entire function and also has only real zeros.*

Proof. The Fourier integral formula has the exponential form

$$G(t) = \frac{1}{2\pi}\lim_{s\to\infty}\int_{-s}^{s} e^{iyt}\int_{-\infty}^{\infty} G(\tau)e^{-iy\tau}d\tau dy, (-\infty < t < \infty). \qquad (3.6)$$

Therefore, for $q > \alpha$, the Eqn. (3.6) may be written by

$$G(t) = \frac{1}{2\pi}\lim_{s\to\infty}\int_{-s}^{s} e^{iyt}\int_0^\infty F(\tau)e^{-(q+iy)\tau}d\tau dy, (-\infty < t < \infty). \qquad (3.7)$$

Then, for $z = q + iy$ and for all real t, from Eqn. (3.7), we have an entire function

$$e^{qt}G(t) = \frac{1}{2\pi i}\lim_{s\to\infty}\int_{q-is}^{q+is} e^{zt}f(z)dz \qquad (3.8)$$

Again due to the statement of the Theorem 3.2, the inverse Laplace transformation of $f(p)$ is given by

$$F(t) = \frac{1}{2\pi i}\lim_{s\to\infty}\int_{q-is}^{q+is} e^{zt}f(z)dz \qquad (3.9)$$

Apply Eqn. (1.4) in the Eqn. (3.9), we may write

$$F(t-b) = \frac{e^c}{2\pi i}\lim_{s\to\infty}\int_{q-is}^{q+is} e^{zt}\prod_{n=1}^{\infty}(1+\frac{z}{z_n})dz \qquad (3.10)$$

where b and all z_n are positive.

Now, make an appeal to the result of Erde'lyi et al [7, Vol. II, p. 98, Eqn. (1.1)], (see also Srivastava and Manocha [26, p. 253, Eqn. (3.3)]), given by

$$e^{zt}=\frac{\Gamma(\nu)}{(\frac{1}{2}t)^{\nu}}\sum_{m=0}^{\infty}(\nu+m)I_{\nu+m}(t)C_m^{\nu}(z),\nu>0$$

and v is not integer, and the transformation formula,

$$I_{\nu}(z)=e^{-\frac{1}{2}\nu\pi i}J_{\nu}(iz),(-\pi<arg(z)<\frac{\pi}{2}),$$

due to Abramowitz and Stegun [1, p. 377, Eqn. (9.6.3)],

Or such that for $(-\pi<arg(z)<\frac{\pi}{2})$, we have

$$\begin{aligned}e^{zt}=e^{(-iz)(it)}&=\frac{\Gamma(\nu)}{(\frac{1}{2}it)^{\nu}}\sum_{m=0}^{\infty}(\nu+m)I_{\nu+m}(it)C_m^{\nu}(-iz)\\&=\frac{\Gamma(\nu)e^{-i\nu\pi}}{(\frac{1}{2}t)^{\nu}}\sum_{m=0}^{\infty}(\nu+m)e^{-\frac{1}{2}m\pi i}J_{\nu+m}(-t)C_m^{\nu}(-iz),\end{aligned}$$

in the integrand of Eqn. (3.10), we find that

$$F(t)=\frac{\Gamma(\nu)e^{-\nu\pi i}}{(\frac{1}{2}(t+b))^{\nu}}\sum_{m=0}^{\infty}(\nu+m)e^{-\frac{1}{2}m\pi i}J_{\nu+m}(-(t+b))H_m^{\nu,c}. \tag{3.11}$$

Here in Eqn. (3.11), we let the integral function

$$H_m^{\nu,c}=\frac{e^c}{2\pi i}\lim_{s\to\infty}\int_{q-is}^{q+is}C_m^{\nu}(-iz)\prod_{n=1}^{\infty}(1+\frac{z}{z_n})dz, \tag{3.12}$$

provided that $-\pi<arg(z)<\frac{\pi}{2}$.

