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MITTAG-LEFFLER POLYNOMIAL DIFFERENTIAL EQUATION

By

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Abstract

By using the monomiality principle and general results on Sheffer polynomial sets, the differential equation satisfied by the Mittag-Leffler polynomial is shown.

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Keywords and phrases: Mittag-Leffler polynomial, Generating functions, Monomiality principle, Differential equations.

1 Introduction

In recent articles the notion of adjoint Sheffer polynomial sets has been introduced [12] and applied in several cases [5, 10, 11, 13]. In particular, the differential equations satisfied by the considered polynomials have been derived. It is worth to note that all the obtained differential equations are of infinite order, but they reduce to differential equations of finite order when acting on polynomials. This is also the case of the differential equation satisfied by the Mittag-Leffler polynomials as it will be shown in the following.

The Mittag-Leffler polynomials were considered in a paper by H. Bateman [9]. See also S. Roman [14], Sect. 4.1.6, pp. 75-78 and 127. Their basic properties are summarized in [18].

This approach is based on the monomiality principle introduced and widely applied by G. Dattoli [6, 7, 8] and on a general result by Y. Ben Cheikh [2] which gives explicitly the differential equation satisfied by a Sheffer polynomial set in terms of the increasing and lowering operators for the same set.

2 Sheffer polynomials

The Sheffer polynomials $\{s_n(x)\}$ are introduced [15] by means of the exponential generating function [16] of the type:

$$A(t)\exp(xH(t)) = \sum_{n=0}^{\infty} s_n(x)\,\frac{t^n}{n!}\ , \tag{2.1}$$

where

$$\begin{aligned} A(t) &= \sum_{n=0}^{\infty} a_n\,\frac{t^n}{n!}\ , \qquad (a_0 \neq 0)\,, \\ H(t) &= \sum_{n=0}^{\infty} h_n\,\frac{t^n}{n!}\ , \qquad (h_0 = 0)\,. \end{aligned} \tag{2.2}$$

According to a different characterization (see [14, p. 18]), the same polynomial sequence can be defined by means of the pair $(g(t), f(t))$, where $g(t)$ is an invertible series and $f(t)$ is a delta series:

$$\begin{aligned} g(t) &= \sum_{n=0}^{\infty} g_n \frac{t^n}{n!}\,, \qquad (g_0 \neq 0)\,, \\ f(t) &= \sum_{n=0}^{\infty} f_n \frac{t^n}{n!}\,, \qquad (f_0 = 0,\ f_1 \neq 0)\,. \end{aligned} \tag{2.3}$$

Denoting by $f^{-1}(t)$ the compositional inverse of $f(t)$ (i.e. such that $f\left(f^{-1}(t)\right) = f^{-1}\left(f(t)\right) = t$), the exponential generating function of the sequence $\{s_n(x)\}$ is given by

$$\frac{1}{g[f^{-1}(t)]} \exp\left(x f^{-1}(t)\right) = \sum_{n=0}^{\infty} s_n(x) \frac{t^n}{n!}\,, \tag{2.4}$$

so that

$$A(t) = \frac{1}{g[f^{-1}(t)]}\,, \qquad H(t) = f^{-1}(t)\,. \tag{2.5}$$

When $g(t) \equiv 1$, the Sheffer sequence corresponding to the pair $(1, f(t))$ is called the associated Sheffer sequence $\{\sigma_n(x)\}$ for $f(t)$, and its exponential generating function is given by

$$\exp\left(x f^{-1}(t)\right) = \sum_{n=0}^{\infty} \sigma_n(x) \frac{t^n}{n!}\,. \tag{2.6}$$

A list of known Sheffer polynomial sequences and their associated ones can be found in [4].

3 The Mittag-Leffler polynomials

We recall that the Mittag-Leffler polynomials are a special case of associated Sheffer polynomials, defined by the generating function

$$\begin{aligned} &A(t) = 1, \qquad H(t) = \log\frac{1+t}{1-t}\,, \\ &G(t,x) = \left(\frac{1+t}{1-t}\right)^x = \exp\left(x\,\log\frac{1+t}{1-t}\right) = \sum_{n=0}^{\infty} \mathcal{M}_n(x) \frac{t^n}{n!}\,. \end{aligned} \tag{3.1}$$

3.1 Shift operators

We recall that a polynomial set $\{p_n(x)\}$ is called quasi-monomial if and only if there exist two operators $\hat{P}$ and $\hat{M}$ such that

$$\hat{P}\left(p_n(x)\right) = n p_{n-1}(x)\,, \qquad \hat{M}\left(p_n(x)\right) = p_{n+1}(x)\,, \qquad (n = 1, 2, \dots). \tag{3.2}$$

$\hat{P}$ is called the *derivative* operator and $\hat{M}$ the *multiplication* operator, as they act in the same way of classical operators on monomials.

This definition traces back to a paper by J.F. Steffensen [17], recently improved by G. Dattoli [6] and widely used in several applications.

Y. Ben Cheikh [2] proved that every polynomial set is quasi-monomial under the action of suitable derivative and multiplication operators. In particular, in the same article (Corollary 3.2), the following result is proved.

Theorem 3.1. *Let $(p_n(x))$ denote a Boas-Buck polynomial set, that is a set defined by the generating function*

$$A(t)\psi(xH(t)) = \sum_{n=0}^{\infty} p_n(x)\,\frac{t^n}{n!}\ , \tag{3.3}$$

where

$$\begin{aligned} A(t) &= \sum_{n=0}^{\infty} \tilde{a}_n t^n\ , \qquad (\tilde{a}_0 \neq 0)\,, \\ \psi(t) &= \sum_{n=0}^{\infty} \tilde{\gamma}_n t^n\ , \qquad (\tilde{\gamma}_n \neq 0 \quad \forall n)\,, \end{aligned} \tag{3.4}$$

with $\psi(t)$ not a polynomial, and lastly

$$H(t) = \sum_{n=0}^{\infty} \tilde{h}_n\, t^{n+1}\ , \qquad (\tilde{h}_0 \neq 0)\,. \tag{3.5}$$

Let $\sigma \in \Lambda^{(-)}$ the lowering operator defined by

$$\sigma(1) = 0\,, \qquad \sigma(x^n) = \frac{\tilde{\gamma}_{n-1}}{\tilde{\gamma}_n}\ x^{n-1}\,, \qquad (n = 1, 2, \dots). \tag{3.6}$$

Put

$$\sigma^{-1}(x^n) = \frac{\tilde{\gamma}_{n+1}}{\tilde{\gamma}_n}\ x^{n+1} \quad (n = 0, 1, 2, \dots). \tag{3.7}$$

Denoting, as before, by $f(t)$ the compositional inverse of $H(t)$, the Boas-Buck polynomial set $\{p_n(x)\}$ is quasi-monomial under the action of the operators

$$\hat{P} = f(\sigma)\,, \qquad \hat{M} = \frac{A'[f(\sigma)]}{A[f(\sigma)]} + xD_x H'[f(\sigma)]\sigma^{-1}\,, \tag{3.8}$$

where prime *denotes the ordinary derivatives with respect to t.*

Note that in our case we are dealing with a Sheffer polynomial set, so that since we have $\psi(t) = e^t$, the operator σ defined by equation (3.6) simply reduces to the derivative operator D_x. Furthermore, we have

$$\frac{A'(t)}{A(t)} = 0\,, \qquad H'(t) = \frac{2}{1-t^2}\,, \qquad H^{-1}(t) = f(t) = \frac{e^t - 1}{e^t + 1}\,, \tag{3.9}$$

so that, for the Mittag-Leffer polynomials, we have

$$\begin{aligned} \hat{P} &= \frac{e^{D_x} - 1}{e^{D_x} + 1} = \tanh\left(\frac{D_x}{2}\right)\,, \\ \hat{M} &= x\,\frac{(e^{D_x} + 1)^2}{2\,e^{D_x}} = x\,\left[1 + \cosh(D_x)\right]\,. \end{aligned} \tag{3.10}$$

Remark 3.1. *The definition of $\hat{P}$ can be obtained as follows.*

First, note that the following expansion holds:

$$\frac{e^t - 1}{e^t + 1} = \tanh\left(\frac{t}{2}\right) = \sum_{k=0}^{\infty} a_k\,\frac{t^k}{k!}\,, \tag{3.11}$$

where the coefficients a_k are solution of the following system

$$\begin{aligned} & a_0 = 0 \\ & a_k + \sum_{h=1}^{k} \binom{k}{h} a_h = 1 \,, \qquad (k \geq 1)\,, \end{aligned} \tag{3.12}$$

so that, for the first few values, we find

$$\begin{aligned} & a_{2h} = 0\,, \qquad \forall h \geq 0\,, \\ & a_1 = \tfrac{1}{2}\,, \qquad && a_3 = -\tfrac{1}{4}\,, \\ & a_5 = \tfrac{1}{2}\,, && a_7 = -\tfrac{17}{8}\,, \\ & a_9 = \tfrac{31}{2}\,, && a_{11} = -\tfrac{691}{4}\,, \\ & a_{13} = \tfrac{5461}{2}\,, && a_{13} = -\tfrac{929569}{16}\,. \end{aligned}$$

Further values can be easily achieved by using Wolfram Alpha©.

Therefore, we can put:

$$\hat{P} = \frac{e^{D_x} - 1}{e^{D_x} + 1} = \tanh\left(\frac{D_x}{2}\right) = \sum_{k=0}^{\infty} a_k \, \frac{D_x^k}{k!}\,. \tag{3.13}$$

In a similar way the operator $\hat{M}$ can be defined. In fact, we have:

$$\hat{M} = x\,[1 + \cosh(D_x)] = x\left[1 + \sum_{k=0}^{\infty} \frac{D_x^{2k}}{(2k)!}\right]. \tag{3.14}$$

3.2 Differential equation

According to the results of monomiality principle [6], the quasi-monomial polynomials $\{p_n(x)\}$ satisfy the differential equation

$$\hat{M}\hat{P}\,p_n(x) = n\,p_n(x)\,. \tag{3.15}$$

In the present case, according to the identity:

$$[1 + \cosh x]\,\tanh(x/2) = \sinh x\,,$$

we can write

$$\hat{M}\,\hat{P} = x\,\frac{e^{2\,D_x} - 1}{2\,e^{D_x}} = x\,\sinh(D_x)\,, \tag{3.16}$$

so that we have the theorem

Theorem 3.2. *The Mittag-Leffler polynomials $\{\mathcal{M}_n(x)\}$ satisfy the differential equation*

$$x\,\sinh(D_x)\,\mathcal{M}_n(x) = n\,\mathcal{M}_n(x)\,, \tag{3.17}$$

that is

$$x\sum_{k=0}^{\infty} \frac{D_x^{2k+1}}{(2k+1)!}\mathcal{M}_n(x) = n\mathcal{M}_n(x), \tag{3.18}$$

or

$$x\sum_{k=0}^{\left[\frac{n-1}{2}\right]} \frac{D_x^{2k+1}}{(2k+1)!}\,\mathcal{M}_n(x) = n\,\mathcal{M}_n(x)\,. \tag{3.19}$$

where $\left[\frac{n-1}{2}\right]$ denotes the integral part *of* $(n-1)/2$.

Proof. It is sufficient to expand in series the operator (3.16). Equation (3.19) follows because, for any fixed n, the series expansion in equation (3.18) reduces to a finite sum when applied to a polynomial of degree n. □

Remark 3.2. The first few values of the Mittag-Leffler polynomials are as follows:

$$\begin{aligned}
&\mathcal{M}_0(x) = 1\,,\\
&\mathcal{M}_1(x) = 2x\,,\\
&\mathcal{M}_2(x) = 4x^2\,,\\
&\mathcal{M}_3(x) = 8x^3 + 4x\,,\\
&\mathcal{M}_4(x) = 16x^4 + 32x^2\,,\\
&\mathcal{M}_5(x) = 32x^5 + 160x^3 + 48x\,,\\
&\mathcal{M}_6(x) = 64x^6 + 640x^4 + 736x^2\,,\\
&\mathcal{M}_7(x) = 128x^7 + 2240x^5 + 6272x^3 + 1440x\,,\\
&\mathcal{M}_8(x) = 256x^8 + 7168x^6 + 39424x^4 + 33792x^2\,.
\end{aligned}$$

(Note that they are different from those reported in [9]).

They satisfy the recurrence relation:

$$\mathcal{M}_{n+1}(x) = 2\,x\,\mathcal{M}_n(x) + n(n-1)\,\mathcal{M}_{n-1}(x)\,, \tag{3.20}$$

as a consequence of the equation $(1-t^2)\,\partial G/\partial t = 2xG(t,x)$, (see [1]), verified by the generating function (3.1).

4 Conclusion

In the framework of a general technique based on the monomiality property, and by using of a preceding result by Y. Ben Cheihk, the differential equation satisfied by the Mittag-Leffler polynomials has been derived.

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THE p^{th} FUNDAMENTAL FORM OF TANGENT RIEMANNIAN HYPERSURFACE OF $\beta-$CHANGED FINSLER SPACE BY h-VECTOR

By

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Abstract

The purpose of this paper is to obtain the relation between p^{th} fundamental forms of tangent Riemannian hypersurfaces of (M^n, L) and (M^n, L^*), where $L^* = f(L, \beta)$ with f as positively homogeneous function of degree one in L and $\beta = b_i(x, y)\, y^i$ and $b_i(x, y)$ are components of an h-vector satisfying $\dot{\partial}_j b_i = \rho h_{ij}$.

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Keywords and phrases: Fundamental forms,Tangent Riemannian Hypersurface, $\beta-$change, Finsler space, h-vector.

1 Introduction

Let (M^n, L) be an n-dimensional Finsler space with fundamental metric function $L(x, y)$. In general $L(x, y)$ is a function of point $x = (x^i)$ and element of support $y = (y^i)$ and positively homogeneous of degree one in y^i. Let (M^n, L') and (M^n, L'') be Finsler spaces whose metric functions $L'(x, y)$ and $L''(x, y)$ are obtained from L by the relations

$$L'^2(x, y) = L^2(x, y) + (b_i(x)y^i)^2, \tag{1.1}$$

$$L''(x, y) = L(x, y) + b_i(x)y^i, \tag{1.2}$$

where b_i are components of a covariant vector, which is a function of position alone. These two transformations have been introduced by Matsumoto [2] which have the geometrical properties stated therein.

If $L(x, y)$ is the metric function of Riemannian manifold, then the function $L''(x, y)$ reduces to the metric function of Randers space. Such a space was first introduced by Randers [6] from the stand-point of general relativity and was applied to the theory of electron microscope by Ingarden [1].

The first fundamental tensor of a hypersurface determines metric of the hypersurface. The second fundamental tensor of the hypersurface determines the normal curvature of the hypersurface. These fundamental tensors are associated with the respective fundamental forms.

The n-fundamental forms of a Riemannian hypersurface have been defined and their properties have been studied by Rund [7]. Prasad [3], [4] has obtained the relation between n-fundamental forms of tangent Riemannian hypersurfaces of (M^n, L) and (M^n, L') and those of (M^n, L) and (M^n, L''). Let $L'''(x, y) = f(L, \beta)$, where $\beta(x, y) = b_i(x)y^i$ and $f(L, \beta)$ is a positively homogeneous function of degree one in L and β. Let $(M^{n-1}, \overline{L})$ and $(M^{n-1}, \overline{L}''')$ be the hypersurfaces of (M^n, L) and (M^n, L''') respectively, represented by the same equation.

A relation between the p^{th} fundamental forms of tangent Riemannian spaces to (M^{n-1}, L) and (M^{n-1}, L''') has been obtained by H.S. Shukla, O.P. Pandey and Khageshwar Mandal [10].

We now assume that b_i in β is an h-vector, i.e. $\beta = b_i(x,y)y^i$ and $\dot{\partial}_j b_i = \rho h_{ij}$ [9]. The purpose of this paper is to obtain the relation between p^{th} fundamental forms of tangent Riemannian hypersurfaces of (M^n, L) and (M^n, L^*), where the metric function L^* is given by

$$L^* = f(L, \beta), \tag{1.3}$$

f being a positively homogeneous function of degree one in L and β, $\beta = b_i(x,y)y^i$ and b_i being an h-vector.

2 Hypersurface of (M^n, L)

Let $(M^{n-1}, \overline{L})$ be a hypersurface of (M^n, L) given by

$$x^i = x^i(u^\alpha). \tag{2.1}$$

Let us suppose that the functions (2.1) are at least of class C^3 in u^α and the projection factor $B^j_\gamma = \frac{\partial x^j}{\partial u^\gamma}$ are such that their matrix has maximal rank $(n-1)$. The fundamental metric function $\overline{L}(u,v)$ of the hypersurface is given by

$$\overline{L}(u^\alpha, v^\alpha) = L(x^i(u^\alpha), B^i_\alpha v^\alpha),$$

where v^α is the element of support for the hypersurface for which

$$y^i = B^i_\alpha v^\alpha. \tag{2.2}$$

If $g_{hj}(x,y)$ denotes the metric tensor of (M^n, L), the induced metric tensor of $(M^{n-1}, \overline{L})$ is given by

$$g_{\alpha\gamma}(u,v) = g_{hj}(x,y) B^h_\alpha B^j_\gamma. \tag{2.3}$$

The inverse of (2.3) is denoted by $g^{\alpha\gamma}(u,v)$, by means of which we define the quantities

$$B^\alpha_i(u,v) = g^{\alpha\gamma}(u,v) g_{ij}(x,y) B^j_\gamma. \tag{2.4}$$

The unit normal vector $N^j(x,y)$ of $(M^{n-1}, \overline{L})$ is determined by the relations

$$\begin{aligned} & g_{hj}(x,y) B^h_\gamma N^j(x,y) = 0 \\ \text{and} \quad & g_{hj}(x,y) N^h(x,y) N^j(x,y) = 1. \end{aligned} \tag{2.5}$$

From (2.3), (2.4) and (2.5), we have the following identities:

$$\begin{aligned} B^\alpha_j B^j_\gamma &= \delta^\alpha_\gamma, \\ B^j_\alpha B^\alpha_h + N^j N_h &= \delta^j_h, \end{aligned} \tag{2.6}$$

where $N_h = g_{hj}(x,y) N^j$. If $C_{hik}(x,y)$ denotes the Cartan tensor of (M^n, L), the induced Carton tensor $C_{\alpha\delta\gamma}(u,v)$ of $(M^{n-1}, \overline{L})$ is given by

$$C_{\alpha\delta\gamma}(u,v) = C_{hjk}(x,y) B^h_\alpha B^j_\delta B^k_\gamma, \tag{2.7}$$

from which we obtain

$$C^\alpha_{\delta\gamma} = B^\alpha_i C^i_{jk} B^j_\delta B^k_\gamma. \tag{2.8}$$

The mixed v-covariant derivative of the projection factor B^j_δ is defined as

$$Z^j_{\delta\gamma} = B^j_\delta|_\gamma = C^j_{hk} B^h_\delta B^k_\gamma - B^j_\alpha C^\alpha_{\delta\gamma}. \tag{2.9}$$

From (2.8) it follows that $Z^j_{\delta\gamma}$ is normal to $(M^{n-1}, \overline{L})$. Therefore, we may write

$$Z^j_{\delta\gamma} = M_{\delta\gamma} N^j. \tag{2.10}$$

From (2.9) it is clear that $M_{\delta\gamma}$ is symmetrical in δ and γ and $M_{\delta\gamma}v^\delta = 0 = M_{\gamma\delta}v^\delta$.

Now the tangent vector space M^{n-1}_x to M^{n-1} at every point $x^i(= u^\alpha)$ of the hypersurface is considered as the Riemannian space $(M^{n-1}_x, \overline{g}_x)$ with the Riemannian metric $\overline{g}_x = g_{\alpha\gamma}(u,v)dv^\alpha\, dv^\gamma$. The components of the Cartan tensor $C^\alpha_{\delta\gamma}$ will be the Christoffel symbols associated with $\overline{g}_x$.

If M^n_x is the tangent vector space to M^n at $x^i(= u^\alpha)$, then $(M^{n-1}_x, \overline{g}_x)$ will be the hypersurface of (M^n_x, g_x) given by equation (2.2), where $g_x = g_{ij}(x,y)dy^i dy^j$ is the Riemannian metric on M^n_x. The factor of proportionality of equation (2.10) will be considered as the coefficient of second fundamental form of tangent Riemannian space $(M^{n-1}_x, \overline{g_x})$.

In general, the coefficients of the p^{th} fundamental form of $(M^{n-1}_x, \overline{g_x})$ are defined as [8]:

$$\begin{aligned}
C_{(1)\alpha\gamma} &= g_{\alpha\gamma},\\
C_{(2)\alpha\gamma} &= M_{\alpha\gamma},\\
C_{(p)\alpha\gamma} &= g_{(p-1)\alpha\epsilon} M^\epsilon_\gamma, \qquad (3 \le p \le n),
\end{aligned} \tag{2.11}$$

where $M^\epsilon_\gamma = g^{\alpha\epsilon}M_{\alpha\gamma}$.

3 An *h*-vector

Let $b_i(x,y)$ be the components of a covariant vector in the Finsler space (M^n, L). It is called an h-vector if there exists a scalar function ρ such that

$$\frac{\partial b_i}{\partial y^j} = \rho h_{ij} \tag{3.1}$$

where h_{ij} are components of angular metric tensor given by

$$h_{ij} = g_{ij} - l_i l_j = L\frac{\partial^2 L}{\partial y^i \partial y^j}.$$

Differentiating (3.1) with respect to y^k, we get

$$\dot{\partial}_j \dot{\partial}_k b_i = (\dot{\partial}_k \rho)h_{ij} + \rho L^{-1}\{L^2 \dot{\partial}_i \dot{\partial}_j \dot{\partial}_k L + h_{ij} l_k\},$$

where $\dot{\partial}_i$ stands for $\frac{\partial}{\partial y^i}$.