Due to occurrence of Bessel functions in above Eqn. (3.11), the function $F(t)$ has real zeros due to Bessel functions of order $\nu,\nu+1,\nu+2,\dots,\ \nu$ is not an integer and use of Hurwitz Theorem (Titchmarsh [27, p.119]). Again then equate it to the Eqn. (3.8), we say that the function $G(t)$ also has real zeros. □

Applications

Now, here, we consider the Mittag-Leffler functions of two variables defined in Section 2 in following form and apply above theory and the integral transform methods given in Theorems 3.1 and 3.2 to achieve our goal as:

Theorem 3.3. *If*

$$F_1(t)=t^{\gamma_1-1}E_1(xt^{\alpha_2},yt^{\beta_2})=t^{\gamma_1-1}E_1\left(\begin{matrix}\delta_1,\alpha_1;\delta_2,\beta_1\\ \gamma_1,\alpha_2,\beta_2;\gamma_2,\alpha_3;\gamma_3,\beta_3\end{matrix}\ \middle|\ \begin{matrix}xt^{\alpha_2}\\ yt^{\beta_2}\end{matrix}\right),\ t>0,$$

where two variable generalized Mittag-Leffler function $E_1(x,y)$ is defined in Eqn. (2.1), then for $\gamma_1>0,\gamma_2=2\alpha_3-1,\gamma_3=2\beta_3-1,\delta_1,\delta_2,x,y\in\mathbb{C},\min\{\alpha_1,\alpha_2,\beta_1,\beta_2,\}>0,\alpha_3>2,\beta_3>2,\Re(\delta_j)>0,\forall j=1,2,\alpha_1\le\alpha_3,\beta_1\le\beta_3,F_1(t)$ has real zeros for positive t and negative x or y.

Proof. The Laplace transformation of the function $F_1(t)=t^{\gamma_1-1}E_1(xt^{\alpha_2},yt^{\beta_2})$ is obtained by

$$f_1(p;\alpha_1,\alpha_3,\beta_1,\beta_3,\gamma_1,\gamma_2,\gamma_3,\delta_1,\delta_2,x,y)=\frac{1}{p^{\gamma_1}}E_{\alpha_3,\gamma_2}^{\delta_1,\alpha_1}(\frac{x}{p^{\gamma_1}})E_{\beta_3,\gamma_3}^{\delta_2,\beta_1}(\frac{y}{p^{\gamma_1}}),\Re(p)>0, \tag{3.13}$$

where, the generalized Mittag-Leffler function $E^{\delta,q}_{\beta,\gamma}(.)$is defined in Eqn. (1.11).

Now, make an appeal to the Eqns. (3.9) and (3.12) to get

$$F_1(t) = \frac{1}{2\pi i}\lim_{s\to\infty}\int_{q-is}^{q+is} e^{pt}p^{-\gamma_1}E^{\delta_1,\alpha_1}_{\alpha_3,\gamma_2}(\frac{x}{p^{\gamma_1}})E^{\delta_2,\beta_1}_{\beta_3,\gamma_3}(\frac{y}{p^{\gamma_1}})dp \tag{3.14}$$

Due to occurrence of the product of generalized Mittag-Leffler functions

$$E^{\delta_1,\alpha_1}_{\alpha_3,\gamma_2}(\frac{x}{p^{\gamma_1}})E^{\delta_2,\beta_1}_{\beta_3,\gamma_3}(\frac{y}{p^{\gamma_1}})$$

in the Eqn. (3.13), we appeal to the Theorem 1.2 and the theory of Popov [20] and Popov and Sedletski [21], that for the given conditions, $p > 0, \gamma_1 > 0, \gamma_2 = 2\alpha_3 - 1, \gamma_3 = 2\beta_3 - 1$, $\delta_1, \delta_2, x, y \in \mathbb{C}$, $\min\{\alpha_1, \alpha_2, \beta_1, \beta_2,\} > 0, \alpha_3 > 2, \beta_3 > 2, \Re(\delta_j) > 0, \forall j = 1, 2, \alpha_1 \le \alpha_3, \beta_1 \le \beta_3$, f_1 has real and negative zeros for x or y. Then, in Eqn. (3.14) use the Theorems 3.1 and 3.2, $F_1(t)$ has real zeros for positive t and negative x or y. □

Example 3.1. *In Theorem 3.3, if we take* $\delta_1 = 0.5, \alpha_1 = 2.4, \alpha_3 = 2.5$, *then,* $\gamma_2 = 4$, *and again, take* $\delta_2 = 0.4, \beta_1 = 2.4, \beta_3 = 2.5$, *then,* $\gamma_3 = 4$, *again set* $x = 5, t \in (0, 10), y \in (-10, 10)$, *in Eqn. (3.14), then in the Figure 3.1,* $F_1(t)$ *seems to be zero when* $1 < t < 4$ *and for negative values of* y. *Although for negative values of* t *it gives no figure.*