The skew-symmetric part of the above equation in j and k gives

$$(\dot{\partial}_k \rho + \rho L^{-1} l_k)h_{ij} - (\dot{\partial}_j \rho + \rho L^{-1} l_j)h_{ik} = 0.$$

Contracting this equation by g^{ij}, we get

$$(n-2)[\dot{\partial}_k \rho + \rho L^{-1} l_k] = 0,$$

which for $n > 2$ gives

$$\dot{\partial}_k \rho = -\frac{\rho}{L} l_k \tag{3.2}$$

where we have used the fact that ρ is positively homogeneous of degree -1 in y^i, i.e.

$$\frac{\partial \rho}{\partial y^j} y^j = -\rho.$$

We shall frequently use equation (3.2) without quoting it in the subsequent sections.

4 The relation between fundamental quantities of (M^n, L^*) and (M^n, L)

Let $(M^{n-1}, \overline{L}^*)$ be a hypersurface of (M^n, L^*) given by the equation (2.1). We shall denote the quantities of $(M^{n-1}, \overline{L}^*)$ and (M^n, L^*) by starred letters. From $L^* = f(L, \beta)$, where $\beta = b_i(x, y)y^i$, and b_i are components of an h-vector, it follows that the metric tensors and Cartan tensors of (M^n, L) and (M^n, L^*) are related by [5]:

$$\begin{aligned} g^*_{ij} =& \frac{fp}{L} g_{ij} - L^{-1}\{\beta(f_1 f_2 - f\beta L w) + L\rho f f_2\} l_i l_j + (fL^2 w + f_2^2) b_i b_j \\ &+ (f_1 f_2 - f\beta L w)(l_i b_j + l_j b_i), \end{aligned} \tag{4.1}$$

where $p = (f_1 + f_2 L\rho)$, $f_1 = \frac{\partial f}{\partial L}$, $f_2 = \frac{\partial f}{\partial \beta}$, $w = \frac{1}{L^2}\frac{\partial^2 f}{\partial L^2}$, $l_i = \frac{\partial L}{\partial y^i}$;

$$g^{*ij} = \frac{L}{fp} g^{ij} + \frac{Lv}{f^3 pt} l^i l^j - \frac{L^4 w}{fpt} b^i b^j - \frac{L^2 u}{f^2 pt}(l^i b^j + l^j b^i), \tag{4.2}$$

where we put $b^i = g^{ij} b_j$, $l^i = g^{ij} l_j$, $b^2 = g^{ij} b_i b_j$, $u = f_1 f_2 - f\beta L w + L\rho f_2^2$,

$$v = (f_1 f_2 - f\beta L w)(f\beta - \triangle f_2 L^2) + L\rho f_2\{f(f + L^2\rho f_2) + L^2 \triangle (f_2^2 + fL^2 w)\},$$

$$t = (f_1 + L^3 w \triangle + L f_2 \rho), \qquad \triangle = b^2 - \frac{b^2}{L^2};$$

$$C^*_{ijk} = \frac{fp}{L} C_{ijk} + \frac{q}{2L}(h_{ij} m_k + h_{jk} m_i + h_{ki} m_j) + \frac{L^2 s}{2} m_i m_j m_k, \tag{4.3}$$

where $s = 3 f_2 w + f w_2$, $m_i = b_i - \frac{\beta}{L} l_i$,

$$m_i l^i = 0, \quad m_i m^i = \triangle = b^i m_i, \quad h_{ij} m^j = m_i, \quad m^i = g^{ij} m_j = b^2 - \frac{\beta}{L} l^i;$$

$$\begin{aligned} C^{*i}_{jk} =& C^i_{jk} + \frac{q}{2fp}(h_{jk} m^i + h^i_j m_k + h^i_k m_j) + \frac{sL^3}{2fp} m_j m_k m^i \\ &- \frac{L}{ft} C_{jk} n^i - \frac{Lq\triangle}{2f^2 pt} h_{jk} n^i - \frac{(2Lq + L^4 \triangle s)}{2f^2 pt} m_j m_k n^i, \end{aligned} \tag{4.4}$$

where $n^i = fL^2 w b^i + u l^i$, $C_{ijk} m^i = C_{jk}$, $C_{ijk} n^i = fL^2 w C_{jk}$, $m_i n^i = fL^2 w \triangle$, $C^h_{ij} h_{hk} = C_{ijk}$, $h^i_k h^k_j = h^i_j$, $h_{ij} n^i = fL^2 w m_j$.

5 Relation between p^{th} fundamental forms of hypersurfaces of (M^n, L^*) and (M^n, L)

Relations (4.1), (4.2) and (2.3) give

$$\begin{aligned} g^*_{\alpha\gamma} =& \frac{fp}{L} g_{\alpha\gamma} - L^{-1}\{\beta(f_1 f_2 - f\beta L w) + L\rho f f_2\} l_\alpha l_\gamma + (fL^2 w + f_2^2) b_\alpha b_\gamma \\ &+ (f_1 f_2 - f\beta L w)(l_\alpha b_\gamma + l_\gamma b_\alpha), \end{aligned} \tag{5.1}$$

$$g^{*\alpha\gamma} = \frac{L}{fp} g^{\alpha\gamma} + \frac{Lv}{f^3 pt} l^\alpha l^\gamma - \frac{L^4 w}{fpt} b^\alpha b^\gamma - \frac{L^2 u}{f^2 pt}(l^\alpha b^\gamma + l^\gamma b^\alpha), \tag{5.2}$$

where

$$b^\alpha = g^{\alpha\gamma} b_\gamma, \qquad l^\alpha = g^{\alpha\gamma} l_\gamma, \qquad \overline{b}^2 = g^{\alpha\gamma} b_\alpha b_\gamma. \tag{5.3}$$

From (5.1) and (5.2), we have

$$C^*_{\alpha\delta\gamma} = \frac{fp}{L} C_{\alpha\delta\gamma} + \frac{q}{2L}(h_{\alpha\delta} m_\gamma + h_{\delta\gamma} m_\alpha + h_{\gamma\alpha} m_\delta) + \frac{L^2 s}{2} m_\alpha m_\delta m_\gamma, \tag{5.4}$$

where $m_\gamma = m_k B^k_\gamma$;

$$\begin{aligned} C^{*\alpha}_{\delta\gamma} =& C^\alpha_{\delta\gamma} + \frac{q}{2fp}(h_{\delta\gamma}m^\alpha + h^\alpha_\delta m_\gamma + h^\alpha_\gamma m_\delta) + \frac{sL^3}{2fp} m_\delta m_\gamma m^\alpha \\ &- \frac{L}{ft} C_{\gamma\delta} n^\alpha - \frac{Lq\triangle}{2f^2pt} h_{\delta\gamma} n^\alpha - \frac{(2Lq + L^4\triangle s)}{2f^2pt} m_\delta m_\gamma n^\alpha, \end{aligned} \tag{5.5}$$

where

$$\begin{aligned} &h_{\delta\gamma} = g_{\delta\gamma} - l_\delta l_\gamma, && n^\alpha = fL^2wb^\alpha + ul^\alpha, && C_{\alpha\delta\gamma}m^\alpha = C_{\delta\gamma}, \\ &C_{\alpha\delta\gamma}n^\alpha = fL^2wC_{\delta\gamma}, && m_\alpha n^\alpha = fL^2w\triangle, && m^\alpha = B^\alpha_i n^i. \end{aligned} \tag{5.6}$$

In general the vector b^i is not tangential to the hypersurface. However, we may write

$$b^i = b^\alpha B^i_\alpha + \sigma N^i \tag{5.7}$$

where $\sigma = b_i N^i = b^i N_i$. The equation (5.7) may also be written as $b_i = b_\alpha B^\alpha_i + \sigma N_i$.

If $\sigma = 0$, b^i is tangential to the hypersurface $(M^{n-1}, \overline{L})$. In the following we have assumed that $\sigma = 0$. From (2.6) ,(5.3) and (5.7), we get

$$b^2 = \overline{b}^2 \tag{5.8}$$

where $b^2 = g^{ij}b_ib_j$, $\overline{b}^2 = g^{\alpha\gamma}b_\alpha b_\gamma$.

It is to be noted that if N^j is a unit normal vector to $(M^{n-1}, \overline{L})$, then it is not unit normal to $(M^{n-1}, \overline{L}^*)$. We may write

$$N^j = T^\alpha B^j_\alpha + KN^{*j}. \tag{5.9}$$

To obtain T^α and K, we use (4.1), (2.5)and (5.3). We get

$$g^*_{hj} B^h_\alpha N^j = 0, \tag{5.10}$$

$$g^*_{hj} N^h N^j = \frac{fp}{L}. \tag{5.11}$$

From (2.3), (5.9), (5.10) and (5.11) it follows that

$$g^*_{\alpha\delta} T^\delta = 0, \tag{5.12}$$

$$K^2 = \frac{fp}{L}. \tag{5.13}$$

Thus if b^i is tangential to the hypersurface $(M^{n-1}, \overline{L})$, then $\sigma = 0$, $T^\alpha = 0$ and $K = \sqrt{\frac{fp}{L}}$.

From (5.9), it follows that $N^i = \sqrt{\frac{fp}{L}} N^{*i}$. Hence we have the following theorem:

Theorem 5.1. *Let (M^n, L^*) be a Finsler space obtained from the Finsler space (M^n, L) by the $\beta-$change (1.3) of Finsler metric. If $(M^{n-1}, \overline{L}^*)$ and $(M^{n-1}, \overline{L})$ be hypersurfaces of these spaces and b^i is tangential to the hypersurface $(M^{n-1}, \overline{L})$ then the vector normal to $(M^{n-1}, \overline{L})$ is also normal to $(M^{n-1}, \overline{L}^*)$.*

Next, we establish the following theorem:

Theorem 5.2. *Let (M^n, L^*) be a Finsler space obtained from the Finsler space (M^n, L) by the $\beta-$change (1.3) of Finsler metric. Let $(M^{n-1}, \overline{L}^*)$ and $(M^{n-1}, \overline{L})$ be hypersurfaces of (M^n, L^*) and (M^n, L) respectively. If b^i is tangential to the hypersurface $(M^{n-1}, \overline{L})$ and (M^n_x, g_x), (M^n_x, g^*_x), $(M^{n-1}_x, \overline{g}_x)$, $(M^{n-1}_x, \overline{g}^*_x)$ are tangent Riemannian spaces to (M^n, L), (M^n, L^*), $(M^{n-1}, \overline{L})$ and $(M^{n-1}, \overline{L}^*)$ respectively, then we have:*

(i) *Second fundamental forms of* $(M_x^{n-1}, \overline{g}_x)$ *and* $(M_x^{n-1}, \overline{g}_x^*)$ *are proportional.*
(ii) *Every asymptotic direction of* $(M_x^{n-1}, \overline{g}_x)$ *is an asymptotic direction of* $(M_x^{n-1}, \overline{g}_x^*)$.
(iii) *The* $p^{th}-$*fundamental tensors of* $(M_x^{n-1}, \overline{g}_x)$ *and* $(M_x^{n-1}, \overline{g}_x^*)$ *are related by*

$$C^*_{(2)\alpha\gamma} = \sqrt{\frac{fp}{L}} C_{(2)\alpha\gamma}.$$

$$C^*_{(p)\alpha\gamma} = \left(\frac{L}{fp}\right)^{(p-3)/2} [C_{(p)\alpha\gamma} - \frac{L^3 w}{t} \sum_{m=2}^{p-1} A_{(m)\alpha} C_{(p-m+1)\gamma}], \quad 3 \leq p \leq n, \tag{5.14}$$

where

$$C_{(p)\alpha} = C_{(p)\alpha\gamma} b^\gamma, \quad C_{(p)} = C_{(p)\alpha\gamma} b^\alpha b^\gamma, \quad 2 \leq p \leq n$$
$$A_{(2)\alpha} = C_{(2)\alpha}$$
$$A_{(m)\alpha} = C_{(m)\alpha} - \frac{L^3 w}{t} \sum_{\lambda=2}^{m-1} A_{(\lambda)\alpha} C_{m-\lambda+1}, \quad 3 \leq m \leq (p-1).$$

Proof. (i) If $Z^{*j}_{\delta\gamma}$ denote the tensor derivative of B^j_δ in tangent Riemannian hypersurface $(M_x^{n-1}, \overline{g}_x^*)$ of the tangent Riemannian space (M_x^n, g_x^*), then from (2.9), (4.4), (5.5) and (5.7), we have

$$Z^{*j}_{\delta\gamma} = Z^j_{\delta\gamma}. \tag{5.15}$$

In view of (5.9) and (2.10) the relation (5.15) yields

$$M^*_{\delta\gamma} = K M_{\delta\gamma}. \tag{5.16}$$

This proves (i).
(ii) A direction t^α for which $M_{\delta\gamma} t^\delta t^\gamma = 0$ is said to be an asymptotic direction in $(M_x^{n-1}, \overline{g}_x)$. In view of this definition and (i), we get (ii).
(iii) The validity of relation (5.14) is established by induction. From (5.2) and (5.16) for $\sigma = 0$, we have

$$M^{*\epsilon}_\delta = M_{\alpha\delta} g^{*\alpha\epsilon} = \sqrt{\frac{L}{fp}}\, M^\epsilon_\delta - \frac{L}{t}\sqrt{\frac{L}{fp}}\, (L^2 w b^\epsilon + \frac{p}{f} l^\epsilon) M_{\lambda\delta} b^\lambda. \tag{5.17}$$

Since $C_{\alpha\delta\gamma} l^\alpha = 0$, $C_{ijk} l^i = 0$, so $C_{(p)\alpha\delta} l^\alpha = 0$, $\forall p = 1, 2, \ldots, n$.
The relations (2.11) and (5.17) yield

$$C^*_{(3)\alpha\gamma} = C_{(3)\alpha\gamma} - \frac{L^3 w}{t} A_{(2)\alpha} C_{(2)\gamma}. \tag{5.18}$$

From (5.17) and (5.16), it is evident that (5.14) holds for $p = 3$.

Suppose that the required result is true for a given value of the integer s with $3 \leq s \leq n-1$, then we have

$$C^*_{(s)\alpha\gamma} = \left(\frac{L}{fp}\right)^{(s-3)/2} - \frac{L^3 w}{t} \sum_{m=2}^{s-1} A_{(m)\alpha} C_{(s-m+1)\gamma}. \tag{5.19}$$

Therefore from (5.16), (5.17) and (5.19), we have

$$\begin{aligned}
C^*_{(s+1)\alpha\gamma} &= C^*_{(s)\alpha\epsilon} M^{*\epsilon}_{\gamma} \\
&= \left(\frac{L}{fp}\right)^{(s-3)/2} \big[C_{(s)\alpha\epsilon} - \frac{L^3 w}{t} \sum_{m=2}^{s-1} A_{(m)\alpha} C_{(s-m+1)\epsilon}\big] \times \\
&\quad \sqrt{\frac{L}{fp}} \big[M^{\epsilon}_{\gamma} - \big(\frac{L^3 w}{t} b^{\epsilon} + \frac{Lp}{ft} l^{\epsilon}\big) M_{\lambda\gamma} b^{\lambda}\big] \\
&= \left(\frac{L}{fp}\right)^{\{(s+1)-3\}/2} \big[C_{(s+1)\alpha\gamma} - \frac{L^3 w}{t} \sum_{m=2}^{s-1} A_{(m)\alpha} C_{(s-m+2)\gamma} \\
&\quad - \frac{L^3 w}{t} \big\{C_{(s)\alpha} - \frac{L^3 w}{t} \sum_{m=2}^{s-1} A_{(m)\alpha} C_{(s-m+1)\gamma}\big\} C_{(2)\gamma}\big] \\
&= \left(\frac{L}{fp}\right)^{\{(s+1)-3\}/2} \big[C_{(s+1)\alpha\gamma} - \frac{L^3 w}{t} \sum_{m=2}^{s-1} A_{(m)\alpha} C_{(s-m+2)\gamma} + A_{(s)\alpha} C_{(2)\gamma}\big],
\end{aligned}$$

or,

$$C^*_{(s+1)\alpha\gamma} = \left(\frac{L}{fp}\right)^{\{(s+1)-3\}/2} \big[C_{(s+1)\alpha\gamma} - \frac{L^3 w}{t} \sum_{m=2}^{s} A_{(m)\alpha} C_{(s-m+2)\gamma}\big].$$

This shows that (5.14) is valid for $p = s + 1$ also, which completes the proof of (iii). □

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FIXED POINTS FOR HYBRID MAPPINGS SATISFYING AN IMPLICIT RELATION IN PARTIAL METRIC SPACES

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Abstract

In this paper, we prove a fixed point theorem for hybrid mappings in partial metric spaces. The theorem contains an altering distance function and involves an implicit relation satisfying the (E.A) - property. In doing so, we generalize a theorem by Popa and Patriciu.
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1 Introduction and Preliminaries

The study of fixed point theorems involving implicit relations was initiated by Popa [5]. These types of theorems cater for a large class of fixed point theorems on mappings that obey a given set of conditions. Popa and Patriciu [6] proved a fixed point theorem for metric spaces for a pair of hybrid mappings involving altering distance and satisfying an implicit relation. In this study, we modify this theorem so that it applies for partial metric spaces.

The following preliminaries will be useful in the course of our work.

The partial metric, which is a generalization of the metric with the zero-self distance axiom relaxed is defined here.

Definition 1.1. [4] *A partial metric on a nonempty set X is a function $p : X \times X \to \mathbb{R}_+$ such that for all $x, y, z \in X$,*
(P1) $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$,
(P2) $p(x, x) \leq p(x, y)$,
(P3) $p(x, y) = p(y, x)$,
(P4) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.
A partial metric space is a pair (X, p) such that X is a nonempty set and p is a partial metric on X.

Let $CB^p(X)$ be a family of all non-empty, closed and bounded subsets of a partial metric space (X, p), induced by the partial metric p. The set A is said to be a bounded subset in (X, p) if there exists $x_0 \in X$ and $M \geq 0$ such that for all $a \in A$, we have $a \in B_p(x_0, M)$.

The Hausdorff partial metric is defined for multivalued mappings in partial metric spaces. We make use of the following definitions given by Aydi *et al.* [2].

Definition 1.2. [2] *For all $A, B \in CB^p(X)$ and $x \in X$, we define*
(i) $\quad p(x, A) = \inf \{p(x, a), a \in A\}$,
(ii) $\quad \delta_p(A, B) = \sup \{p(a, B) : a \in A\}$,
(iii) $\quad \delta_p(B, A) = \sup \{p(b, A) : b \in B\}$,
(iv) $\quad H_p(A, B) = \max \{\delta_p(A, B), \delta_p(B, A)\}$.
The mapping $H_p : CB^p \times CB^p \to [0, +\infty)$ is called the partial Hausdorff metric.

We now state some properties of mappings δ_p and H_p.

Lemma 1.1. [2] *Let (X, p) be a partial metric space. For any $A, B \in CB^p(X)$ we have*
(i) $\quad \delta_p(A, A) = \sup\{p(a, a) : a \in A\}$,
(ii) $\quad \delta_p(A, A) \leq \delta_p(A, B)$,
(iii) $\quad \delta_p(A, B) = 0$ *implies that* $A \subseteq B$,
(iv) *If* $b \in B$, *then* $p(a, B) \leq p(a, b) \leq \delta(a, B)$
(h1) $H_p(A, A) \leq H_p(A, B)$,
(h2) $H_p(A, B) = H_p(B, A)$,
(h3) $H_p(A, B) = 0$ *implies* $A = B$.

Let f and T be mappings with $f : (X, p) \to (X, p)$ and $T : (X, p) \to CB^p(X)$. A point $x \in X$ is said to be a coincidence point of f and T if $fx \in Tx$. The set of all coincidence points of f and T is denoted by $C(f, T)$.

Inspired by Aamri and Moutawakil [1] and Kamran [3] we define the (E.A)-property in the context of hybrid mappings in partial metric spaces.

Definition 1.3. *Let (X, p) be a partial metric space. The mappings $f : X \to X$ and $T : X \to CB^p(X)$ are said to satisfy the (E.A)-property if there exists a sequence $\{x_n\}$ in X such that* $\lim fx_n = t \in A = \lim Tx_n$ *and* $p(t, t) = 0$.

Definition 1.4. *An altering distance is a mapping $\varphi : [0, \infty) \to [0, \infty)$ which satifies:*

(φ_1)*: $\varphi(t)$ is increasing and continuous,*
(φ_2)*: $\varphi(t) = 0$ if and only if $t = 0$.*

The function space $\mathfrak{F}_a$ is defined as follows in Popa and Patriciu [6].

Let $\mathfrak{F}_a$ be the set of all continuous functions $F(t_1, ..., t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ satisfying the following
(F1) : F is nondecreasing in variable t_1,
(F2) : $F(t, 0, 0, t, t, 0) \leq 0$ implies $t = 0$.

Examples of functions in $\mathfrak{F}_a$ are hereby stated.

Example 1.1. [6] $F(t_1, ..., t_6) = t_1 - \max\{t_2, (t_3 + t_4)/2, (t_5 + t_6)/2\}$.
(F1)*: Obviously.*
(F2) *:* $F(t, 0, 0, t, t, 0) = t/2 \leq 0$ *implies* $t = 0$.

Example 1.2. [6] $F(t_1, ..., t_6) = t_1 - at_2 - b(t_3 + t_4) - c(t_5 + t_6)$, *where* $a, b, c \geq 0$ *and* $b + c < 1$.
(F1)*: Obviously.*
(F2) *:* $F(t, 0, 0, t, t, 0) = t(1 - b) \leq 0$ *implies* $t = 0$.