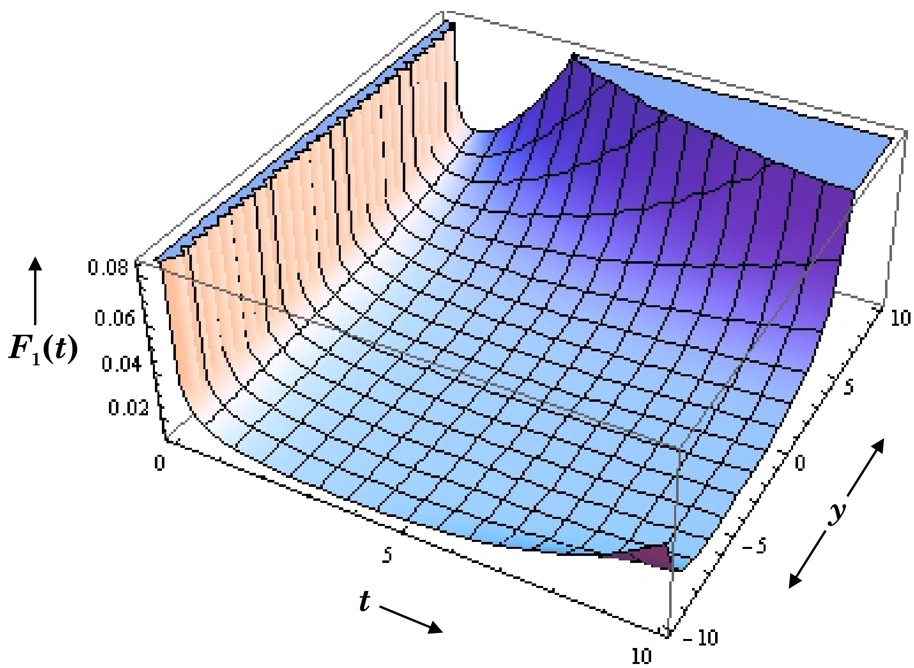

Figure 3.1

Example 3.2. *In Theorem 3.3, if we take* $\delta_1 = 0.5, \alpha_1 = 2.4, \alpha_3 = 2.5$, *then,*$\gamma_2 = 4$, *and again, take* $\delta_2 = 0.4, \beta_1 = 2.4, \beta_3 = 2.5$, *then,* $\gamma_3 = 4$, *again set* $t = 5, x \in (-10, 10), y \in (-10, 10)$, *in Eqn. (3.14), then in the Figure 3.2,* $F_1(t)$ *seems to be zero when* x *and* y *are negative.*

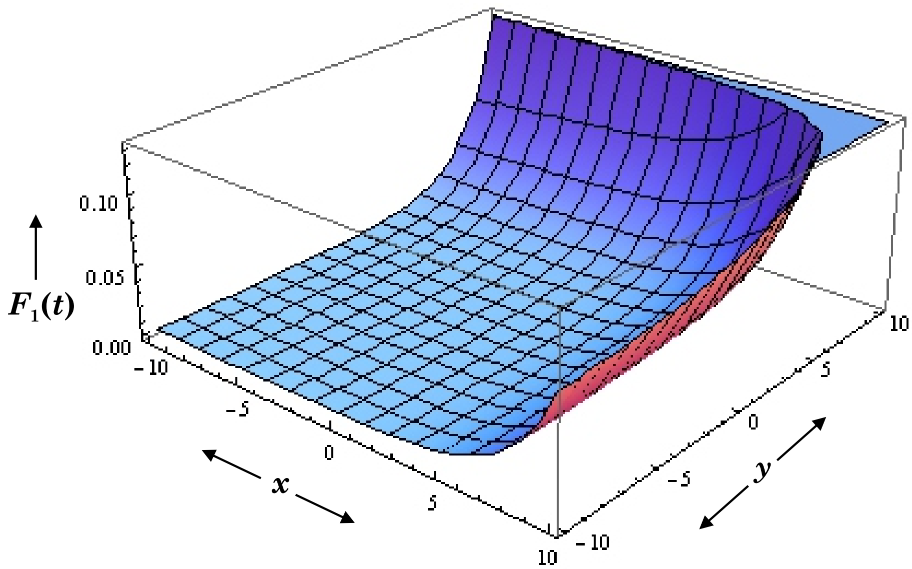

Figure 3.2

Theorem 3.4. *If*

$$F_1(t) = t^{\gamma_1 - 1} E_1(xt^{\alpha_2}, yt^{\beta_2}) = t^{\gamma_1 - 1} E_1 \left(\begin{matrix} \delta_1, \alpha_1; \delta_2, \beta_1 \\ \gamma_1, \alpha_2, \beta_2; \gamma_2, \alpha_3; \gamma_3, \beta_3 \end{matrix} \middle| \begin{matrix} xt^{\alpha_2} \\ yt^{\beta_2} \end{matrix} \right), \; t > 0,$$