Popa and Patriciu [6] proved the following theorem:

Theorem 1.1. *Let $f:(X,d)\to(X,d)$ and $T:(X,d)\to CB(X)$ be such that*

$$\begin{aligned}F\big(&\phi(H(Tx,Ty)),\phi(d(fx,fy)),\phi(d(fx,Tx)),\\&\phi(d(fy,Ty)),\phi(d(fx,Ty)),\phi(d(fy,Tx))\big)\le 0\end{aligned}\tag{1.1}$$

for all $x,y\in X$, where $F\in\mathfrak{F}_a$ and $\phi(t)$ is an altering distance. If $f(X)$ is a closed subset of X and (f,T) satisfy $(E.A)-$ property, then $C(f,T)\neq 0$. Moreover, if $fv=ffv$ for $v\in C(f,T)$, then f and T have a common fixed point.

In this study we modify Theorem 1.1 so that it applies to partial metric spaces.

2 Main Results

Now, we will extend Theorem 1.1 to partial metric spaces as follows:

Theorem 2.1. *Let (X,p) be a complete partial metric space. Let $f:(X,p)\to(X,p)$ and $T:(X,p)\to CB^p(X)$ be continuous mappings such that*

$$\begin{aligned}&F(\varphi(H_p(Tx,Ty)),\varphi(p(fx,fy)),\varphi(p(fx,Tx)),\\&\varphi(p(fy,Ty)),\varphi(p(fx,Ty)),\varphi(p(fy,Tx)))\le 0\end{aligned}\tag{2.1}$$

for all $x,y\in X$, where $F\in\mathfrak{F}_a$ and $\varphi(t)$ is an altering distance. If $f(X)$ is a closed subset of X and (f,T) satisfy $(E.A)-$ property, then $C(f,T)\neq\emptyset$. Moreover, if $fv=ffv$ for $v\in C(f,T)$, then f and T have a common fixed point.

Proof. From the Definition 1.3, there exists a sequence $\{x_n\}$ in X such that $\lim fx_n=t\in A=\lim Tx_n$, with $p(t,t)=0$. Since f,T are continuous mappings, we have $t=fx\in Tx$ where $x=lim_{n\to\infty}x_n$. This makes $x\in C(f,T)$, implying $C(f,T)\neq\emptyset$.

We have,

$$t=fx=ffx=ft\in Tx.$$

By (2.1) we have

$$\begin{aligned}&F(\varphi\big(H_p(Tx,Tt)),\varphi(p(fx,ft)),\varphi(p(fx,Tx)),\varphi(p(ft,Tt)),\\&\qquad\qquad\varphi(p(fx,Tt)),\varphi(p(ft,Tx))\big)\le 0\\\Rightarrow\ &F\big(\varphi(H_p(Tx,Tt)),\varphi(p(t,t)),\varphi(p(t,Tx)),\varphi(p(t,Tt)),\\&\qquad\qquad\varphi(p(t,Tt)),\varphi(p(t,Tx))\big)\le 0.\\\Rightarrow\ &F\big(\varphi(H_p(Tx,Tt)),0,0,\varphi(p(t,Tt)),\varphi(p(t,Tt)),0\big)\le 0.\end{aligned}\tag{2.2}$$

As $t\in Tx$ we have $p(t,Tt)\le H_p(Tx,Tt)$. Since F is non-decreasing in the first variable, (2.2) becomes

$$F\big(\varphi(p(t,Tt)),0,0,\varphi(p(t,Tt)),\varphi(p(t,Tt)),0\big)\le 0.\tag{2.3}$$

Using the (F_2) property we get

$$p(t,Tt)=0\ \Rightarrow t=ft\in Tt,$$

making $\lim fx_n=t$ a common fixed point of f and T. □

If $\varphi(t)=t$ we get the following corollary.

Corollary 2.1. *Let (X, p) be a complete partial metric space. Let $f : (X, p) \to (X, p)$ and $T : (X, p) \to CB^p(X)$ be continuous mappings such that*

$$\begin{aligned} F(H_p(Tx, Ty), p(fx, fy), p(fx, Tx), p(fy, Ty), \\ p(fx, Ty), p(fy, Tx)) \leq 0 \end{aligned} \tag{2.4}$$

for all $x, y \in X$, where $F \in \mathfrak{F}_a$. If $f(X)$ is a closed subset of X and (f, T) satisfy $(E.A)-$property, then $C(f, T) \neq \emptyset$. Moreover, if $fv = ffv$ for $v \in C(f, T)$, then f and T have a common fixed point.

We state an example for the use of this theorem.

Example 2.1. *Consider the implicit function F defined in Example* 1.1*. Let $\varphi(t) = t$. This leads to the following theorem.*

Theorem 2.2. *Let (X, p) be a complete partial metric space. Let $f : (X, p) \to (X, p)$ and $T : (X, p) \to CB^p(X)$ be continuous mappings such that*

$$\begin{aligned} H_p(Tx, Ty) \leq \max \big\{ p(fx, fy), (p(fx, Tx) + p(fy, Ty))/2, \\ (p(fx, Ty) + p(fy, Tx))/2 \big\}. \end{aligned} \tag{2.5}$$

for all $x, y \in X$. If $f(X)$ is a closed subset of X and (f, T) satisfy $(E.A)-$ property, then $C(f, T) \neq \emptyset$. Moreover, if $fv = ffv$ for $v \in C(f, T)$, then f and T have a common fixed point.

Proof. In view of the $(E.A)-$ property on mappings f and T described in Definition 1.3, there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n \to +\infty} fx_n = t \in A = \lim_{n \to +\infty} Tx_n \text{ and } p(t, t) = 0. \tag{2.6}$$

As f and T are continuous mappings, we have $fx = t \in Tx$ where $x = \lim_{n \to +\infty} x_n$. This makes x a coincidence point of f and T. Hence $C(f, T) \neq \emptyset$.

As x is a coincidence point, from the assumption we have

$$t = fx = ffx = ft \in Tx, \tag{2.7}$$

making t a fixed point of f.

To complete the proof, we need to show that t is also a fixed point of T, that is $t \in Tt$. We consider the following:

$$\begin{aligned} &H_p(Tx, Tt) \\ &\quad \leq \max \left\{ p(fx, ft), \frac{1}{2}\big(p(fx, Tx) + p(ft, Tt)\big), \frac{1}{2}\big(p(fx, Tt) + p(ft, Tx)\big) \right\} \\ &\quad = \max \left\{ p(t, t), \frac{1}{2}\big(p(t, Tx) + p(t, Tt)\big), \frac{1}{2}\big(p(t, Tt) + p(t, Tx)\big) \right\} \\ &\quad = \frac{1}{2}\big(p(t, Tx) + p(t, Tt)\big). \end{aligned} \tag{2.8}$$

From (2.6), we have $p(t, t) = 0$. In addition, from (2.7) we have $t \in Tx$ which implies $p(t, Tx) = 0$. Thus (2.8) becomes

$$H_p(Tx, Tt) \leq \frac{1}{2} p(t, Tt). \tag{2.9}$$

As $t \in Tx$, we have

$$\begin{aligned} p(t, Tt) &\le H_p(Tx, Tt) \le \frac{1}{2} p(t, Tt) \\ &\Rightarrow p(t, Tt) = 0 \\ &\Rightarrow t \in Tt. \end{aligned}$$

Hence t is also a fixed point of T. We have shown that t is a common fixed point of f and T. We have also shown that $p(t, t) = 0$. □

Compliance with ethical standards

Conflict of interest

The authors declare that they have no conflicts of interest.

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ON A SYMMETRIC UNIMODAL DISTRIBUTION ON THE CIRCLE

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Abstract

For an absolutely continuous circular random variable, a probability distribution is called a circular distribution if its total probability is concentrated on the circumference of a unit circle. Several circular distributions have been introduced by various authors and researchers. In this paper, we consider a symmetric unimodal distribution on the circle used in the modelling of the circular data by discussing its several distributional properties and characterizations.

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1 Introduction

As pointed out by Jammalamadaka and SenGupta [12], a probability distribution is called a circular distribution if its total probability is concentrted on the circumference of a unit circle. Further, according to them, since each point on the circumference represents a direction, such a distribution is a way of assigning probabilities to different directions or defining a directional distribution. The range of a circular random variable X, measured in radians, may be taken to be $0 \leq x < 2\pi$ or $-\pi \leq x < \pi$. For an absolutely continuous circular random variable X, (with respect to the Lebesgue measure on the circumference), a probability density function $f(x)$ exists and has the following basic properties:
(i) $f(x) \geq 0$,
(ii) $\int_0^{2\pi} f(x)dx = 1$,
(iii) $f(x) = f(x + 2\pi k)$, for any integer k (that is, $f(x)$ is periodic).

For an absolutely continuous circular random variable X, several circular distributions have been introduced by various authors and researchers; see, for example, Mardia and Jupp [15], and Jammalamadaka and SenGupta [12], among others. A general family of symmetric unimodal distributions on the circle was introduced by Jones and Pewsey [13] for an absolutely continuous circular random variable X, that is, $X \sim JP(\mu, \Psi)$, which has the following probability density function (pdf):

$$f_\Psi(x) \propto [1 + \tanh(\kappa\psi)\cos(x - \mu)]^{\frac{1}{\Psi}}, \mu - \pi < x \leq \mu + \pi, \tag{1.1}$$

where μ is the location parameter, $\kappa \geq 0$ is a concentration parameter, and $\psi \in \Re$. For details, the interested readers are referred to Jones and Pewsey [13]. Following Jones and Pewsey [13], without loss of generality, considering the case, when $\psi = 1$ and employing the parametrization $0 \leq \frac{\tanh(\kappa)}{2} \leq \frac{1}{2}$, we observe that the pdf (1.1) of the general family of

Jones and Pewsey symmetric unimodal distributions on the circle reduces to the following pdf:

$$f_\psi(x) = C[1 + \tanh(\kappa)\cos(x-\mu)], \mu - \pi < x \le \mu + \pi \tag{1.2}$$

where C is the normalizing constant to be determined. Thus integrating (1.2) with respect to x in the interval $\mu - \pi < x \le \mu + \pi$, it is easy to see that

$$\begin{aligned} C &= \frac{1}{\int_{\mu-\pi}^{\mu+\pi}[1+\tanh(\kappa)\cos(x-\mu)]dx} \\ &= \frac{1}{\int_{-\pi}^{\pi}[1+\tanh(\kappa)\cos(t)]dt}, \quad \text{(substituting } x-\mu = t\text{)}, \\ &= \frac{1}{2\int_{0}^{\pi}[1+\tanh(\kappa)\cos(t)]dt}, \text{(since } 1+\tanh(\kappa)\cos(t) \text{ is an even function of } t\text{)}, \\ &= \frac{1}{2\pi}, \end{aligned}$$

in view of which the pdf (1.2) of the general family of Jones and Pewsey symmetric unimodal distributions on the circle reduces to the following pdf:

$$f_\psi(x) = \frac{1}{2\pi}[1 + \tanh(\kappa)\cos(x-\mu)], \quad \mu - \pi < x \le \mu + \pi \tag{1.3}$$

For the sake of simplicity, taking $\tanh(\kappa) = 2\xi$ in (1.3), we obtain the following pdf of the general family of Jones and Pewsey symmetric unimodal distributions on the circle:

$$f(x) = \frac{1}{2\pi}[1 + 2\xi\cos(x-\mu)] \tag{1.4}$$

where $0 \le \xi \le \frac{1}{2}$, and ξ and μ are concentration and mean direction parameters respectively. From now onward, for the sake of simplicity, we will denote above-said circular distribution of X by $X \sim JP(\mu, \xi)$ and call it the Jones and Pewsey standard symmetric unimodal distribution on the circle. The polar representation of (1.4) is a cardioid curve or heart-shaped. It is interesting to note that the Jones and Pewsey standard symmetric unimodal distribution on the circle was also considered by Jeffreys [11], and is used in the modelling of the circular data. In view of this, $X \sim JP(\mu, \xi)$ is also referred as the Jeffreys circular distribution of X, that is, $X \sim JD(\mu, \xi)$. For $\xi = 0$, the Jones and Pewsey symmetric unimodal distribution on the circle reduces to a circular uniform distribution. The cumulative distribution (cdf) $F(x)$ corresponding to the pdf (1.4) is easily given by

$$\begin{aligned} F(x) &= \int_{\mu-\pi}^{x} \frac{1}{2\pi}[1 + 2\xi\cos(t-\mu)]dt \\ &= \frac{1}{2\pi}[x - \mu + \pi + 2\xi\sin(x-\mu)], \end{aligned} \tag{1.5}$$

where $\mu - \pi < x \le \mu + \pi$, and $0 \le \xi \le \frac{1}{2}$. It is easily seen that the mean of$X \sim JP(\mu, \xi)$ is given by

$$\text{Mean} = E(X) = \int_{\mu-\pi}^{\mu+\pi} x\left[\frac{1}{2\pi}[1 + 2\xi\cos(x-\mu)]\right]dx = \mu.$$

Using Maple software, the plots of the pdf (1.4) and cdf (1.5), for some values of the parameters, are provided below in Figures 1.1-1.2.

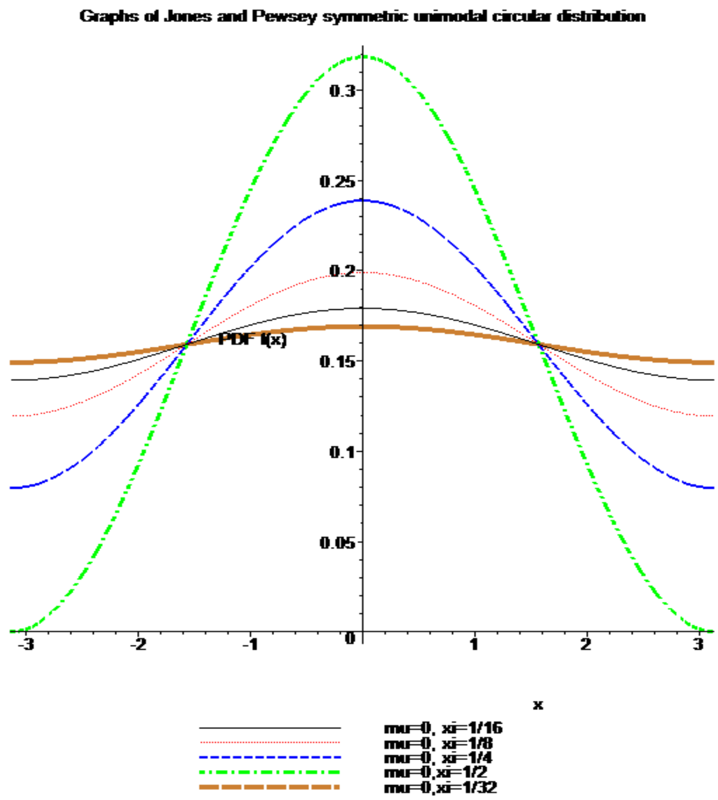

Figure 1.1: PDF $f(x)$ of $X \sim JP(\mu, \xi)$.

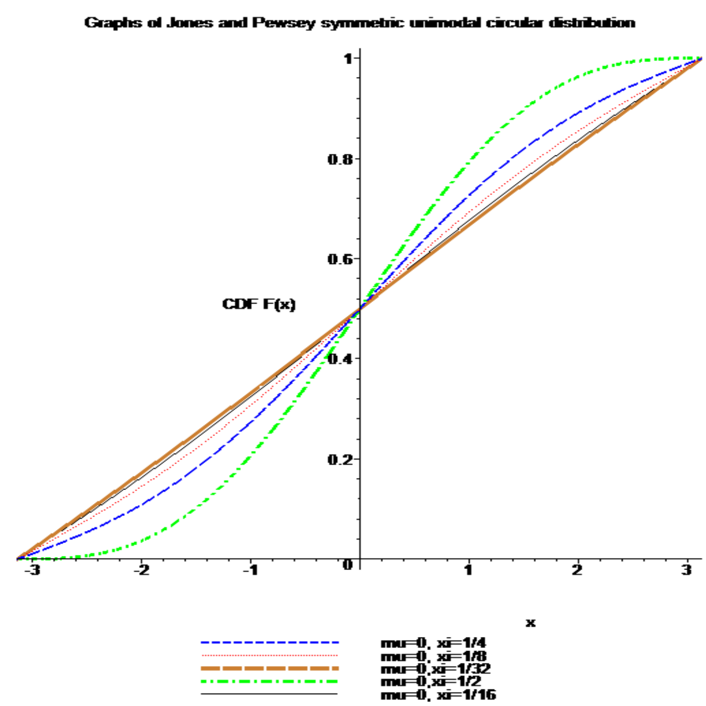

Figure 1.2: CDF $F(x)$ of $X \sim JP(\mu, \xi)$

The effects of the parameters can easily be seen from these graphs. Similarly, we can draw graphs of the pdf (1.4) and cdf (1.5) for other values of the parameters. We note that, since the pdf in (1.4) is an even function, it is easily seen that it is symmetric about the mean, $E(X) = \mu$, for different values of the parameters.

2 Distributional Properties

We present some distributional properties of $X \sim JP(\mu, \xi)$ in this section.

2.1 Median

Since the graph of the pdf of $X \sim JP(\mu, \xi)$ is symmetric about the mean $E(X) = \mu$, its median *MD* is also be given by *MD*$= \mu$.

2.2 Mode

Further, the mode of the Jones and Pewsey symmetric unimodal distribution on the circle, $X \sim JP(\mu, \xi)$, is the value of $x = x_m$ (say), for which its pdf (1.4), that is,

$$f(x) = \frac{1}{2\pi}[1 + 2\xi \cos(x - \mu)],$$

is maximum. Now, differentiating $f(x)$ with respect to x, we have

$$\frac{df(x)}{dx} = -\frac{\xi}{\pi} \sin(x - \mu),$$

which, when equated to 0, and solving for x, easily gives $x = x_m = \mu$. It can be easily seen that $\frac{d^2 f(x)}{dx^2} = -\frac{\xi}{\pi} \cos(x - \mu)$ is < 0, when $x = x_m = \mu$, and thus the mode of the Jones and Pewsey symmetric unimodal distribution on the circle, $X \sim JP(\mu, \xi)$, is $x_m = \mu$, and the maximum value of the pdf (1.4) is given by $f_X(x_m) = f_X(\mu) = \frac{1}{2\pi}(1+2\xi)$. Clearly, the Jones and Pewsey circular distribution, $X \sim JP(\mu, \xi)$, is unimodal.

2.3 Inflection Points

By solving the following equation

$$\frac{d^2 f(x)}{dx^2} = -\frac{\xi}{\pi} \cos(x - \mu) = 0$$

for x, it can easily be seen that the pdf $f(x)$ given by (1.4) is concave up in the intervals $(-\pi, \mu - \frac{\pi}{2}) \cup (\mu + \frac{\pi}{2}, \pi)$ and concave down in the interval $(\mu - \frac{\pi}{2}, \mu + \frac{\pi}{2})$, and, hence, its inflection points are given by $x = \mu \pm \frac{\pi}{2}$.

2.4 Moment Generating Function

The moment generating function $M_X(t)$ of $X \sim JP(\mu, \xi)$ is given by

$$M_X(t) = E(e^{Xt}) = \int_{\mu-\pi}^{\mu+\pi} e^{tx} \frac{1}{2\pi}[1 + 2\xi \cos(x - \mu)]dx$$

from which, on substituting $x - \mu = z$, and using the well-known integral formula

$$\int e^{ax} \cos bx dx = \frac{e^{ax}}{a^2 + b^2}(a \cos bx + b \sin bx) + C,$$

and noting that $\sinh(z) = \frac{1}{2}(e^z - e^{-z})$, it is easy to see that, after simplification, the moment generating function $M_X(t)$ is given by

$$M_X(t) = \begin{cases} 1, \quad t = 0 \\ \frac{[1+(1-2\xi)t^2]e^{\mu t} \sinh(\pi t)}{\pi t(t^2+1)}, \quad t \neq 0 \end{cases} \tag{2.1}$$

Remark 2.1. *Using the above expression* (1.6) *for the MGF,* $M_X(t)$*, and by applying the L'Hospital's rule, it is easily seen that*

$$E(X) = \lim_{t \to 0} \frac{dM_X(t)}{dt} = \mu,$$

which is true since we already have $E(X) = \mu$*, as shown above.*

2.5 Characteristic Function

Following in the same manner as above, the characteristic function $\Phi_X(t)$ of $X \sim JP(\mu, \xi)$ is easily obtained as follows:

$$\Phi_X(t) = E(e^{itX}), -\infty < t < \infty, i = \sqrt{-1},$$

or,

$$\begin{aligned} \Phi_X(t) &= \int_{\mu-\pi}^{\mu+\pi} e^{itx} \frac{1}{2\pi}[1 + 2\xi \cos(x - \mu)]dx \\ &= \begin{cases} 1, & t = 0 \\ \frac{[1+(1-2\xi)i^2t^2]e^{i\mu t} \sinh(i\pi t)}{\pi it(i^2t^2+1)}, & t \neq 0 \end{cases} \\ &= \begin{cases} 1, & t = 0 \\ \frac{[1+(1-2\xi)i^2t^2]e^{i\mu t} \sin(\pi t)}{\pi t(i^2t^2+1)}, & t \neq 0, \end{cases} \end{aligned} \tag{2.2}$$

since $\sinh(i\pi t) = -i \sin(i(i\pi t))$, where $i = \sqrt{-1}$.