where two variable generalized Mittag-Leffler function $E_1(x, y)$ is defined in Eqn. (2.1), then for $\gamma_1, \gamma_2, \gamma_3, \delta_1, \delta_2,\ x, y \in \mathbb{C}, \min\{\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3\} > 0. F_1(t)$ has real zeros.

Proof. In Eqn. (3.14), use the techniques of the Theorem 3.2 to get

$$F_1(t) = \frac{\Gamma(\nu) e^{-\nu\pi i}}{(\frac{1}{2}(t))^{\nu}} \sum_{m=0}^{\infty} (\nu + m) J_{\nu+m}(-(t)) e^{-\frac{1}{2} m\pi i} H_m^{\nu,0} \tag{3.15}$$

where, $H_m^{\nu,0} = \frac{1}{2\pi i} \lim_{s\to\infty} \int_{q-is}^{q+is} C_m^{\nu}(-ip) p^{-\gamma_1} E_{\alpha_3,\gamma_2}^{\delta_1,\alpha_1}(\frac{x}{p^{\gamma_1}}) E_{\beta_3,\gamma_3}^{\delta_2,\beta_1}(\frac{y}{p^{\gamma_1}}) dp.$ $\nu > 0$ and is not an integer and provided that $-\pi < arg(p) < \frac{\pi}{2}$.

Here, in right hand side of the Eqn. (3.15), each term of the series has the Bessel functions of order $\nu, \nu + 1, \nu + 2, \ldots$, where ν is not an integer, so that on applying the theory of the Theorem 3.2, we have that for $\gamma_1, \gamma_2, \gamma_3, \delta_1, \delta_2,\ x, y \in \mathbb{C}, \min\{\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3\} > 0$, $F_1(t)$ has real zeros whenever $\nu > 0$, and is not an integer, also $e^{-\frac{1}{2}(m+2\nu)\pi i} H_m^{\nu,0}$ is a real number. □

Example 3.3. *In Theorem 3.4, if we consider $e^{-(2\nu+m)\frac{\pi}{2}i} = (2 + \sqrt{3})$ (any real number), $\nu = .5, t \in (-100, 100)$, then in Figure 3.3, $F_1(t)$ seems to be zero for real values of t.*

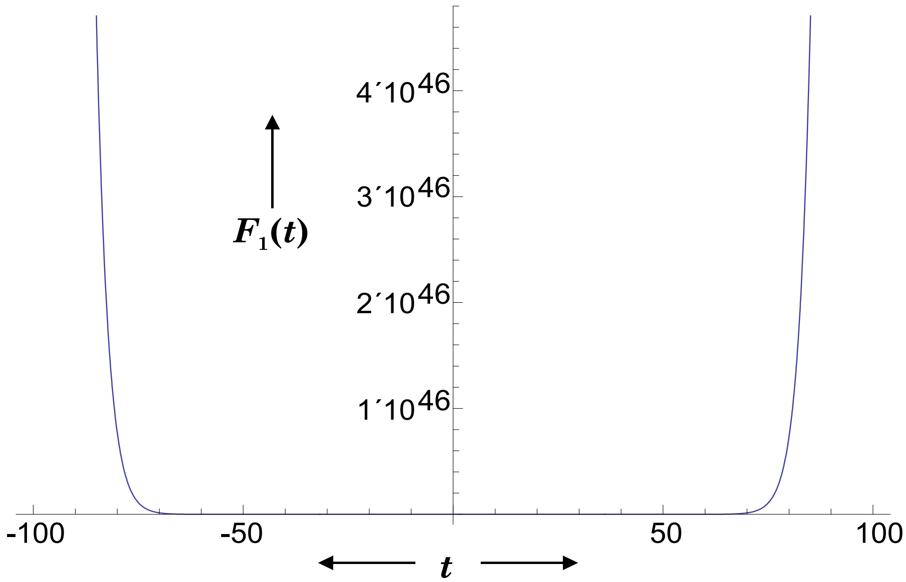

Figure 3.3

Theorem 3.5. *If*

$$F_2(t) = \Gamma(\delta_1) t^{\gamma_1 - 1} \frac{1}{2\pi i} \int_{-\infty}^{0^+} e^u u^{-\delta_1} E_2(x \frac{t^{\alpha_3}}{u^{\alpha_1}}, y \frac{t^{\beta_2}}{u^{\beta_1}}) du, \tag{3.16}$$