2.6 n^{th} Moment

When n is a positive integer, using the pdf (1.4) of $X \sim JP(\mu, \xi)$, we have the following expression for the n^{th} Moment:

$$E(X^n) = \int_{\mu-\pi}^{\mu+\pi} x^n [\frac{1}{2\pi}[1 + 2\xi \cos(x - \mu)]]dx, \tag{2.3}$$

which cannot be evaluated analytically in closed form. Hence, (2.3) should be solved numerically by some appropriate numerical quadrature rules, such as the Newton-Cotes or Gaussian quadrature formulas. However, using the expression (2.2) for the characteristic function of $X \sim JP(\mu, \xi)$, we can easily find all of its moments. Thus, in view of the following definition of the n^{th} moment in terms of the characteristic function, that is,

$$E(X^n) = .(-i)^n \frac{d^n \Phi_X(t)}{dt^n}|_{t=0}, \tag{2.4}$$

we can easily obtain all of the moments of $X \sim JP(\mu, \xi)$ by the differentiation of (2.2). Hence, in view of (2.2) and (2.4), and noting that $i = \sqrt{-1}$ and $i^2 = -1$, the first moment, second moment, and variance of $X \sim JP(\mu, \xi)$ are easily obtained, as described below.

Using the above expression (2.4) for the n^{th} Moment, $E(X^n)$, and by applying the L'Hospital's rule, it is easily seen that

First Moment: $E(X) = \lim_{t\to 0}[(-i)\frac{d\Phi_X(t)}{dt}] = (-i)i\mu = \mu$, (since $i^2 = -1$), which is true since we already have $E(X) = \mu$, as shown above.

Second Moment:

$$\begin{aligned} E(X^2) &= \lim_{t\to 0}[(-i)^2 \frac{d^2\Phi_X(t)}{dt^2}] = (-i)^2(\frac{1}{3}i^2\pi^2 + i^2\mu^2 - 4i^2\xi) \\ &= \frac{1}{3}\pi^2 + \mu^2 - 4\xi, \end{aligned}$$

and

Variance: $Var(X) = E(X^2) - [E(X)]^2 = \frac{1}{3}\pi^2 - 4\xi.$

Similarly, we can obtain other moments using (2.2) and (2.4).

2.7 Entropy

An entropy provides an excellent tool to quantify the amount of information (or uncertainty) contained in a random observation regarding its parent distribution (population). A large value of entropy implies the greater uncertainty in the data. As proposed by Shannon [18], entropy of an absolutely continuous random variable X having the probability density function $\phi_X(x)$ is defined as

$$H[X] = E[-\ln\{\phi_X(x)\}] = -\int_s \phi_x(x)\ln\{\phi_x(x)\}dx,$$

where $S = \{x|\varphi_X(x) > 0\}$. Thus, Shannon entropy of the Jones and Pewsey symmetric unimodal distribution on the circle, $X \sim JP(\mu, \xi)$, with the pdf (1.4), is given by

$$\begin{aligned} H[X] &= E[-\ln\{f(x)\}] \\ &= -\frac{1}{2\pi}\int_{\mu-\pi}^{\mu+\pi} [1+2\xi\cos(x-\mu)]\ln\{\frac{1}{2\pi}[1+2\xi\cos(x-\mu)]\}dx, \end{aligned}$$

which cannot be evaluated analytically in closed form, and hence should be solved numerically by some appropriate numerical quadrature rules, such as the Newton-Cotes or Gaussian quadrature formulas. However, Shannon entropy, $H[X]$, of the Jones and Pewsey symmetric unimodal distribution on the circle, $X \sim JP(\mu, \xi)$, are computed for some selected values of the parameters by using Maple software, which are provided in the Table 2.1.

Table 2.1: Shannon Entropy, $H[X]$, of $X \sim JP(\mu, \xi)$

Paramters	**Shannon Entropy,** $H[X]$
$\mu = 0, \xi = 0.03125$	1.836900027
$\mu = 0, \xi = 0.0625$	1.833963148
$\mu = 0, \xi = 0.125$	1.822127376
$\mu = 0, \xi = 0.25$	1.773238935
$\mu = 0, \xi = 0.5$	1.531024247

The effects of the parameters on Shannon entropy, $H[X]$, of the Jones and Pewsey symmetric unimodal distribution on the circle, $X \sim JP(\mu, \xi)$, can easily be seen from the above Table 2.1. It is obvious from these computations that, when $\mu = 0$, Shannon entropy, $H[X]$, decreases as ξ increases, that is, Shannon entropy, $H[X]$, is a decreasing function of ξ, when $\mu = 0$. Similarly, we can compute the Shannon entropy, $H[X]$, of the Jones and Pewsey symmetric unimodal distribution on the circle, $X \sim JP(\mu, \xi)$, for other values of the parameters, and study the effects of the parameters on Shannon entropy.

3 Reliability

Recalling the definition of the hazard (or failure) rate for non-repairable populations as the instantaneous rate of failure for the survivors to time, say, x, during the next instant of time, the survival (or reliability), the hazard (or failure) rate functions, and cumulative hazard function $H(x)$ of $X \sim JP(\mu, \xi)$ are respectively given by

$$R(x) = 1 - F(x) = 1 - \frac{1}{2\pi}[x - \mu + \pi + 2\xi\sin(x-\mu)],$$

$$h(x) = \frac{f(x)}{1-F(x)} = \frac{\frac{1}{2\pi}[1+2\xi\cos(x-\mu)]}{1-\frac{1}{2\pi}[x-\mu+\pi+2\xi\sin(x-\mu)]},$$

and

$$H(x) = -\ln(R(x)) = -\ln[1 - \frac{1}{2\pi}[x - \mu + \pi + 2\xi \sin(x - \mu)]],$$

where $\mu - \pi < x \leq \mu + \pi$, and $0 \leq \xi \leq \frac{1}{2}$. Further, $X \sim JP(\mu, \xi)$ has an increasing failure rate (IFR), since it is easily seen that its hazard rate, $h(x)$, has the following property: $h'(x) \geq 0$, that is, $f'(x)[1 - F(x)] + [f(x)]^2 \geq 0$, for all x in $\mu - \pi < x \leq \mu + \pi$, and for all values of the parameters. Also, we observe that the hazard rate, $h(x)$ of$X \sim JP(\mu, \xi)$, is concave up, that is, bathtub shaped, since it can easily be seen that $h''(x) > 0$, for all x in $\mu - \pi < x \leq \mu + \pi$, and for all values of the parameters. Using Maple software, the graphs of the reliability function $R(x)$, hazard function $h(x)$, and cumulative hazard function $H(x)$ of $X \sim JP(\mu, \xi)$, are sketched for some selected values of parameters in the following Figures 3.1 - 3.3, respectively. Similarly, we can draw these graphs for other values of the parameters. The effects of the parameters are obvious from these figures. The increasing and bathtub shape behaviors of the hazard (or failure rate) function, $h(x)$, are also evident from the Figure 3.2.

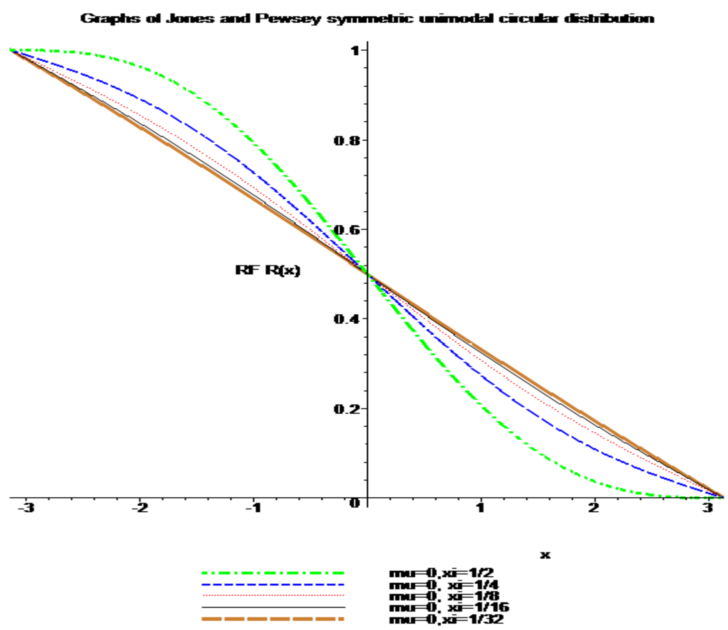

Figure 3.1: Reliability Function, $R(x)$, of $X \sim JP(\mu, \xi)$.

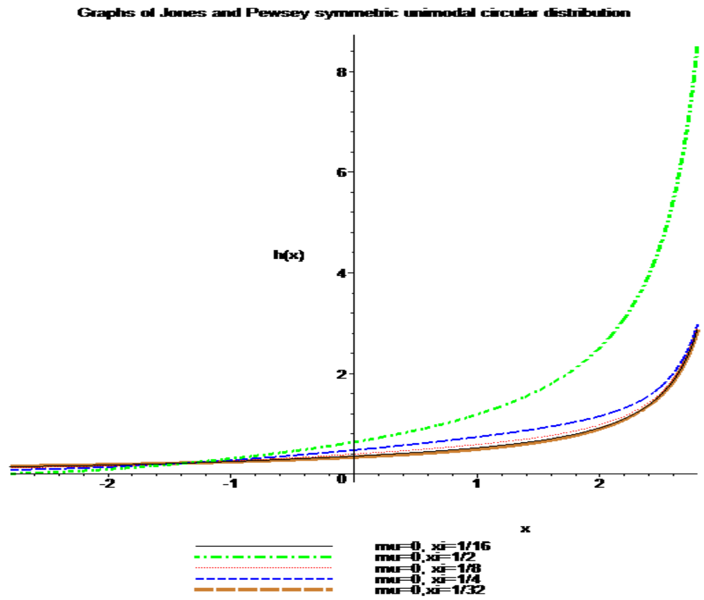

Figure 3.2: Hazard Function, $h(x)$, of $X \sim JP(\mu, \xi)$.

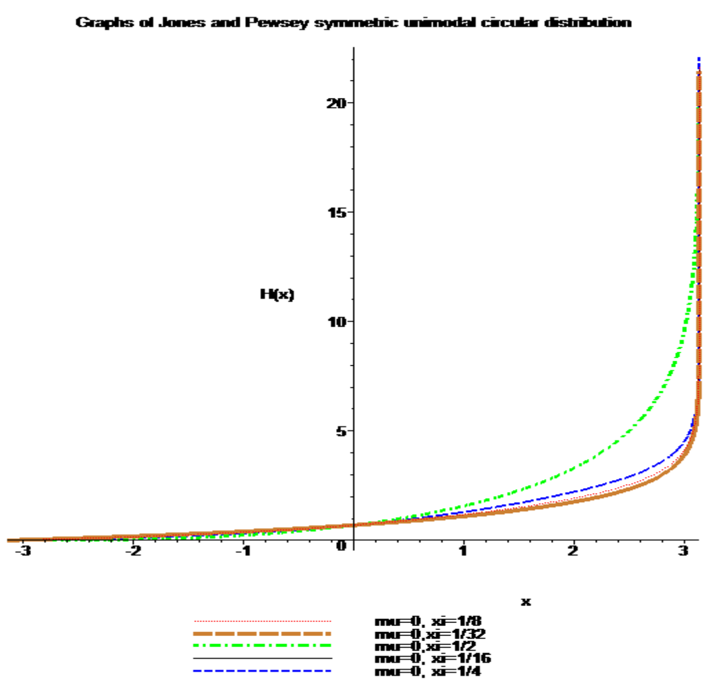

Figure 3.3: Cumulative Hazard Function, $H(x)$, of $X \sim JP\ (\mu,\ \xi)$.

4 Percentile Points

Here we compute the percentile points of $X \sim JP(\mu, \xi)$, with the pdf (1.4) and cdf (1.5). For any $0 < p < 1$, the $100pth$ percentile (also called the quantile of order p) of $X \sim JP(\mu, \xi)$, is a number x_p such that the area under $f_X(x)$ to the left of x_p is p. That is, x_p is any root of the equation given by $F(x_p) = \int_{\mu-\pi}^{x_p} f_X(u)du = p$. The percentile points x_p associated with the cdf of $X \sim JP(\mu, \xi)$, that is, $F(x) = \frac{1}{2\pi}[x - \mu + \pi + 2\xi \sin(x - \mu)]$, are computed for some selected values of the parameters by using Maple software, which are provided in the Table 4.1 below.

Table 4.1: Percentile Points of $X \sim JP(\mu, \xi)$

Percentiles p / Parameters		**0.6**	**0.7**	**0.75**	**0.8**	**0.85**	**0.9**	**0.95**	**0.99**
$\mu = 0, \xi = 0$	x_p	0.62832	1.25664	1.57080	1.88500	2.19911	2.51327	2.82743	3.07876
$\mu = 0, \xi = 0.0625$	x_p	0.56174	1.14291	1.44676	1.76224	2.09063	2.43182	2.78364	3.06980
$\mu = 0, \xi = 0.125$	x_p	0.50694	1.04092	1.32812	1.63548	1.96864	2.33233	2.72665	3.05785
$\mu = 0, \xi = 0.1875$	x_p	0.46138	0.95132	1.21880	1.51063	1.83736	2.21298	2.65063	3.04116
$\mu = 0, \xi = 0.25$	x_p	0.42305	0.87338	1.12061	1.39285	1.70351	2.07565	2.54760	3.01626
$\mu = 0, \xi = 0.3125$	x_p	0.39044	0.80578	1.03377	1.28526	1.57412	1.92765	2.40981	2.97532
$\mu = 0, \xi = 0.375$	x_p	0.36242	0.74704	0.95749	1.18897	1.45421	1.77956	2.23850	2.89751
$\mu = 0, \xi = 0.4375$	x_p	0.33810	0.69578	0.89055	1.10369	1.34611	1.64040	2.05166	2.72347
$\mu = 0, \xi = 0.5$	x_p	0.31680	0.65081	0.83171	1.02845	1.25010	1.51484	1.87264	2.41277

5 Characterizations

Since fitting of a particular probability distribution to the real-world data is also an important area of research, it becomes necessary to justify whether the given probability distribution satisfies the underlying requirements by its characterizations. The characterizations of probability distributions have been investigated by many researchers; see, for example, Ahsanullah [2], Ahsanullah and Shakil [3], Ahsanullah et al. [4, 5, 6], Galambos and Kotz [8], Glänzel [9], Glänzel et al. [10], Kotz and Shanbhag [14], and Nagaraja [16], among others. As pointed out by Glänzel [9], these characterizations may serve as a basis for parameter estimation, and may also be useful in developing some goodness-of-fit tests of distributions by using data whether they satisfy certain properties given in the characterizations of distributions. It appears from the literature that no attention has been paid to the characterizations of the Jones and Pewsey symmetric unimodal distribution on the circle, $X \sim JP(\mu, \psi)$. Therefore, in view of the importance of the characterization problem in the fields of probability and statistics, in this section, we present some new characterizations of the Jones and Pewsey symmetric unimodal distribution on the circle, $X \sim JP(\mu, \psi)$, by considering its special case when $\psi = 1$ and employing the parametrization

$0 \leq \frac{\tanh(\kappa)}{2} \leq \frac{1}{2}$, that is, $0 \leq \xi \leq \frac{1}{2}$, where $\tanh(\kappa) = 2\xi$, which we refer as the Jones and Pewsey symmetric unimodal distribution on the circle, that is, $X \sim JP(\mu, \xi)$. In order to prove our main results (Theorems 5.1-5.5) on characterizations, we will need some assumption and lemmas, which are provided as Assumption 1, and Lemmas 1 and 2, in the Appendix I.

5.1 Characterizations by Truncated Moment

Here, we will provide the characterizations by the truncated moment method.

Theorem 5.1. *Suppose that X is an absolutely continuous random variable with* cdf $F(x)$ *with $F(\mu - \pi) = 0$, and $F(\mu + \pi) = 1$, and $E(X)$ exists. Then, $E(X|X \leq x) = g(x)\tau(x)$, where*

$$g(x) = \frac{\int_{\mu-\pi}^{x} \theta[1 + 2\xi \cos(\theta - \mu)]d\theta}{[1 + 2\xi \cos(\theta - \mu)]},$$

and $\tau(x) = \frac{f(x)}{F(x)}$, if and only if $f(x) = \frac{1}{2\pi}[1 + 2\xi \cos(x - \mu)]$, which is the pdf *of the Jones and Pewsey symmetric unimodal distribution on the circle, $X \sim JP(\mu, \xi)$.*

Proof. Since $E(X|X \leq x) = \frac{\int_{\gamma}^{x} \theta f(\theta)d\theta}{F(x)}$ and $\tau(x) = \frac{f(x)}{F(x)}$, we have $g(x) = \frac{\int_{\gamma}^{x} \theta f(\theta)d\theta}{f(x)}$. Now, if the random variable X satisfies the Assumption 1 and has the Jones and Pewsey symmetric unimodal distribution on the circle, $X \sim JP(\mu, \xi)$,with the pdf $f(x) = \frac{1}{2\pi}[1 + 2\xi \cos(x - \mu)]$, then we have

$$g(x) = \frac{\int_{\mu-\pi}^{x} \theta f(\theta)d\theta}{f(x)} = \frac{\int_{\mu-\pi}^{x} \theta[1 + 2\xi \cos(\theta - \mu)]d\theta}{[1 + 2\xi \cos(\theta - \mu)]}.$$

Consequently, the proof of "if" part of the Theorem 5.1 follows from Lemma 1. Conversely, we will now prove the "only if" condition of Theorem 5.1. Suppose that

$$g(x) = \frac{\int_{\mu-\pi}^{x} \theta[1 + 2\xi \cos(\theta - \mu)]d\theta}{[1 + 2\xi \cos(\theta - \mu)]},$$

from which, after differentiation and simplification, we easily have

$$g'(x) = x + g(x)\frac{2\xi \sin(x - \mu)}{1 + 2\xi \cos(x - \mu)},$$

or,

$$\frac{x - g'(x)}{g(x)} = -\frac{2\xi \sin(x - \mu)}{1 + 2\xi \cos(x - \mu)}.$$

Consequently, by using Lemma 1, we obtain

$$\frac{f'(x)}{f(x)} = \frac{x - g'(x)}{g(x)} = -\frac{2\xi \sin(x - \mu)}{1 + 2\xi \cos(x - \mu)},$$

which, on integrating with respect to x, we obtain $f(x) = c[1 + 2\xi \cos(x - \mu)]$, where c is a constant to be determined. Using the boundary conditions $F(\mu - \pi) = 0$, and $F(\mu + \pi) = 1$, we have $c = \frac{1}{2\pi}$, and thus $f(x) = \frac{1}{2\pi}[1 + 2\xi \cos(x - \mu)]$, which is the pdf of the Jones and Pewsey symmetric unimodal distribution on the circle, $X \sim JP(\mu, \xi)$. This completes the proof of Theorem 5.1. □

Theorem 5.2. *If the random variable X satisfies the Assumption 1 with $\gamma = \mu - \pi$ and $\delta = \mu + \pi$, then $E(X|X \geq x.) = h(x)r(x)$, where $r(x) = \frac{f(x)}{1-F(x)}$, and*

$$h(x) = \frac{\int_{x}^{\mu+\pi} \theta f(\theta)d\theta}{f(x)} = \frac{\int_{x}^{\mu+\pi} \theta[1 + 2\xi \cos(\theta - \mu)]d\theta}{[1 + 2\xi \cos(\theta - \mu)]},$$

if and only if $f(x) = \frac{1}{2\pi}[1+2\xi \cos(x-\mu)]$, which is the pdf *of the Jones and Pewsey symmetric unimodal distribution on the circle, $X \sim JP(\mu, \xi)$.*

Proof. The proof is similar to the Theorem 5.1, and easily follows from Lemma 2. □

5.2 Characterizations by Order Statistics

Here, we will provide the characterizations based on order statistics, for which we first recall the following well-known results.

Let$X_1, X_2, ..., X_n$ be n independent copies of the random variable X having absolutely continuous distribution function $F(x)$ and pdf $f(x)$. Suppose that $X_{1,n} \leq X_{2,n} \leq ... \leq X_{n,n}$are the corresponding order statistics. It is known that $X_{j,n}|X_{k,n} = x$,for $1 \leq k < j \leq n$,is distributed as the $(j-k)th$ order statistics from $(n-k)$ independent observations from the random variable V having the pdf$f_V(v|x)$ where$f_V(v|x) = \frac{f(v)}{1-F(x)}$, $0 \leq v < x$, see, for example, Arnold et al. [1], chapter 2, among others.Further, $X_{i.,n}|X_{k,n} = x, 1 \leq i < k \leq n$,is distributed as ith order statistics from k independent observations from the random variable W having the pdf $f_W(w|x)$ where $f_W(w|x) = \frac{f(w)}{F(x),}, w < x$. Let $S_{k-1} = \frac{1}{k-1}(X_{1,n} + X_{2,n} + ... + X_{k-1,n})$, and $T_{k,n} = \frac{1}{n-k}(X_{k+1,n} + X_{k+2,n} + ... + X_{n.n})$.

In the following two theorems, we will provide the characterizations the Jones and Pewsey symmetric unimodal distribution on the circle, $X \sim JP(\mu, \xi)$,based on order statistics.

Theorem 5.3. *Suppose the random variable X satisfies the Assumption 1 with $\gamma = \mu - \pi$ and $\delta = \mu + \pi$, then $E(S_{k-1}|X_{k,n} = x) = g(x)\tau(x)$, where $\tau(x) = \frac{f(x)}{F(x)}$ and*

$$g(x) = \frac{\int_{\mu-\pi}^{x} \theta[1 + 2\xi \cos(\theta - \mu)]d\theta}{[1 + 2\xi \cos(\theta - \mu)]},$$

if and only if $f(x) = \frac{1}{2\pi}[1 + 2\xi \cos(x - \mu)]$.

Proof. It is known, see David and Nagaraja [7], that

$$E(S_{k-1}|X_{k,n} = x) = E(X|X \leq x).$$

Thus the result follows from Theorem 5.1. □

Theorem 5.4. *Suppose the random variable X satisfies the Assumption 1 with $\gamma = \mu - \pi$ and $\delta = \mu + \pi$, then $E(T_{k,n}|X_{k,n} = x) = h(x)r(x)$, where $r(x) = \frac{f(x)}{1 - F(x)}$ and*

$$h(x) = \frac{\int_{x}^{\mu + \pi} \theta [1 + 2\xi \cos(\theta - \mu)] d\theta}{[1 + 2\xi \cos(\theta - \mu)]},$$

if and only if

$$f(x) = \frac{1}{2\pi} [1 + 2\xi \cos(x - \mu)].$$

Proof. It is known, see David and Nagaraja [7], that

$$E(T_{k,n}|X_{k,n} = x) = E(X|X \geq x).$$

Thus the result follows from Theorem 5.2. □

5.3 Characterization by Upper Record Values

Here, we will provide the characterizations based on upper record values, for which we first recall the following definitions. Suppose that $X_1, X_2, ...$ is a sequence of independent and identically distributed absolutely continuous random variables with distribution function $F(x)$ and pdf $f(x)$. Let $Y_n = \max(X_1, X_2, ..., X_n)$ for $n \geq 1$. We say that X_j is an upper record value of $\{X_n, n \geq 1\}$ if $Y_j > Y_{j-1}, j > 1$. The indices at which the upper records occur are given by the record times $\{U(n) > \min(j|j > U(n+1), X_j > X_{U(n-1)}, n > 1)\}$ and $U(1) = 1$. We will denote the nth upper record value as $X(n) = X_{U(n)}$. In the following theorem, we will provide the characterization of the Jones and Pewsey symmetric unimodal distribution on the circle, $X \sim JP(\mu, \xi)$, based on upper record values.

Theorem 5.5. *Suppose the random variable X satisfies the Assumption 1 with $\gamma = \mu - \sigma$ and $\delta = \mu + \sigma$, then $E(X(n+1)|X(n) = x) = h(x)r(x)$, where $r(x) = \frac{f(x)}{1-F(x)}$ and*

$h(x) = \frac{\int_x^{\mu+\pi} \theta[1+2\xi\cos(\theta-\mu)]d\theta}{[1+2\xi\cos(\theta-\mu)]}$,

if and only if $f(x) = \frac{1}{2\pi}[1 + 2\xi\cos(x - \mu)]$.

Proof. It is known, see Nevzorov [17], that $E(X(n+1)|X(n) = x) = E(X|X \geq x)$. Thus the result follows from Theorem 5.2. □

6 Concluding Remarks

Several circular distributions have been introduced by various authors and researchers. A general family of symmetric unimodal distributions on the circle was introduced by Jones and Pewsey [13] for an absolutely continuous circular random variable X, that is, $X \sim JP(\mu, \psi)$, which has the probability density function (pdf), given by (1.1), that is,

$$f_\psi(x) \propto [1 + \tanh(\kappa\psi)\cos(x - \mu)]^{\frac{1}{\psi}}, \quad \mu - \pi < x \leq \mu + \pi.$$

In this paper, without loss of generality, we considered a special case of the Jones and Pewsey symmetric unimodal distribution on the circle, $X \sim JP(\mu, \psi)$, that is, when $\psi = 1$ and employing the parametrization $0 \leq \frac{\tanh(\kappa)}{2} \leq \frac{1}{2}$, that is, $0 \leq \xi \leq \frac{1}{2}$, where $\tanh(\kappa) = 2\xi$, which we referred as the Jones and Pewsey symmetric unimodal distribution on the circle, that is, $X \sim JP(\mu, \xi)$. We discussed several distributional properties and characterizations of $X \sim JP(\mu, \xi)$. It is hoped that the findings of the paper will be useful for researchers in the fields of probability, statistics, and other applied sciences.

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APPENDIX I

Assumption 1: Suppose the random variable X is absolutely continuous with cumulative distribution function (*cdf*) $F(x)$ and probability density function (*pdf*) $f(x)$. We assume that $\gamma = \{x|F(x) > 0\}$ and $\delta = \inf\{x|F(x) < 1\}$. We further assume that $E(X)$ exists.
Lemma 1: Under the Assumption 1, if $E(X|X \leq x) = g(x)\tau(x)$, where $\tau(x) = \frac{f(x)}{F(x)}$ and $g(x)$ is a continuous differentiable function of x with the condition that $\int_\gamma^x \frac{u-g'(u)}{g(u)}du$ is finite for all x, $\gamma < x < \delta$, then $f(x) = ce^{\int_\gamma^x \frac{u-g'(u)}{g(u)}du} = ce^{\int_\gamma^x \frac{f'(u)}{f(u)}du}$, where $\frac{u-g'(u)}{g(u)} = \frac{f'(u)}{f(u)}$, and c is determined by the condition $\int_\gamma^\delta f(x)dx = 1$.
Proof. For proof, see Ahsanullah and Shakil [3].
Lemma 2: Under the Assumption 1, if $E(X|X \geq x) = g(x)r(x)$, where $r(x) = \frac{f(x)}{1-F(x)}$ and $g(x)$ is a continuous differentiable function of x with the condition that $\int_x^\delta \frac{u+g'(u)}{g(u)}du$ is finite for all x, $\gamma < x < \delta$, then $f(x) = ce^{-\int_x^\delta \frac{u+g'(u)}{g(u)}du} = ce^{\int_x^\delta \frac{f'(u)}{f(u)}du}$, where $-\frac{u+g'(u)}{g(u)} = \frac{f'(u)}{f(u)}$, and c is determined by the condition $\int_\gamma^\delta f(x)dx = 1$.
Proof. For proof, see Ahsanullah and Shakil [3].

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REAL ZEROS OF GENERALIZED MITTAG-LEFFLER FUNCTIONS OF TWO VARIABLES THROUGH INTEGRAL TRANSFORMS

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Abstract

In this work, we study the distribution of real zeros of two variables Mittag-Leffler functions and the integral functions involving some generalized Mittag-Leffler functions of two variables through integral transforms and use them in computation.
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1 Introduction

The gradual development of this work is divided into two subsections. In subsection 1.1, we present the study of distribution of real zeros of some integral functions found in the literature like gamma function, $\xi(z)$ function, one and two parameter Mittag-Leffler functions and also certain theorems to obtain the real zeros of integral functions. In subsection 1.2, the theorems concerning obtaining real zeros through integral transform methods are introduced.

1.1 Real zeros of integral functions

Titchmarsh [27, p.268] had written that a number of important functions have no complex zeros; for example, all the zeros of $\frac{1}{\Gamma(z)}, z \in \mathbb{C}$, are real. Here, $\frac{1}{\Gamma(z)}$ is a meromorphic function defined in following Weierstrass product form (see also Conway [4, p. 176])

$$\frac{1}{\Gamma(z)} = ze^{\gamma z}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}}, \text{where,} \gamma \text{ is chosen so that } \Gamma(1)=1. \tag{1.1}$$

Clearly, the function given in Eqn. (1.1) has zeros at $z = 0, -1, -2, -3, \ldots$.

On the other hand it is some time very difficult to decide whether the zeros are real or not; for example it was conjectured by Riemann, in 1859, that all the zeros of the function $\Xi(z)$ defined by (see in [27, p. 265])

$$\Xi(z) = \xi(\frac{1}{2}+iz), i = \sqrt{(-1)} \text{ and } \xi(s) = \frac{s}{2}(s-1)\pi^{-\frac{1}{2}s}\Gamma(\frac{1}{2}s)\zeta(s),$$

where, the Riemann zeta function $\zeta(s)$ is defined as

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}, \Re(s) > 1, \tag{1.2}$$

are real, but this has never been proved. Again, Titchmarsh [27, p. 269] suggested that following theorems are applicable to check that the function has real zeros:

Theorem 1.1. ***(The Laguerre's theorem)***. (See also, Kumar and Pathan [13, p. 67]):

Suppose that the function $\Phi(w)$ is having negative real non-zero zeros and that is given by

$$\Phi(w) = ae^{kw} \prod_{n=1}^{\infty} (1 + \frac{w}{\alpha_n}) e^{-\frac{w}{\alpha_n}}, \tag{1.3}$$

where, a, and α_n being all positive, k is any constant.

Also, let $f(z)$ be an integral function of the form

$$f(z) = e^{bz+c} \prod_{n=1}^{\infty} \left(1 + \frac{z}{z_n}\right), \tag{1.4}$$

b and all z_n are positive.

If

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \; a_n = \frac{f^{(n)}(0)}{n!}, \tag{1.5}$$

then,

$$g(z) = \sum_{n=0}^{\infty} a_n \Phi(n) z^n \tag{1.6}$$

is an integral function, all of whose zeros are real and negative.

It is remarked that the integral functions or entire functions are those meromorphic functions which have only zeros, but not poles. For example, Titchmarsh [27, p. 285] had pointed out that the generalized hypergeometric function defined by the formula

$$ {}_pF_q(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_p)_n}{(\beta_1)_n \ldots (\beta_q)_n} \frac{z^n}{n!},$$
$$(\alpha)_n = \alpha(\alpha+1) \ldots (\alpha+n-1), (\alpha)_0 = 1,$$

is an integral function, if $q \geq p$, which has no poles.

Making an application of above theorem of Laguerre (Theorem 1.1), Peresyolkova [19] has derived that the function defined by

$$\varphi_{\rho_1,\rho_2}(z, \mu_1, \mu_2) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\mu_1 + \frac{n}{\rho_1})\Gamma(\mu_2 + \frac{n}{\rho_2})} \tag{1.7}$$

is an integral function whose all zeros are real and negative as the Mittag-Leffler function given by

$$E_{\rho_1}(z, \mu_1) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\mu_1 + \frac{n}{\rho_1})} \tag{1.8}$$

has real and negative non-zero zeros when $\rho_1 \leq \frac{1}{2}$, and also that may be represented in infinite product form as

$$E_{\rho_1}(z, \mu_1) = \frac{1}{\Gamma(\mu_1)} \prod_{n=1}^{\infty} (1 + \frac{z}{z_n}), \tag{1.9}$$

where all z_n are positive and that is an entire function having its non -zero real and negative zeros.

On the other hand, Craven and Csordas [5] have defined Laguerre multiplier sequences and then prove that for $\beta > 2$ and $\gamma > 0$, a positive integer $m_0 \in m_0(\beta, \gamma)$ and $m \geq m_0$, the meromorphic function defined by

$$E_{\beta,\gamma}^{(m)}(z) = \sum_{n=0}^{\infty} \frac{\Gamma(m+n+1)}{\Gamma(\beta(m+n)+\gamma)} \frac{z^n}{n!} \tag{1.10}$$

is a meromorphic function of Laguerre-Polya class and has only real zeros. In this connection, the Po'lya-Benz and Po'lya-Schur type theorems are very useful to study the real zeros of polynomials (see Aleman et al. [2]).

Again, from Eqn. (1.8) we have the relation $E_{\frac{1}{\beta}}(z, 1) = E_{\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\beta n + 1)}$. The complex zeros of this Mittag-Leffler function $E_{\beta}(z)$ were studied by Wiman [28] and found that the asymptotic curve along which the zeros are located for $0 < \beta < 2$ and showed that they fall on the negative real axis for $\beta \geq 2$.

Hilfer and Seybold [10] also studied the location of zeros of generalized Mittag Leffler function $E_{\beta}(z)$ as function of β, for the case $1 < \beta < 2$ and noted that with increasing β more and more pairs of zeros collapse onto the negative real axis.

Hanneken et al. [9, pp. 15-26] enumerated the complex zeros of the Mittag-Leffler function $E_{\beta}(z)$ when $0 < \beta < 1$, the finite real zeros of the $E_{\beta}(z)$ for the case $1 < \beta < 2$ and that infinite real zeros when $\beta \geq 2$.

Further, for a generalized Mittag-Leffler function, defined by the formula $E_{\beta,\gamma}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\beta n + \gamma)}$, Popov [20] and Popov and Sedletski [21] have worked on its distribution of real zeros and found that all zeros of this function $E_{\beta,\gamma}(z)$ are real negative and simple when $\beta > 2$ and $0 < \gamma \leq (2\beta - 1)$. Also, in the work of [20], it has shown that all zeros of the function $E_{4,9}(z)$ are real, negative and simple.

Recently, Kumar and Pathan [13] have developed another method of sequence of integrals on making an appeal to the Theorem 1.1 and the techniques of Kumar, Pathan and Chandel [15] and Kumar and Srivastava [16] to obtain the distribution of real non-zero zeros of generalized Mittag-Leffler function of Shukla and Prajapati [22].

In our work, we use the techniques due to ([13], [15] and [16]) and consider the conditions due to the work of Popov [20] and Popov and Sedletski [21], and prove that:

Theorem 1.2. *For $z \in \mathbb{C}, \Re(\beta) > 2, 0 < \Re(\gamma) \leq 2\beta - 1, \Re(\delta) > 0, q \in (0,1) \cup \mathbb{N}, q \leq \beta$ and all z_n be positive. The generalized Mittag-Leffler function defined due to Shukla and Prajapati [22] given by*

$$E_{\beta,\gamma}^{\delta,q}(z) = \sum_{n=0}^{\infty} \frac{(\delta)_{qn}}{\Gamma(\beta n + \gamma) n!} z^n, \tag{1.11}$$

here,

$$(\delta)_{qn} = \frac{\Gamma(\delta + qn)}{\Gamma(\delta)} \text{ and when } q \in \mathbb{N}, (\delta)_{qn} = \prod_{r=1}^{q} (\frac{\delta + r - 1}{q})_n, \tag{1.12}$$

may be represented by Weierstrass product form (see Kumar and Pathan [13])

$$E_{\beta,\gamma}^{\delta,q}(z) = \frac{1}{\Gamma(\gamma)} \exp[\frac{(\delta)_q \Gamma(\gamma)}{\Gamma(\gamma + \beta)} z] \prod_{n=1}^{\infty} ((1 + \frac{z}{z_n}) \exp[-\frac{z}{z_n}]) \tag{1.13}$$

Also that the function $E^{\delta,q}_{\beta,\gamma}(z)$ defined in Eqns. (1.11)-(1.13) is an entire function and it has simple real non-zero and negative zeros.

It is remarkable that on making specialization of the parameters of the function given in Eqns. (1.11)-(1.13), we get the results identical due to the Popov [20] and Popov and Sedletski [21].

1.2 Real zeros due to integral transforms

Csordas and Varga [6] have presented the theorem of Po'lya such that:

Theorem 1.3. *If $K(t)$ and $|K(t)|$ are integrable over $\mathbb{R}$ and $K(t) = O(\exp(-|t|^{2+\alpha})), \alpha > 0$, $t \to \infty$, and $K : \mathbb{R} \to \mathbb{R}$, is real analytic on an interval about origin such that*

$$K(t) = \sum_{n=0}^{\infty} c_n t^n, t \in (-r, r), r > 0, c_n \in \mathbb{R}, n = 0, 1, 2, \ldots. \tag{1.14}$$

Then

$$H(z) = \int_0^\infty t^{z-1} K(t) dt \tag{1.15}$$

is a meromorphic function and if $H(z)$ has only real negative zeros, then the entire function

$$F_q(z) = \int_{-\infty}^{\infty} K(t^{2q}) e^{izt} dt \ (q = 1, 2, 3, \ldots) \tag{1.16}$$

has only real zeros.

Here, in our investigations, we study the distribution of real zeros of generalized Mittag-Leffler functions of two variables through above said integral transforms methods and make its computation.

2 Generalized Mittag-Leffler functions of two variables

In this section, we present following generalized Mittag-Leffler functions of two variables:

The two variable generalized Mittag-Leffler functions due to Garg et al. [8] are defined in the form

$$\begin{aligned} E_1(x, y) &= E_1\left(\begin{matrix} \delta_1, \alpha_1; \delta_2, \beta_1 \\ \gamma_1, \alpha_2, \beta_2; \gamma_2, \alpha_3; \gamma_3, \beta_3 \end{matrix} \middle| \begin{matrix} x \\ y \end{matrix}\right) \\ &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\delta_1)_{\alpha_1 m} (\delta_2)_{\beta_1 n}}{\Gamma(\gamma_1 + \alpha_2 m + \beta_2 n)} \frac{x^m}{\Gamma(\gamma_2 + \alpha_3 m)} \frac{y^n}{\Gamma(\gamma_3 + \beta_3 n)} \end{aligned} \tag{2.1}$$

provided that $\gamma_1, \gamma_2, \gamma_3, \delta_1, \delta_2, x, y \in \mathbb{C}, \min\{\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3\} > 0$.

$$\begin{aligned} E_2(x, y) &= E_2\left(\begin{matrix} \delta_1, \alpha_1, \beta_1; \delta_2, \alpha_2 \\ \gamma_1, \alpha_3, \beta_2; \gamma_2, \alpha_4; \gamma_3, \beta_3 \end{matrix} \middle| \begin{matrix} x \\ y \end{matrix}\right) \\ &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\delta_1)_{\alpha_1 m + \beta_1 n} (\delta_2)_{\alpha_2 m}}{\Gamma(\gamma_1 + \alpha_3 m + \beta_2 n)} \frac{x^m}{\Gamma(\gamma_2 + \alpha_4 m)} \frac{y^n}{\Gamma(\gamma_3 + \beta_3 n)} \end{aligned} \tag{2.2}$$

provided that $\gamma_1, \gamma_2, \gamma_3, \delta_1, \delta_2, x, y \in \mathbb{C}, \min\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3\} > 0$.

Recently, Kumar and Pathan [14] have found a two variable generalized Mittag-Leffler function in the study of statistical characteristics, operational formulae of special functions and anomalous diffusion in the form

$$E^{(\delta)}_{\alpha,\beta}(x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\delta)_{m+n} x^m y^n}{m! n! \Gamma(\alpha m + \beta n + 1)}, \forall x, y, \delta \in \mathbb{C}, \alpha > 0, \beta > 0, \Re(\delta) > 0. \tag{2.3}$$

Then, Kumar and Pathan [13] has obtained the distribution of non-zero zeros of generalized Mittag-Leffler function of two variables through its infinite product formula (1.13) and with the help of Eqn. (1.10) and they [13] defined other generalized Mittag-Leffler function of two variables in the form

$$E^{(\delta)}_{\alpha,\beta;\gamma}(x,y)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\delta)_{m+n}x^m y^n}{m!n!\Gamma(\alpha m+\beta n+\gamma)} \tag{2.4}$$

provided that $x, y, \delta, \gamma \in \mathbb{C}, \alpha > 0, \beta > 0,\ \Re(\gamma) > 0, \Re(\delta) > 0$.

Very recently, Pathan and Kumar [18] have obtained an infinite product formula for the distribution of non-zero zeros of multi-index generalized Mittag-Leffler function.

Kumar [12] has generalized the two variable Mittag-Leffler function given in Eqn. (2.4) in the form

$$E^{(\delta;\alpha_1,\beta_1)}_{\alpha_2,\beta_2;\gamma_1:\alpha_3;\gamma_2:\beta_3;\gamma_3}(x,y)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\delta)_{\alpha_1 m+\beta_1 n}x^m y^n}{\Gamma(\alpha_2 m+\beta_2 n+\gamma_1)\Gamma(\alpha_3 m+\gamma_2)\Gamma(\beta_3 n+\gamma_3)}, \tag{2.5}$$

provided that $x, y, \delta, \gamma_i \in \mathbb{C}, \alpha_i > 0, \beta_i > 0,\ \Re(\gamma_i) > 0, \Re(\delta) > 0, i = 1, 2, 3$, and studied its convergence conditions and some of its relations to various special functions.

3 Real zeros of Mittag-Leffler functions and integral functions involving generalized Mittag-Leffler functions of two variables through integral transforms

In this section, to obtain the distribution of real zeros of Mittag-Leffler functions of two variables and the integral functions involving two variables Mittag-Leffler functions, we present some theorems.