where, two variable generalized Mittag-Leffler function

$$E_2(x, y) = E_2 \left(\begin{array}{c|c} \delta_1, \alpha_1, \beta_1; \delta_2, \alpha_2 & x \\ \gamma_1, \alpha_3, \beta_2; \gamma_2, \alpha_4; \gamma_3, \beta_3 & y \end{array} \right)$$

is defined in Eqn. (2.2), then for $\gamma_1 > 0, \gamma_2 = 2\alpha_4 - 1, \gamma_3 = 2\beta_3 - 1, \delta_1, \delta_2, x, y \in \mathbb{C}, \Re(\delta_j) > 0, \forall j = 1, 2,$ $min\{\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2\} > 0, \alpha_4 > 2, \beta_3 > 2, \alpha_2 \leq \alpha_4, F_2(t)$ *has real zeros.*

Proof. In right hand side of Eqn. (3.16), define $E_2(x, y)$ due to Eqn. (2.2) and then make an application of the formula of Erde'lyi et al. [7, Vol. I] given by

$$\frac{1}{\Gamma(\lambda + m)} = \frac{1}{2\pi i} \int_{-\infty}^{0^+} e^u u^{-\lambda - m} du, \Re(\lambda) > 0, \ m = 0, 1, 2, 3, \ldots, \tag{3.17}$$

we get

$$F_2(t) = t^{\gamma_1 - 1} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\delta_2)_{\alpha_2 m}}{\Gamma(\gamma_1 + \alpha_3 m + \beta_2 n)} \frac{(x t^{\alpha_3})^m}{\Gamma(\gamma_2 + \alpha_4 m)} \frac{(y t^{\beta_2})^n}{\Gamma(\gamma_3 + \beta_3 n)}. \tag{3.18}$$

Then, take Laplace transformation of $F_2(t)$ given in Eqn. (3.18) to get

$$f_2(p; \alpha_1, \alpha_3, \beta_1, \beta_3, \gamma_1, \gamma_2, \gamma_3, \delta_1, \delta_2, x, y) = \frac{1}{p^{\gamma_1}} E_{\alpha_4, \gamma_2}^{\delta_2, \alpha_2} (\frac{x}{p^{\gamma_1}}) E_{\beta_3, \gamma_3} (\frac{y}{p^{\gamma_1}}), \Re(p) > 0 \tag{3.19}$$

where, the generalized Mittag-Leffler function $E_{\beta,\gamma}(.)$ is defined in fourth paragraph of Eqn. (1.10). Now, then use the techniques of Theorem 3.3 in Eqn. (3.19), we may say that $F_2(t)$, defined in Eqn. (3.18), has real zeros. □

Example 3.4. *In Eqn. (3.19) of Theorem 3.5, set* $\gamma_1 = .5, \delta_2 = .5, \alpha_2 = 2.5, \alpha_4 = 3, \beta_3 = 2.5$*, then* $\gamma_2 = 5$*, and* $\gamma_3 = 4$*,* $x \in (-10, 10)$*,* $y \in (-10, 10)$ $F_2(t)$*, defined in Eqn. (3.16), has real zeros in Figure 3.4.*