Motivated by above work, Theorem 1.3 and the work of Cardon [3], in our investigations, we present following theorems:

Theorem 3.1. *If $K(t)$ and $|K(t)|$ are integrable over $\mathbb{R}$ and $K(t) = O(\exp(-|t|^{2+\alpha})), \alpha > 0, t \to \infty$, and $K : \mathbb{R} \to \mathbb{R}$, is real analytic on an interval about origin such that*

$$K(t)=\sum_{n=0}^{\infty}c_n t^n, t\in(-r,r), r>0, c_n\in\mathbb{R}, n=0,1,2,\ldots.$$

Also, if we let

$$G(t)=e^{pt}K(t), p>0. \tag{3.1}$$

Then

$$H(p,z)=\int_0^{\infty}e^{-pt}t^{z-1}G(t)dt \tag{3.2}$$

is a meromorphic function and if $H(p, z)$ has only real negative zeros, then the entire function

$$F_q(p,z)=\int_{-\infty}^{\infty}e^{-pt^{2q}}G(t^{2q})e^{izt}dt \quad (p>0, q=1,2,3,\ldots) \tag{3.3}$$

has only real zeros and hence then inverses also have only real zeros for all $q = 1, 2, 3, ...$

Proof. On using Theorem 1.3 we easily satisfy the Eqns. (3.1) and (3.2). Again Eqn. (3.2) shows that $H(p, z)$ is Laplace transform of the function $t^{z-1}G(t)$. Therefore the inverse Laplace transform of $H(p, z)$ is an entire function and it has negative real zeros as when $H(p, z)$ has negative real zeros. On applying the result for Dirac delta function such that

$\frac{1}{2\pi i}\lim_{\beta\to\infty}\int_{\gamma-i\beta}^{\gamma+i\beta} e^{(t-\theta)p}dp = \delta(t-\theta)$, (see Kanwal [11, p. 68]), and Eqn. (3.2), this implies that the inverse Laplace transform of $H(p,z)$ i.e.

$$\begin{aligned}
L^{-1}\{H(p,z)\} &= \frac{1}{2\pi i}\lim_{\beta\to\infty}\int_{\gamma-i\beta}^{\gamma+i\beta} e^{tp}H(p,z)dp \\
&= \frac{1}{2\pi i}\lim_{\beta\to\infty}\int_{\gamma-i\beta}^{\gamma+i\beta} e^{tp}\int_0^\infty e^{-p\theta}\theta^{z-1}G(\theta)d\theta dp \\
&= \int_0^\infty \theta^{z-1}G(\theta)\frac{1}{2\pi i}\lim_{\beta\to\infty}\int_{\gamma-i\beta}^{\gamma+i\beta} e^{(t-\theta)p}dpd\theta \\
&= \int_0^\infty \theta^{z-1}G(\theta)\delta(t-\theta)d\theta = \begin{cases} 0, \text{when}, t=0, z>1, \\ 0, \text{when}, \theta\neq t. \end{cases} \qquad (3.4)
\end{aligned}$$

In the similar manner, the inverse Laplace transform of $F_q(p,z)$ is an entire function and it has real zeros. That is

$$L^{-1}\{F_q(p,z)\} = \int_{-\infty}^{\infty} G(\theta^{2q})e^{iz\theta}\delta(t-\theta^{2q})d\theta, \qquad (3.5)$$

has real zeros for $q = 1,2,3,...$ □

On the other hand, we also present:

Theorem 3.2. *If F be of order $e^{\alpha t}, t>0$ and for some $\alpha>0$ that is F be of $O(e^{\alpha t})$ for some α and if the image $f(p)=\int_0^\infty e^{-pt}F(t)dt, \Re(p)>\alpha$, is an analytic and entire function of the type given in Eqn. (1.4), of Theorem 1.1, and has negative real non-zero zeros and then, for any $q\geq\alpha$ the function $G(t)=e^{-qt}F(t), t>0, G(t)=0, t<0$ be an entire function and also has only real zeros.*

Proof. The Fourier integral formula has the exponential form

$$G(t) = \frac{1}{2\pi}\lim_{s\to\infty}\int_{-s}^{s} e^{iyt}\int_{-\infty}^{\infty} G(\tau)e^{-iy\tau}d\tau dy, (-\infty<t<\infty). \qquad (3.6)$$

Therefore, for $q>\alpha$, the Eqn. (3.6) may be written by

$$G(t) = \frac{1}{2\pi}\lim_{s\to\infty}\int_{-s}^{s} e^{iyt}\int_0^{\infty} F(\tau)e^{-(q+iy)\tau}d\tau dy, (-\infty<t<\infty). \qquad (3.7)$$

Then, for $z=q+iy$ and for all real t, from Eqn. (3.7), we have an entire function

$$e^{qt}G(t) = \frac{1}{2\pi i}\lim_{s\to\infty}\int_{q-is}^{q+is} e^{zt}f(z)dz \qquad (3.8)$$

Again due to the statement of the Theorem 3.2, the inverse Laplace transformation of $f(p)$ is given by

$$F(t) = \frac{1}{2\pi i}\lim_{s\to\infty}\int_{q-is}^{q+is} e^{zt}f(z)dz \qquad (3.9)$$

Apply Eqn. (1.4) in the Eqn. (3.9), we may write

$$F(t-b) = \frac{e^c}{2\pi i}\lim_{s\to\infty}\int_{q-is}^{q+is} e^{zt}\prod_{n=1}^{\infty}(1+\frac{z}{z_n})dz \qquad (3.10)$$

where b and all z_n are positive.

Now, make an appeal to the result of Erde'lyi et al [7, Vol. II, p. 98, Eqn. (1.1)], (see also Srivastava and Manocha [26, p. 253, Eqn. (3.3)]), given by

$$e^{zt} = \frac{\Gamma(\nu)}{(\frac{1}{2}t)^{\nu}} \sum_{m=0}^{\infty} (\nu + m) I_{\nu+m}(t) C_m^{\nu}(z), \nu > 0$$

and v is not integer, and the transformation formula,

$$I_{\nu}(z) = e^{-\frac{1}{2}\nu\pi i} J_{\nu}(iz), (-\pi < arg(z) < \frac{\pi}{2}),$$

due to Abramowitz and Stegun [1, p. 377, Eqn. (9.6.3)],

Or such that for $(-\pi < arg(z) < \frac{\pi}{2})$, we have

$$\begin{aligned} e^{zt} = e^{(-iz)(it)} &= \frac{\Gamma(\nu)}{(\frac{1}{2}it)^{\nu}} \sum_{m=0}^{\infty} (\nu + m) I_{\nu+m}(it) C_m^{\nu}(-iz) \\ &= \frac{\Gamma(\nu) e^{-i\nu\pi}}{(\frac{1}{2}t)^{\nu}} \sum_{m=0}^{\infty} (\nu + m) e^{-\frac{1}{2}m\pi i} J_{\nu+m}(-t) C_m^{\nu}(-iz), \end{aligned}$$

in the integrand of Eqn. (3.10), we find that

$$F(t) = \frac{\Gamma(\nu) e^{-\nu\pi i}}{(\frac{1}{2}(t+b))^{\nu}} \sum_{m=0}^{\infty} (\nu + m) e^{-\frac{1}{2}m\pi i} J_{\nu+m}(-(t+b)) H_m^{\nu,c}. \tag{3.11}$$

Here in Eqn. (3.11), we let the integral function

$$H_m^{\nu,c} = \frac{e^c}{2\pi i} \lim_{s\to\infty} \int_{q-is}^{q+is} C_m^{\nu}(-iz) \prod_{n=1}^{\infty} (1 + \frac{z}{z_n}) dz, \tag{3.12}$$

provided that $-\pi < arg(z) < \frac{\pi}{2}$.

Due to occurrence of Bessel functions in above Eqn. (3.11), the function $F(t)$ has real zeros due to Bessel functions of order $\nu, \nu+1, \nu+2, \ldots$, ν is not an integer and use of Hurwitz Theorem (Titchmarsh [27, p.119]). Again then equate it to the Eqn. (3.8), we say that the function $G(t)$ also has real zeros. □

Applications

Now, here, we consider the Mittag-Leffler functions of two variables defined in Section 2 in following form and apply above theory and the integral transform methods given in Theorems 3.1 and 3.2 to achieve our goal as:

Theorem 3.3. *If*

$$F_1(t) = t^{\gamma_1 - 1} E_1(xt^{\alpha_2}, yt^{\beta_2}) = t^{\gamma_1 - 1} E_1 \left(\begin{matrix} \delta_1, \alpha_1; \delta_2, \beta_1 \\ \gamma_1, \alpha_2, \beta_2; \gamma_2, \alpha_3; \gamma_3, \beta_3 \end{matrix} \middle| \begin{matrix} xt^{\alpha_2} \\ yt^{\beta_2} \end{matrix} \right), \ t > 0,$$

where two variable generalized Mittag-Leffler function $E_1(x, y)$ is defined in Eqn. (2.1), then for $\gamma_1 > 0, \gamma_2 = 2\alpha_3 - 1, \gamma_3 = 2\beta_3 - 1, \delta_1, \delta_2, x, y \in \mathbb{C}, \min\{\alpha_1, \alpha_2, \beta_1, \beta_2, \} > 0, \alpha_3 > 2, \beta_3 > 2, \Re(\delta_j) > 0, \forall j = 1, 2, \alpha_1 \leq \alpha_3, \beta_1 \leq \beta_3, F_1(t)$ has real zeros for positive t and negative x or y.

Proof. The Laplace transformation of the function $F_1(t) = t^{\gamma_1 - 1} E_1(xt^{\alpha_2}, yt^{\beta_2})$ is obtained by

$$f_1(p; \alpha_1, \alpha_3, \beta_1, \beta_3, \gamma_1, \gamma_2, \gamma_3, \delta_1, \delta_2, x, y) = \frac{1}{p^{\gamma_1}} E_{\alpha_3, \gamma_2}^{\delta_1, \alpha_1}(\frac{x}{p^{\gamma_1}}) E_{\beta_3, \gamma_3}^{\delta_2, \beta_1}(\frac{y}{p^{\gamma_1}}), \Re(p) > 0, \tag{3.13}$$

where, the generalized Mittag-Leffler function $E^{\delta,q}_{\beta,\gamma}(.)$is defined in Eqn. (1.11).

Now, make an appeal to the Eqns. (3.9) and (3.12) to get

$$F_1(t) = \frac{1}{2\pi i}\lim_{s\to\infty}\int_{q-is}^{q+is} e^{pt}p^{-\gamma_1}E^{\delta_1,\alpha_1}_{\alpha_3,\gamma_2}(\frac{x}{p^{\gamma_1}})E^{\delta_2,\beta_1}_{\beta_3,\gamma_3}(\frac{y}{p^{\gamma_1}})dp \tag{3.14}$$

Due to occurrence of the product of generalized Mittag-Leffler functions

$$E^{\delta_1,\alpha_1}_{\alpha_3,\gamma_2}(\frac{x}{p^{\gamma_1}})E^{\delta_2,\beta_1}_{\beta_3,\gamma_3}(\frac{y}{p^{\gamma_1}})$$

in the Eqn. (3.13), we appeal to the Theorem 1.2 and the theory of Popov [20] and Popov and Sedletski [21], that for the given conditions, $p > 0, \gamma_1 > 0, \gamma_2 = 2\alpha_3 - 1, \gamma_3 = 2\beta_3 - 1$, $\delta_1, \delta_2, x, y \in \mathbb{C}$, $\min\{\alpha_1, \alpha_2, \beta_1, \beta_2,\} > 0, \alpha_3 > 2, \beta_3 > 2, \Re(\delta_j) > 0, \forall j = 1, 2, \alpha_1 \le \alpha_3, \beta_1 \le \beta_3$, f_1 has real and negative zeros for x or y. Then, in Eqn. (3.14) use the Theorems 3.1 and 3.2, $F_1(t)$ has real zeros for positive t and negative x or y. □

Example 3.1. *In Theorem 3.3, if we take* $\delta_1 = 0.5, \alpha_1 = 2.4, \alpha_3 = 2.5$, *then,* $\gamma_2 = 4$, *and again, take* $\delta_2 = 0.4, \beta_1 = 2.4, \beta_3 = 2.5$, *then,* $\gamma_3 = 4$, *again set* $x = 5, t \in (0, 10), y \in (-10, 10)$, *in Eqn. (3.14), then in the Figure 3.1,* $F_1(t)$ *seems to be zero when* $1 < t < 4$ *and for negative values of* y. *Although for negative values of* t *it gives no figure.*

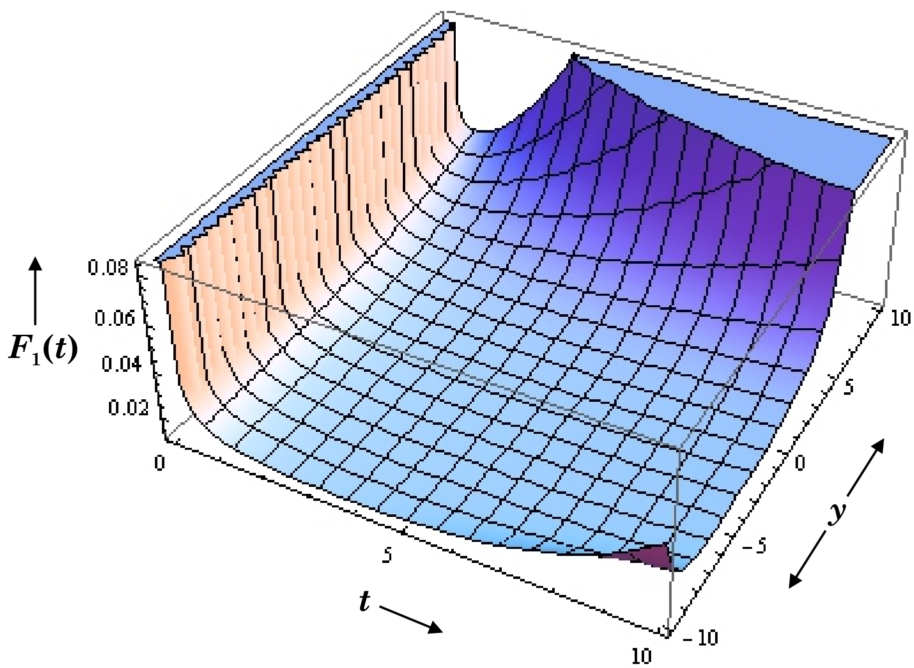

Figure 3.1

Example 3.2. *In Theorem 3.3, if we take* $\delta_1 = 0.5, \alpha_1 = 2.4, \alpha_3 = 2.5$, *then,*$\gamma_2 = 4$, *and again, take* $\delta_2 = 0.4, \beta_1 = 2.4, \beta_3 = 2.5$, *then,* $\gamma_3 = 4$, *again set* $t = 5, x \in (-10, 10), y \in (-10, 10)$, *in Eqn. (3.14), then in the Figure 3.2,* $F_1(t)$ *seems to be zero when* x *and* y *are negative.*

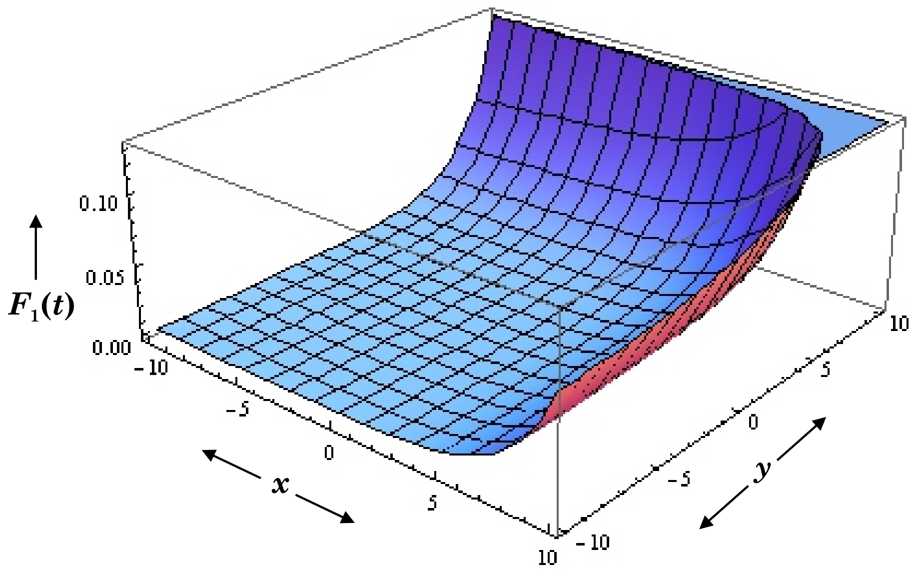

Figure 3.2

Theorem 3.4. *If*

$$F_1(t) = t^{\gamma_1-1}E_1(xt^{\alpha_2}, yt^{\beta_2}) = t^{\gamma_1-1}E_1\left(\begin{array}{c|c} \delta_1, \alpha_1; \delta_2, \beta_1 & xt^{\alpha_2} \\ \gamma_1, \alpha_2, \beta_2; \gamma_2, \alpha_3; \gamma_3, \beta_3 & yt^{\beta_2} \end{array}\right), \; t > 0,$$

where two variable generalized Mittag-Leffler function $E_1(x, y)$ is defined in Eqn. (2.1), then for $\gamma_1, \gamma_2, \gamma_3, \delta_1, \delta_2,\ x, y \in \mathbb{C}, \min\{\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3\} > 0. F_1(t)$ has real zeros.

Proof. In Eqn. (3.14), use the techniques of the Theorem 3.2 to get

$$F_1(t) = \frac{\Gamma(\nu)e^{-\nu\pi i}}{(\frac{1}{2}(t))^\nu}\sum_{m=0}^{\infty}(\nu+m)J_{\nu+m}(-(t))e^{-\frac{1}{2}m\pi i}H_m^{\nu,0} \tag{3.15}$$

where, $H_m^{\nu,0} = \frac{1}{2\pi i}\lim_{s\to\infty}\int_{q-is}^{q+is}C_m^\nu(-ip)p^{-\gamma_1}E_{\alpha_3,\gamma_2}^{\delta_1,\alpha_1}(\frac{x}{p^{\gamma_1}})E_{\beta_3,\gamma_3}^{\delta_2,\beta_1}(\frac{y}{p^{\gamma_1}})dp.$ $\nu > 0$ and is not an integer and provided that $-\pi < arg(p) < \frac{\pi}{2}$.

Here, in right hand side of the Eqn. (3.15), each term of the series has the Bessel functions of order $\nu, \nu+1, \nu+2, \ldots$, where ν is not an integer, so that on applying the theory of the Theorem 3.2, we have that for $\gamma_1, \gamma_2, \gamma_3, \delta_1, \delta_2,\ x, y \in \mathbb{C}, \min\{\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3\} > 0$, $F_1(t)$ has real zeros whenever $\nu > 0$, and is not an integer, also $e^{-\frac{1}{2}(m+2\nu)\pi i}H_m^{\nu,0}$ is a real number. $\square$

Example 3.3. *In Theorem 3.4, if we consider $e^{-(2\nu+m)\frac{\pi}{2}i} = (2+\sqrt{3})$ (any real number), $\nu = .5, t \in (-100, 100)$, then in Figure 3.3, $F_1(t)$ seems to be zero for real values of t.*

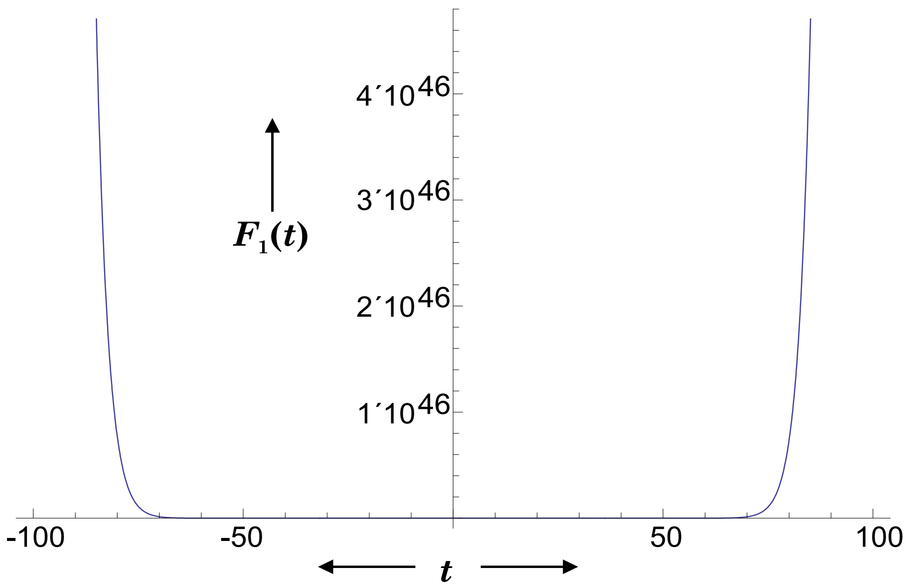

Figure 3.3

Theorem 3.5. *If*

$$F_2(t) = \Gamma(\delta_1) t^{\gamma_1 - 1} \frac{1}{2\pi i} \int_{-\infty}^{0^+} e^u u^{-\delta_1} E_2(x \frac{t^{\alpha_3}}{u^{\alpha_1}}, y \frac{t^{\beta_2}}{u^{\beta_1}}) du, \tag{3.16}$$

where, two variable generalized Mittag-Leffler function

$$E_2(x, y) = E_2 \left(\begin{array}{c|c} \delta_1, \alpha_1, \beta_1; \delta_2, \alpha_2 & x \\ \gamma_1, \alpha_3, \beta_2; \gamma_2, \alpha_4; \gamma_3, \beta_3 & y \end{array} \right)$$

is defined in Eqn. (2.2), then for $\gamma_1 > 0, \gamma_2 = 2\alpha_4 - 1, \gamma_3 = 2\beta_3 - 1, \delta_1, \delta_2, x, y \in \mathbb{C}, \Re(\delta_j) > 0, \forall j = 1, 2,$ $min\{\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2\} > 0, \alpha_4 > 2, \beta_3 > 2, \alpha_2 \leq \alpha_4, F_2(t)$ *has real zeros.*

Proof. In right hand side of Eqn. (3.16), define $E_2(x, y)$ due to Eqn. (2.2) and then make an application of the formula of Erde'lyi et al. [7, Vol. I] given by

$$\frac{1}{\Gamma(\lambda + m)} = \frac{1}{2\pi i} \int_{-\infty}^{0^+} e^u u^{-\lambda - m} du, \Re(\lambda) > 0, \; m = 0, 1, 2, 3, \ldots, \tag{3.17}$$

we get

$$F_2(t) = t^{\gamma_1 - 1} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\delta_2)_{\alpha_2 m}}{\Gamma(\gamma_1 + \alpha_3 m + \beta_2 n)} \frac{(x t^{\alpha_3})^m}{\Gamma(\gamma_2 + \alpha_4 m)} \frac{(y t^{\beta_2})^n}{\Gamma(\gamma_3 + \beta_3 n)}. \tag{3.18}$$