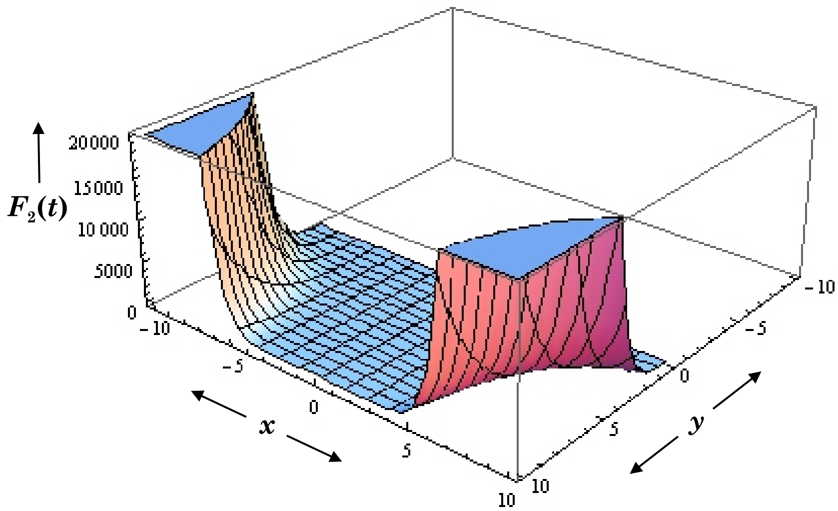

Figure 3.4

Theorem 3.6. *If*

$$F_3(t) = \Gamma(\delta)t^{\gamma_1-1}\frac{1}{2\pi i}\int_{-\infty}^{0^+} e^u u^{-\delta} E^{(\delta;\alpha_1,\beta_1)}_{\alpha_2,\beta_2;\gamma_1:\alpha_3;\gamma_2:\beta_3;\gamma_3}(x\frac{t^{\alpha_2}}{u^{\alpha_1}}, y\frac{t^{\beta_2}}{u^{\beta_1}})du, \tag{3.20}$$

where, two variable generalized Mittag-Leffler function $E^{(\delta;\alpha_1,\beta_1)}_{\alpha_2,\beta_2;\gamma_1:\alpha_3;\gamma_2:\beta_3;\gamma_3}(.,.)$ *is defined in Eqn. (2.5), then for* $\gamma_1 > 0, \gamma_2 = 2\alpha_3 - 1, \gamma_3 = 2\beta_3 - 1, \delta, x, y \in \mathbb{C}, \min\{\alpha_1, \alpha_2, \beta_1, \beta_2\} > 0, \alpha_3 > 2, \beta_3 > 2, \Re(\gamma_i) > 0, \forall i = 1, 2, 3; \Re(\delta) > 0,$ $F_3(t)$ *has real zeros.*

Proof. Use the techniques of Theorem 3.5 in Eqn. (3.20) to get

$$f_3(p;\alpha_1,\alpha_3,\beta_1,\beta_3,\gamma_1,\gamma_2,\gamma_3,\delta_1,\delta_2,x,y) = \frac{1}{p^{\gamma_1}}E_{\alpha_3,\gamma_2}(\frac{x}{p^{\gamma_1}})E_{\beta_3,\gamma_3}(\frac{y}{p^{\gamma_1}}), \Re(p) > 0 \tag{3.21}$$

Then use the techniques of Theorem 3.3 in Eqn. (3.21), we may say that $F_3(t)$, defined in Eqn. (3.20), has real zeros. □

Theorem 3.7. *If*

$$F_2(t) = \Gamma(\delta_1)t^{\gamma_1-1}\frac{1}{2\pi i}\int_{-\infty}^{0^+} e^u u^{-\delta_1} E_2(x\frac{t^{\alpha_3}}{u^{\alpha_1}}, y\frac{t^{\beta_2}}{u^{\beta_1}})du, \tag{3.22}$$

where, two variable generalized Mittag-Leffler function

$$E_2(x,y) = E_2\left(\begin{matrix}\delta_1,\alpha_1,\beta_1;\delta_2,\alpha_2\\ \gamma_1,\alpha_3,\beta_2;\gamma_2,\alpha_4;\gamma_3,\beta_3\end{matrix}\,\middle|\,\begin{matrix}x\\ y\end{matrix}\right)$$

is defined in Eqn. (2.2), then for $\gamma_1, \gamma_2, \gamma_3, \delta_1, \delta_2, x, y \in \mathbb{C}, \min\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3\} > 0, F_2(t)$ *has real zeros.*

Proof. Use the techniques of the Theorem 3.5 in Eqn. (3.22) and make an appeal to the Theorem 3.2, and for $-\pi < arg(p) < \frac{\pi}{2}$, we get

$$F_2(t) = \frac{1}{2\pi i}\lim_{s\to\infty}\int_{q-is}^{q+is} e^{pt}p^{-\gamma_1}E^{\delta_2,\alpha_2}_{\alpha_4,\gamma_2}(\frac{x}{p^{\gamma_1}})E_{\beta_3,\gamma_3}(\frac{y}{p^{\gamma_1}})dp \tag{3.23}$$

Then, use the theory of the Theorem 3.4 in Eqn. (3.23), we may prove that for $\gamma_1, \gamma_2, \gamma_3, \delta_1, \delta_2, x, y \in \mathbb{C}, \min\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3\} > 0, F_2(t)$ has real zeros. □