Then, take Laplace transformation of $F_2(t)$ given in Eqn. (3.18) to get

$$f_2(p; \alpha_1, \alpha_3, \beta_1, \beta_3, \gamma_1, \gamma_2, \gamma_3, \delta_1, \delta_2, x, y) = \frac{1}{p^{\gamma_1}} E^{\delta_2, \alpha_2}_{\alpha_4, \gamma_2}(\frac{x}{p^{\gamma_1}}) E_{\beta_3, \gamma_3}(\frac{y}{p^{\gamma_1}}), \Re(p) > 0 \tag{3.19}$$

where, the generalized Mittag-Leffler function $E_{\beta,\gamma}(.)$ is defined in fourth paragraph of Eqn. (1.10). Now, then use the techniques of Theorem 3.3 in Eqn. (3.19), we may say that $F_2(t)$, defined in Eqn. (3.18), has real zeros. □

Example 3.4. *In Eqn. (3.19) of Theorem 3.5, set* $\gamma_1 = .5, \delta_2 = .5, \alpha_2 = 2.5, \alpha_4 = 3, \beta_3 = 2.5$*, then* $\gamma_2 = 5$*, and* $\gamma_3 = 4$*,* $x \in (-10, 10)$*,* $y \in (-10, 10)$ $F_2(t)$*, defined in Eqn. (3.16), has real zeros in Figure 3.4.*

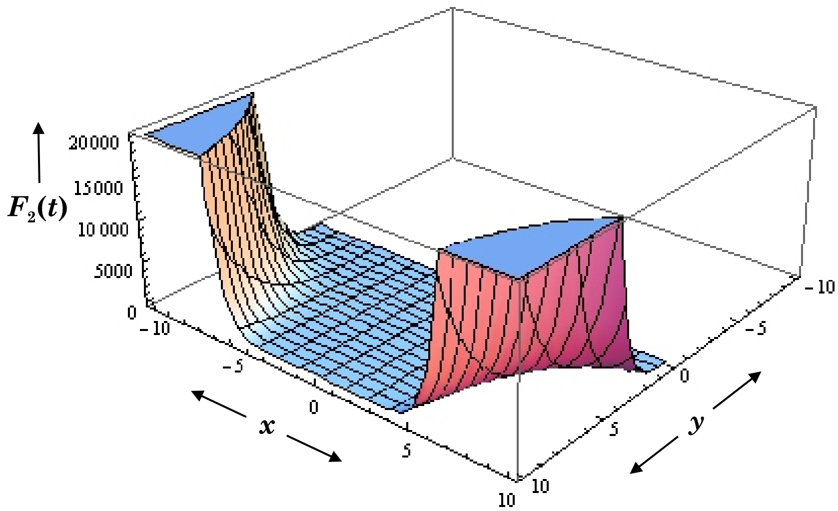

Figure 3.4

Theorem 3.6. *If*

$$F_3(t) = \Gamma(\delta)t^{\gamma_1-1}\frac{1}{2\pi i}\int_{-\infty}^{0^+} e^u u^{-\delta} E^{(\delta;\alpha_1,\beta_1)}_{\alpha_2,\beta_2;\gamma_1:\alpha_3;\gamma_2:\beta_3;\gamma_3}(x\frac{t^{\alpha_2}}{u^{\alpha_1}}, y\frac{t^{\beta_2}}{u^{\beta_1}})du, \tag{3.20}$$

where, two variable generalized Mittag-Leffler function $E^{(\delta;\alpha_1,\beta_1)}_{\alpha_2,\beta_2;\gamma_1:\alpha_3;\gamma_2:\beta_3;\gamma_3}(.,.)$ *is defined in Eqn. (2.5), then for* $\gamma_1 > 0, \gamma_2 = 2\alpha_3 - 1, \gamma_3 = 2\beta_3 - 1, \delta, x, y \in \mathbb{C}, \min\{\alpha_1, \alpha_2, \beta_1, \beta_2\} > 0, \alpha_3 > 2, \beta_3 > 2, \Re(\gamma_i) > 0, \forall i = 1, 2, 3; \Re(\delta) > 0,$ $F_3(t)$ *has real zeros.*

Proof. Use the techniques of Theorem 3.5 in Eqn. (3.20) to get

$$f_3(p; \alpha_1, \alpha_3, \beta_1, \beta_3, \gamma_1, \gamma_2, \gamma_3, \delta_1, \delta_2, x, y) = \frac{1}{p^{\gamma_1}} E_{\alpha_3,\gamma_2}(\frac{x}{p^{\gamma_1}}) E_{\beta_3,\gamma_3}(\frac{y}{p^{\gamma_1}}), \Re(p) > 0 \tag{3.21}$$

Then use the techniques of Theorem 3.3 in Eqn. (3.21), we may say that $F_3(t)$, defined in Eqn. (3.20), has real zeros. □

Theorem 3.7. *If*

$$F_2(t) = \Gamma(\delta_1)t^{\gamma_1-1}\frac{1}{2\pi i}\int_{-\infty}^{0^+} e^u u^{-\delta_1} E_2(x\frac{t^{\alpha_3}}{u^{\alpha_1}}, y\frac{t^{\beta_2}}{u^{\beta_1}})du, \tag{3.22}$$

where, two variable generalized Mittag-Leffler function

$$E_2(x,y) = E_2\left(\begin{matrix} \delta_1, \alpha_1, \beta_1; \delta_2, \alpha_2 \\ \gamma_1, \alpha_3, \beta_2; \gamma_2, \alpha_4; \gamma_3, \beta_3 \end{matrix} \middle| \begin{matrix} x \\ y \end{matrix}\right)$$

is defined in Eqn. (2.2), then for $\gamma_1, \gamma_2, \gamma_3, \delta_1, \delta_2, x, y \in \mathbb{C}, \min\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3\} > 0, F_2(t)$ *has real zeros.*

Proof. Use the techniques of the Theorem 3.5 in Eqn. (3.22) and make an appeal to the Theorem 3.2, and for $-\pi < arg(p) < \frac{\pi}{2}$, we get

$$F_2(t) = \frac{1}{2\pi i}\lim_{s\to\infty}\int_{q-is}^{q+is} e^{pt} p^{-\gamma_1} E^{\delta_2,\alpha_2}_{\alpha_4,\gamma_2}(\frac{x}{p^{\gamma_1}}) E_{\beta_3,\gamma_3}(\frac{y}{p^{\gamma_1}})dp \tag{3.23}$$

Then, use the theory of the Theorem 3.4 in Eqn. (3.23), we may prove that for $\gamma_1, \gamma_2, \gamma_3, \delta_1, \delta_2, x, y \in \mathbb{C}, \min\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3\} > 0, F_2(t)$ has real zeros. □

Theorem 3.8. *If*

$$F_3(t) = \Gamma(\delta) t^{\gamma_1 - 1} \frac{1}{2\pi i} \int_{-\infty}^{0^+} e^u u^{-\delta} E^{(\delta;\alpha_1,\beta_1)}_{\alpha_2,\beta_2;\gamma_1:\alpha_3;\gamma_2:\beta_3;\gamma_3} (x\frac{t^{\alpha_2}}{u^{\alpha_1}}, y\frac{t^{\beta_2}}{u^{\beta_1}}) du, \tag{3.24}$$

where, two variable generalized Mittag-Leffler function $E^{(\delta;\alpha_1,\beta_1)}_{\alpha_2,\beta_2;\gamma_1:\alpha_3;\gamma_2:\beta_3;\gamma_3}(.,.)$ *is defined in Eqn. (2.5), then* $x, y, \delta, \gamma_i \in \mathbb{C}, \alpha_i > 0, \beta_i > 0,\ \Re(\gamma_i) > 0, \Re(\delta) > 0, i = 1, 2, 3, F_3(t)$ *has real zeros.*

Proof. Use the techniques of the Theorem 3.5 in Eqn. (3.24) and make an appeal to the Theorem 3.2, and for $-\pi < arg(p) < \frac{\pi}{2}$, we get

$$F_3(t) = \frac{1}{2\pi i} \lim_{s\to\infty} \int_{q-is}^{q+is} e^{pt} p^{-\gamma_1} E_{\alpha_3,\gamma_2}(\frac{x}{p^{\gamma_1}}) E_{\beta_3,\gamma_3}(\frac{y}{p^{\gamma_1}}) dp \tag{3.25}$$

Then, use the theory of the Theorem 3.4 in Eqn. (3.25), we may prove that for $x, y, \delta, \gamma_i \in \mathbb{C}, \alpha_i > 0, \beta_i > 0,\ \Re(\gamma_i) > 0, \Re(\delta) > 0, i = 1, 2, 3, F_3(t)$ has real zeros. □

Corollary 3.1. *For* $\alpha_2 + \alpha_3 - \alpha_1 > 0$ *and* $\beta_2 + \beta_3 - \beta_1 > 0, x, y, \delta, \gamma_i \in \mathbb{C}, \alpha_i > 0, \beta_i > 0,\ \Re(\gamma_i) > 0, \Re(\delta) > 0, i = 1, 2, 3,$ *the integral function, defined by*

$$F_4(t) = t^{\gamma_1 - 1} \frac{1}{2\pi i} \int_{-\infty}^{0^+} e^u u^{-\delta} S^{1:1;1}_{1:1;1}\left(\begin{matrix} [\delta : \alpha_1, \beta_1] : [1 : 1]; [1 : 1]; \\ [\gamma_1 : \alpha_2, \beta_2] : [\gamma_2 : \alpha_3]; [\gamma_3 : \beta_3]; \end{matrix} \ x\frac{t^{\alpha_2}}{u^{\alpha_1}}, y\frac{t^{\beta_2}}{u^{\beta_1}} \right) du, \tag{3.26}$$

has real non zero zeros, where, in the integrand the function

$$S^{1:1;1}_{1:1;1}\left(\begin{matrix} [\delta : \alpha_1, \beta_1] : [1 : 1]; [1 : 1]; \\ [\gamma_1 : \alpha_2, \beta_2] : [\gamma_2 : \alpha_3]; [\gamma_3 : \beta_3]; \end{matrix} \ .,. \right)$$

is the Srivastava and Daoust function of two variables [23, 24] (also see Martice' [17]).

Proof. Recall the transformation formula due to Kumar [12, Eqn. (1.8)] in Eqn. (3.26), and then make an appeal to the Theorem 3.8, we may obtain that $F_4(t)$ has real zeros. □

4 Conclusion

The integral transforms techniques are seeming very applicable to study the distribution of real zeros of generalized two variables Mittag- Leffler functions, see Figures 3.1-3.4, and also these functions, particularly, have the relation with Mittag-Leffler function of one variable which is used in solving various problems occurring in fractional and differ-integral equations. The Theorems 3.4, 3.7 and 3.8 have Bessel functions which are happening in various problems in the literature and easy to computation work. Therefore, this type of work may be helpful in computation of many scientific problems. Also, on specialization of the parameters of the generalized Mittag-Leffler functions of two variables, we may find various hypergeometric functions of two variables such as Horn functions of two variables and Kampe' de Fe'riet function (see, Srivastava and Manocha [26], Srivastava and Karlsson [25]), Srivastava and Daoust function [23, 24] (also see Martice'[17]) so that we may also compute various problems of hypergeometric functions and many problems related to Mittag-Leffler functions through our above results.

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DHAGE ITERATION METHOD FOR IVPS OF NONLINEAR SECOND ORDER HYBRID FUNCTIONAL INTEGRODIFFERENTIAL EQUATIONS OF NEUTRAL TYPE

By

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Abstract

In this paper we prove an existence and approximation result for a second order initial value problems of nonlinear hybrid functional integrodifferential equations of neutral type via construction of an algorithm. The main results rely on the Dhage iteration method embodied in a recent hybrid fixed point principle of Dhage (2015) and includes the existence and approximation theorems for several functional differential equations considered earlier in the literature. An example is also furnished to illustrate the hypotheses and the abstract result of this paper.

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1 Statement of the Problem

Given the real numbers $r > 0$ and $T > 0$, consider the closed and bounded intervals $I_0 = [-r, 0]$ and $I = [0, T]$ in $\mathbb{R}$ and let $J = [-r, T]$. By $\mathcal{C} = C(I_0, \mathbb{R})$ we denote the space of continuous real-valued functions defined on I_0. We equip the space $\mathcal{C}$ with he norm $\|\cdot\|_{\mathcal{C}}$ defined by

$$\|x\|_{\mathcal{C}} = \sup_{-r\leq\theta\leq 0} |x(\theta)|. \tag{1.1}$$

Clearly, $\mathcal{C}$ is a Banach space with this supremum norm and it is called the **history space** of the functional differential equation in question.

For any continuous function $x : J \to \mathbb{R}$ and for any $t \in I$, we denote by x_t the element of the space $\mathcal{C}$ defined by

$$x_t(\theta) = x(t + \theta),\ -r \leq \theta \leq 0. \tag{1.2}$$

The differential equations involving the history of the dynamic systems are called functional differential equations and the differential equations involving the derivative of history function are called neutral functional differential equations. It has been recognized long back the importance of such problems in the theory of differential equations. Since then, several classes of nonlinear functional differential equations of neutral type have been discussed in the literature for different qualitative properties of the solutions (see Ntouyas [26] and the references therein). Recently, the study of a special class of functional differential equations involving maxima is initiated by Dhage [9], Dhage and Otrocol [22] and Dhage and Dhage [14] via a new Dhage iteration method and established the existence and approximation results along with algorithm of the solutions. Therefore, it is desirable to extend this new method to other classes of functional differential equations involving delay in the arguments. Very recently, the present author in [11] applied this new iteration method to IVPs of nonlinear first order neutral functional differential equations involving a delay. The present paper is also an attempt in this direction and extends the Dhage iteration method to ordinary second order functional differential equations of neutral type.

In this paper, we consider the nonlinear second order hybrid functional differential equations (in short HFIDE) of neutral type

$$\left.\begin{aligned} \frac{d}{dt}\big[x'(t) - f(t, x(t), x_t)\big] &= g\left(t, x_t, \int_0^t k(s, x_s)\, ds\right),\ t \in I, \\ x_0 = \phi,\ x'(0) &= \eta, \end{aligned}\right\} \tag{1.3}$$

where $\phi \in \mathcal{C}$, $k : I \times \mathcal{C} \to \mathbb{R}$, $f : I \times \mathbb{R} \times \mathcal{C} \to \mathbb{R}$ and $g \ : \ I \times \mathcal{C} \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

Definition 1.1. *A function $x \in C^1(J, \mathbb{R})$ is said to be a solution of the HFIDE (1.3) if*
(i) $x_0 \in \mathscr{C}$,
(ii) $x_t \in \mathscr{C}$ *for each* $t \in I$, *and*
(iii) *the function* $t \mapsto \big[x'(t) - f\big(t, x(t), x_t\big)\big]$ *is continuously differentiable on* I *and satisfies the equations in (1.3),*
where $C^1(J, \mathbb{R})$ is the space of continuously differentiable real-valued functions defined on J.

The neutral HFIDE (1.3) is well-known and is a linear perturbation of second type of functional differential equations. See Dhage [4, 5] and the references therein) and can be handled with the hybrid operator theoretic technique involving the sum of two operators in a Banach space (see Dhage [5] and the references therein). It has been discussed in Ntouyas *et al.* [26] with usual known method of Lerray-Schauder fixed point principle and established the existence theorem. The special cases of it are well-known and extensively discussed in the literature for different aspects of the solutions (sSee Hale [24], Dhage [11, 12] and the references therein). There is a vast literature on nonlinear functional differential equations of neutral type for different aspects of the solutions via different approaches and methods. The method of upper and lower solution or monotone method is interesting and well-known, however it requires the existence of both the lower as well as upper solutions as well as certain inequality involving monotonicity of the nonlinearity. In this paper we prove the existence and approximation theorem for the hybrid functional differential equations neutral type (1.3) via a new Dhage iteration method which does not require the existence of both upper and lower solution as well the related monotonic inequality and also obtain the algorithm for the solutions under some natural conditions.

The rest of the paper is organized as follows. Section 2 deals with the preliminary definitions and auxiliary results that will be used in subsequent sections of the paper. The main result and an illustrative example are given in Section 3.

2 Auxiliary Results

Throughout this paper, unless otherwise mentioned, let $(E, \preceq, \|\cdot\|)$ denote a partially ordered normed linear space. Two elements x and y in E are said to be **comparable** if either the relation $x \preceq y$ or $y \preceq x$ holds. A non-empty subset C of E is called a **chain** or **totally ordered** if all the elements of C are comparable. It is known that E is **regular** if $\{x_n\}$ is a nondecreasing (resp. nonincreasing) sequence in E and $x_n \to x^*$ as $n \to \infty$, then $x_n \preceq x^*$ (resp. $x_n \succeq x^*$) for all $n \in \mathbb{N}$. The conditions guaranteeing the regularity of E may be found in Guo and Lakshmikantham [23] and the references therein. Similarly a few details of a partially ordered normed linear space is given in Dhage [5] while orderings defined by different order cones are given in Deimling [1], Guo and Lakshmikantham [23], Heikkilaá and Lakshmikantham [25] and the references therein.

We need the following definitions (see Dhage [4, 5] and the references therein) in what follows.

A mapping $\mathcal{T} : E \to E$ is called **isotone** or **nondecreasing** if it preserves the order relation $\preceq$, that is, if $x \preceq y$ implies $\mathcal{T}x \preceq \mathcal{T}y$ for all $x, y \in E$. Similarly, $\mathcal{T}$ is called **nonincreasing** if $x \preceq y$ implies $\mathcal{T}x \succeq \mathcal{T}y$ for all $x, y \in E$. Finally, $\mathcal{T}$ is called **monotonic** or simply **monotone** if it is either nondecreasing or nonincreasing on E. A mapping $\mathcal{T} : E \to E$ is called **partially continuous** at a point $a \in E$ if for $\epsilon > 0$ there exists a $\delta > 0$ such that $\|\mathcal{T}x - \mathcal{T}a\| < \epsilon$ whenever x is comparable to a and $\|x - a\| < \delta$. $\mathcal{T}$ called partially continuous on E if it is partially continuous at every point of it. It is clear that if $\mathcal{T}$ is partially continuous on E, then it is continuous on every chain C contained in E and vice-versa. A non-empty subset S of the partially ordered Banach space E is called **partially bounded** if every chain C in S is bounded. An operator $\mathcal{T}$ on a partially normed linear space E into itself is called **partially bounded** if $\mathcal{T}(E)$ is a partially bounded subset of E. $\mathcal{T}$ is called **uniformly partially bounded** if all chains C in $\mathcal{T}(E)$ are bounded by a unique constant. A non-empty subset S of the partially ordered Banach space E is called **partially compact** if every chain C in S is a compact subset of E. A mapping $\mathcal{T} : E \to E$ is called **partially compact** if $\mathcal{T}(E)$ is a partially relatively compact subset of E. $\mathcal{T}$ is called **uniformly partially compact** if $\mathcal{T}$ is a uniformly partially bounded and partially compact operator on E. $\mathcal{T}$ is called **partially totally bounded** if for any bounded subset S of E, $\mathcal{T}(S)$ is a partially relatively compact subset of E. If $\mathcal{T}$ is partially continuous and partially totally bounded, then it is called **partially completely continuous** on E.

Remark 2.1. *Suppose that $\mathcal{T}$ is a nondecreasing operator on E into itself. Then $\mathcal{T}$ is a partially bounded or partially compact if $\mathcal{T}(C)$ is a bounded or relatively compact subset of E for each chain C in E.*

Definition 2.1 (Dhage [5])**.** *The order relation $\preceq$ and the metric d on a non-empty set E are said to be $\mathcal{D}$-**compatible** if $\{x_n\}$ is a monotone sequence, that is, monotone nondecreasing or monotone nonincreasing sequence in E and if a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to x^* implies that the original sequence $\{x_n\}$ converges to x^*. Similarly, given a partially ordered normed linear space $(E, \preceq, \|\cdot\|)$, the order relation $\preceq$ and the norm $\|\cdot\|$ are said to be $\mathcal{D}$-compatible if $\preceq$ and the metric d defined through the norm $\|\cdot\|$ are $\mathcal{D}$-compatible. A subset S of E is called **Janhavi** if the order relation $\preceq$ and the metric d or the norm $\|\cdot\|$ are $\mathcal{D}$-compatible in it. In particular, if $S = E$, then E is called a **Janhavi metric** or **Janhavi Banach space**.*

Definition 2.2 (Dhage [5]). *An upper semi-continuous and monotone nondecreasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a $\mathcal{D}$-**function** provided $\psi(0) = 0$. An operator $\mathcal{T} : E \to E$ is called partial **nonlinear $\mathcal{D}$-contraction** if there exists a $\mathcal{D}$-function ψ such that*

$$\|\mathcal{T}x - \mathcal{T}y\| \leq \psi\big(\|x - y\|\big) \tag{2.1}$$

for all comparable elements $x, y \in E$, where $0 < \psi(r) < r$ for $r > 0$. In particular, if $\psi(r) = k\,r$, $k > 0$, $\mathcal{T}$ is called a partial Lipschitz operator with a Lipschitz constant k and moreover, if $0 < k < 1$, $\mathcal{T}$ is called a partial linear contraction on E with a contraction constant k.

The **Dhage iteration method** embodied in the following applicable hybrid fixed point principle of Dhage [9] in a partially ordered normed linear space is used as a key tool for our work contained in this paper. The details of other hybrid fixed point theorems involving the **Dhage iteration principle** and method are given in Dhage [4, 5, 7, 13], Dhage and Dhage [16], Dhage *et.al* [17, 18], Dhage and Otrocol [22] and the references therein.

Theorem 2.1. *Let $\big(E, \preceq, \|\cdot\|\big)$ be a regular partially ordered complete normed linear and let every compact chain C of E be Janhavi. Let $\mathcal{A}, \mathcal{B} : E \to E$ be two nondecreasing operators such that*

(a) $\mathcal{A}$ is a partially bounded and partial nonlinear $\mathcal{D}$-contraction,
(b) $\mathcal{B}$ is partially continuous and partially compact, and
(c) there exists an element $\alpha_0 \in E$ such that $\alpha_0 \preceq \mathcal{A}\alpha_0 + \mathcal{B}\alpha_0$ or $\alpha_0 \succeq \mathcal{A}\alpha_0 + \mathcal{B}\alpha_0$.

Then the operator equation $\mathcal{A}x + \mathcal{B}x = x$ has a solution x^* and the sequence $\{x_n\}$ of successive iterations defined by $x_0 = \alpha_0$, $x_{n+1} = \mathcal{A}x_n + \mathcal{B}x_n$, $n = 0, 1, \ldots$; converges monotonically to x^*.

Remark 2.2. The condition that every compact chain of E is Janhavi holds if every partially compact subset of E possesses the compatibility property with respect to the order relation $\preceq$ and the norm $\|\cdot\|$ in it. This simple fact is used to prove the main existence results of this paper.

Remark 2.3. The regularity of E in above Theorem 2.1 may be replaced with a stronger continuity condition of the operator $\mathcal{A}$ and $\mathcal{B}$ on E which is a result proved in Dhage [4].

3 Main Results

In this section, we prove an existence and approximation result for the neutral HFIDE (1.3) on a closed and bounded interval $J = [a, b]$ under mixed partial Lipschitz and partial compactness type conditions on the nonlinearities involved in it. We place the neutral HFIDE (1.3) in the function space $C(J, \mathbb{R})$ of continuous real-valued functions defined on J. We define a norm $\|\cdot\|$ and the order relation $\preceq$ in $C(J, \mathbb{R})$ by

$$\|x\| = \sup_{t \in J} |x(t)| \tag{3.1}$$

and

$$x \preceq y \iff x(t) \leq y(t) \quad \text{for all } \ t \in J. \tag{3.2}$$

Clearly, $C(J, \mathbb{R})$ is a Banach space with respect to above supremum norm and also partially ordered with respect to the above partially order relation $\preceq$. It is known that the partially ordered Banach space $C(J,\mathbb{R})$ is regular and lattice so that every pair of elements of E has a lower and an upper bound in it (see Dhage [4, 5] and the references therein). The following useful lemma concerning the Janhavi subsets of $C(J,\mathbb{R})$ follows immediately from the Arzelá-Ascoli theorem for compactness (see Dhage [11] and Dhage and Dhage [14]).

Lemma 3.1 (Dhage [11] and Dhage and Dhage [14]). *Let $\big(C(J,\mathbb{R}),\preceq,\|\cdot\|\big)$ be a partially ordered Banach space with the norm $\|\cdot\|$ and the order relation $\preceq$ defined by (3.1) and (3.2) respectively. Then every partially compact subset of $C(J,\mathbb{R})$ is Janhavi.*

We introduce an order relation $\preceq_{\mathcal{C}}$ in $\mathcal{C}$ induced by the order relation $\preceq$ defined in $C(J,\mathbb{R})$. Thus, for any $x,y\in\mathcal{C}$, $x\preceq_{\mathcal{C}} y$ implies $x(\theta)\preceq y(\theta)$ for all $\theta\in I_0$. Moreover, if $x,y\in C(J,\mathbb{R})$ and $x\preceq y$, then $x_t\preceq_{\mathcal{C}} y_t$ for all $t\in I$.

We need the following definition in what follows.

Definition 3.1. *A function $u\in C^1(J,\mathbb{R})$ is said to be a solution of the HFIDE (1.3) if*

(i) $u_t\in\mathcal{C}$ for each $t\in I$, and
(ii) the function $t\mapsto[u'(t)-f(t,u_t)]$ is continuously differentiable on I and satisfies

$$\left.\begin{aligned}\frac{d}{dt}\big[u'(t)-f(t,u(t),u_t)\big]\leq g\left(t,u_t,\int_0^t k(s,u_s)\,ds\right),\ t\in I,\\ u_0\preceq_{\mathcal{C}}\ \phi,,\ u'(0)\leq\eta.\end{aligned}\right\}\tag{$*$}$$

Similarly, a continuously differentiable function $v\in C(J,\mathbb{R})$ is called an upper solution of the neutral HFIDE (1.3) if the above inequality is satisfied with reverse sign.

We consider the following set of assumptions in what follows:

(H_1) There exists a constant $M_f>0$ such that $|f(t,x,y)|\leq M_f$ for all $t\in I$, $x\in\mathbb{R}$ and $y\in\mathcal{C}$.
(H_2) There exists $\mathcal{D}$-function $\varphi\ :\ \mathbb{R}_+\to\mathbb{R}_+$ such that
$$0\leq f(t,x_1,x_2)-f(t,y_1,y_2)\leq\varphi(\max\{x_1-y_2\,,\,\|x-y\|_{\mathcal{C}}\})$$
for all $t\in I$ and $x_1,y_1\in\mathbb{R}$, $x_2,y_2\in\mathcal{C}$, $x_1\geq y_1$, $x_2\succeq_{\mathcal{C}} y_2$. Moreover, $T\,\varphi(r)<r$, $r>0$.
(H_3) The function g is bounded on $I\times\mathcal{C}\times\mathbb{R}$ with bound M_g.
(H_4) The function $k(t,x)$ is nondecreasing in x for each $t\in I$.
(H_5) The function $g(t,x,y)$ is nondecreasing in x and y for each $t\in I$.
(H_6) The neutral HFIDE (1.3) has a lower solution $u\in C(J,\mathbb{R})$.

Lemma 3.2. *A function $x\in C(J,\mathbb{R})$ is a solution of the neutral HFIDE (1.3) if and only if it is a solution of the nonlinear functional integral equation*

$$x(t)=\begin{cases}\phi(0)+\big[\eta-f(0,\phi(0),\phi)\big]t+\displaystyle\int_0^t f(s,x(s),x_s)\,ds\\ \qquad+\displaystyle\int_0^t(t-s)\,g\left(s,x_s,\int_0^s k(\tau,x_\tau)\,d\tau\right)\,ds,\quad\text{if}\ \ t\in I,\\ \phi(t),\quad\text{if}\ \ t\in I_0.\end{cases}\tag{3.3}$$

Theorem 3.1. *Suppose that hypotheses (H_1) through (H_6) hold. Then the neutral HFIDE (1.3) has a solution x^* defined on J and the sequence $\{x_n\}$ of successive approximations defined by*

$$x_0 = u,$$

$$x_{n+1}(t) = \begin{cases} \phi(0) + \big[\eta - f(0,\phi(0),\phi)\big]t + \displaystyle\int_0^t f(s, x_n(s), x_s^n)\,ds \\ \qquad + \displaystyle\int_0^t (t-s)\,g\left(s, x_s^n, \int_0^s k(\tau, x_\tau^n)\,d\tau\right) ds, \quad \textit{if } \ t \in I, \\ \phi(t), \quad \textit{if } \ t \in I_0, \end{cases} \tag{3.4}$$

where $x_s^n(\theta) = x_n(s+\theta)$, $\theta \in I_0$, converges monotonically to x^*.

Proof. Set $E = C(J,\mathbb{R})$. Then, in view of Lemma 3.1, every compact chain C in E possesses the compatibility property with respect to the norm $\|\cdot\|$ and the order relation $\preceq$ so that every compact chain C is Janhavi in E.

Define two operators $\mathcal{A}$ and $\mathcal{B}$ on E by

$$\mathcal{A}x(t) = \begin{cases} \big[\eta - f(0,\phi(0),\phi)\big]t + \displaystyle\int_0^t f(s, x(s), x_s)\,ds, \quad \text{if } \ t \in I, \\ 0, \quad \text{if } \ t \in I_0, \end{cases} \tag{3.5}$$

and

$$\mathcal{B}x(t) = \begin{cases} \phi(0) + \displaystyle\int_0^t (t-s) g\left(s, x_s \int_0^s k(\tau, x_\tau)\,d\tau\right) ds, \quad \text{if } \ t \in I, \\ \phi(t), \quad \text{if } \ t \in I_0. \end{cases} \tag{3.6}$$

From the continuity of the functions f, g and the integral, it follows that $\mathcal{A}$ and $\mathcal{B}$ define the operators $\mathcal{A}, \mathcal{B} : E \to E$. Applying Lemma 3.2, the neutral HFIDE (1.3) is equivalent to the operator equation

$$\mathcal{A}x(t) + \mathcal{B}x(t) = x(t), \ t \in J. \tag{3.7}$$

Now, we show that the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of Theorem 2.1 in a series of following steps.

Step I: *$\mathcal{A}$ and $\mathcal{B}$ are nondecreasing on E.*

let $x, y \in E$ be such that $x \succeq y$. Then $x_t \succeq_{\mathcal{C}} y_t$ for all $t \in I$ and by hypothesis (H_2), we get

$$\begin{aligned} \mathcal{A}x(t) &= \begin{cases} \big[\eta - f(0,\phi(0),\phi)\big]t + \displaystyle\int_0^t f(s, x(s), x_s)\,ds, \quad \text{if } \ t \in I, \\ 0, \quad \text{if } \ t \in I_0, \end{cases} \\ &\geq \begin{cases} \big[\eta - f(0,\phi(0),\phi)\big]t + \displaystyle\int_0^t f(s, y(s), y_s)\,ds, \quad \text{if } \ t \in I, \\ 0, \quad \text{if } \ t \in I_0, \end{cases} \\ &= \mathcal{A}y(t), \end{aligned}$$

for all $t \in J$. This shows that the operator $\mathcal{A}$ is also nondecreasing on E.

Similarly, by hypothesis (H_4), we get

$$\begin{aligned}
\mathcal{B}x(t) &= \begin{cases} \phi(0) + \displaystyle\int_0^t (t-s) g\left(s, x_s \int_0^s k(\tau, x_\tau)\,d\tau\right) ds, & \text{if } \ t \in I, \\ \phi(t), \quad \text{if } \ t \in I_0, \end{cases} \\
&\geq \begin{cases} \phi(0) + \displaystyle\int_0^t (t-s) g\left(s, y_s \int_0^s k(\tau, y_\tau)\,d\tau\right) ds, & \text{if } \ t \in I, \\ \phi(t), \quad \text{if } \ t \in I_0, \end{cases} \\
&= \mathcal{B}y(t),
\end{aligned}$$

for all $t \in J$. This shows that the operator $\mathcal{B}$ is also nondecreasing on E.

Step II: *$\mathcal{A}$ is a nonlinear partial $\mathcal{D}$-contraction on E.*

Let $x, y \in E$ be any two elements such that $x \succeq y$. Then, by hypothesis (H_2),

$$\begin{aligned}
|\mathcal{A}x(t) - \mathcal{A}y(t)| &\leq \int_0^t |f(s, x(s), x_s) - f(s, y(s), y_s)|\,ds \\
&\leq \int_0^t \varphi\big(\max\{x(s) - y(s)\,,\, \|x_s - y_s\|_{\mathcal{C}}\}\big)\,ds \\
&\leq \int_0^t \varphi\big(\max\{|x(s) - y(s)|\,,\, \|x_s - y_s\|_{\mathcal{C}}\}\big)\,ds \\
&\leq T\,\varphi(\|x - y\|) \qquad (3.8)
\end{aligned}$$

for all $t \in J$. Taking the supremum over t, we obtain

$$\|\mathcal{A}x - \mathcal{A}y\| \leq \psi(\|x - y\|)$$

for all $x, y \in E$, $x \succeq y$, where $\psi(r) = T\,\varphi(r) < r$ for $r > 0$. As a result $\mathcal{A}$ is a nonlinear partial $\mathcal{D}$-contraction on E in view of Remark 2.2.

Step III: *$\mathcal{B}$ is partially continuous on E.*

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in a chain C such that $x_n \to x$ as $n \to \infty$. Then $x_s^n \to x_s$ as $n \to \infty$. Since the g is continuous and hypothesis (H_3) holds, by dominated convergence theorem, we obtain

$$\begin{aligned}
\lim_{n\to\infty} \mathcal{B}x_n(t) &= \begin{cases} \phi(0) + \displaystyle\int_0^t (t-s) \left[\lim_{n\to\infty} g\left(s, x_s^n, \int_0^s k(\tau, x_\tau^n)\,d\tau\right)\right] ds, & \text{if } \ t \in I, \\ \phi(t), \quad \text{if } \ t \in I_0, \end{cases} \\
&= \begin{cases} \phi(0) + \displaystyle\int_0^t (t-s) g\left(s, x_s, \int_0^s k(\tau, x_\tau)\,d\tau\right) ds, & \text{if } \ t \in I, \\ \phi(t), \quad \text{if } \ t \in I_0, \end{cases} \\
&= \mathcal{B}x(t)
\end{aligned}$$

for all $t \in J$. This shows that $\mathcal{B}x_n$ converges to $\mathcal{B}x$ pointwise on J.

Now we show that $\{\mathcal{B}x_n\}_{n\in\mathbb{N}}$ is an equicontinuous sequence of functions in E. Now there are three cases:

Case I: Let $t_1, t_2 \in J$ with $t_1 > t_2 \geq 0$. Then we have

$$\begin{aligned}
&\big|\mathcal{B}x_n(t_2) - \mathcal{B}x_n(t_1)\big| \\
&\leq \Big| \int_0^{t_2} (t_2 - s) g\left(s, x_s \int_0^s k(\tau, x_\tau)\, d\tau\right) ds - \int_0^{t_1} (t_1 - s) g\left(s, x_s \int_0^s k(\tau, x_\tau)\, d\tau\right) ds \Big| \\
&\leq \Big| \int_0^{t_2} (t_2 - s) g\left(s, x_s \int_0^s k(\tau, x_\tau)\, d\tau\right) ds - \int_0^{t_1} (t_2 - s) g\left(s, x_s \int_0^s k(\tau, x_\tau)\, d\tau\right) ds \Big| \\
&\quad + \Big| \int_0^{t_1} (t_2 - s) g\left(s, x_s \int_0^s k(\tau, x_\tau)\, d\tau\right) ds - \int_0^{t_1} (t_1 - s) g\left(s, x_s \int_0^s k(\tau, x_\tau)\, d\tau\right) ds \Big| \\
&\leq \left| \int_{t_1}^{t_2} T \left| g\left(s, x_s \int_0^s k(\tau, x_\tau)\, d\tau\right) \right| ds \right| + \int_0^T |t_1 - t_2| \left| g\left(s, x_s \int_0^s k(\tau, x_\tau)\, d\tau\right) \right| ds \\
&\leq 2M_g T\, |t_2 - t_1| \\
&\to 0 \quad \text{as} \quad t_2 \to t_1,
\end{aligned}$$

uniformly for all $n \in \mathbb{N}$.

Case II: Let $t_1, t_2 \in J$ with $t_1 < t_2 \leq 0$. Then we have

$$|\mathcal{B}x_n(t_2) - \mathcal{B}x_n(t_1)| = |\phi(t_2) - \phi(t_1)| \to 0 \quad \text{as} \quad t_2 \to t_1,$$

uniformly for all $n \in \mathbb{N}$.

Case III: Let $t_1, t_2 \in J$ with $t_1 < 0 < t_2$. Then we have

$$|\mathcal{B}x_n(t_2) - \mathcal{B}x_n(t_1)| \leq |\mathcal{B}x_n(t_2) - \mathcal{B}x_n(0)| + |\mathcal{B}x_n(0) - \mathcal{B}x_n(t_1)| \to 0 \ \text{ as } \ t_2 \to t_1.$$

uniformly for all $n \in \mathbb{N}$.

Thus in all three cases, we obtain

$$|\mathcal{B}x_n(t_2) - \mathcal{B}x_n(t_1)| \to 0 \quad \text{as} \quad t_2 \to t_1,$$

uniformly for all $n \in \mathbb{N}$. This shows that the convergence $\mathcal{B}x_n \to \mathcal{B}x$ is uniform and that $\mathcal{B}$ is a partially continuous operator on E into itself.

Step IV: *$\mathcal{B}$ is partially compact operator on E.*

Let C be an arbitrary chain in E. We show that $\mathcal{B}(C)$ is uniformly bounded and equicontinuous set in E. First we show that $\mathcal{B}(C)$ is uniformly bounded. Let $y \in \mathcal{B}(C)$ be any element. Then there is an element $x \in C$ such that $y = \mathcal{T}x$. By hypothesis (H_2),

$$\begin{aligned}
|y(t)| &= |\mathcal{B}x(t)| \\
&\leq \begin{cases} |\phi(0)| + \displaystyle\int_0^t |t - s| \left| g\left(s, x_s, \int_0^s k(\tau, x_\tau)\, d\tau\right) \right| ds, & \text{if } \ t \in I, \\ |\phi(t)|, \quad \text{if } \ t \in I_0, \end{cases} \\
&\leq \|\phi\| + M_f T^2 \\
&= r,
\end{aligned}$$

for all $t \in J$. Taking the supremum over t we obtain $\|y\| \leq \|\mathcal{B}x\| \leq r$ for all $y \in \mathcal{B}(C)$. Hence $\mathcal{B}(C)$ is a uniformly bounded subset of E. Next we show that $\mathcal{B}(C)$ is an equicontinuous set in E. Let $t_1, t_2 \in J$, with $t_1 < t_2$. Then proceeding with the arguments that given in Step II it can be shown that

$$\big|y(t_2) - y(t_1)\big| = |\mathcal{B}x(t_2) - \mathcal{B}x(t_1)| \to 0 \quad \text{as} \quad t_1 \to t_2$$

uniformly for all $y \in \mathcal{B}(C)$. This shows that $\mathcal{B}(C)$ is an equicontinuous subset of E. Now, $\mathcal{B}(C)$ is a uniformly bounded and equicontinuous subset of functions in E and hence it is compact in view of Arzelá-Ascoli theorem. Consequently $\mathcal{B} : E \to E$ is a partially compact operator on E into itself.

Step IV: *u satisfies the operator inequality $u \preceq \mathcal{A}u + \mathcal{B}u$.*

By hypothesis (H_4), the neutral HFIDE (1.3) has a lower solution u defined on J. Then we have

$$\left.\begin{aligned} \frac{d}{dt}\big[u'(t) - f(t, u(t), u_t)\big] \le g\left(t, u_t, \int_0^t k(s, x_s)\, ds\right), \ \ t \in I,\\ u_0 \preceq_{\mathcal{C}} \phi, \ \ u'(0) \le \eta. \end{aligned}\right\}$$

Integrating the above inequality from 0 to t, we get

$$\begin{aligned} u(t) &\le \begin{cases} \phi(0) + \big[\eta - f(0, \phi(0), \phi)\big]t \\ \quad + \displaystyle\int_0^t f(s, u(s), u_s)\, ds + \int_0^t (t-s) g\left(s, u_s, \int_0^s k(\tau, x_\tau)\, d\tau\right) ds, & \text{if} \ \ t \in I, \\ \phi(t), \quad \text{if} \ \ t \in I_0, \end{cases} \\ &= \mathcal{A}u(t) + \mathcal{B}u(t) \end{aligned}$$

for all $t \in J$. As a result we have that $u \preceq \mathcal{A}u + \mathcal{B}u$.

Thus, $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of Theorem 2.1 and so the operator equation $\mathcal{A}x + \mathcal{B}x = x$ has a solution. Consequently the integral equation (3.3), and a fortiori the hybrid functional differential equation (1.3) has a solution x^* defined on J. Furthermore, the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by (3.5) converges monotonically to x^*. This completes the proof. □

Remark 3.1. The conclusion of Theorem 3.1 also remains true if we replace the hypothesis (H_4) and (H_7) with the following ones:

(H_4') The neutral HFIDE (1.3) has an upper solution $v \in C(J, \mathbb{R})$.

The proof of Theorem 3.1 under this new hypothesis is similar and can be obtained by closely observing the same arguments with appropriate modifications.

Example 3.1. Given the closed and bounded intervals $I_0 = [-1, 0]$ and $I = [0, 1]$, consider the neutral HFIDE of neutral type

$$\left.\begin{aligned} \frac{d}{dt}\big[x'(t) - f_1(t, x(t), x_t)\big] = g_1\left(t, x_t, \int_0^t k_1(s, x_s)\, ds\right), \ \ t \in I,\\ x_0 = \phi, \ \ x'(0) = \eta, \end{aligned}\right\} \tag{3.9}$$

where $\phi \in \mathcal{C}$ and $k_1 : I \times \mathcal{C} \to \mathbb{R}$, $f_1 : I \times \mathbb{R} \times \mathcal{C} \to \mathbb{R}$, $g_1 : I \times \mathcal{C} \times \mathbb{R} \to \mathbb{R}$ are continuous functions given by

$$\phi(t) = \sin t, \ \ t \in [-1, 0],$$

$$k_1(t, x) = \begin{cases} 1, & \text{if} \ \ x \preceq_{\mathcal{C}} 0, \\ \ln(1 + \|x\|_{\mathcal{C}}) + 1, & \text{if} \ \ 0 \preceq_{\mathcal{C}} x, \ x \neq 0, \end{cases}$$

$$
f_1(t,x,y) = \begin{cases} \dfrac{1}{2}\left[\dfrac{x}{x+1} + \dfrac{\|y\|_{\mathcal{C}}}{1+\|y\|_{\mathcal{C}}}\right] + 1, & \text{if } \ 0 \le x,\ 0 \preceq_{\mathcal{C}} y,\, x \ne 0,\, y \ne 0, \\ \dfrac{1}{2}\dfrac{x}{x+1} + 1, & \text{if } \ 0 \le x,\, x \ne 0,\, y \preceq_{\mathcal{C}} 0, \\ \dfrac{1}{2}\dfrac{\|y\|_{\mathcal{C}}}{1+\|y\|_{\mathcal{C}}} + 1, & \text{if } \ x \le 0,\, 0 \preceq_{\mathcal{C}} y,\, y \ne 0, \\ 1, & \text{if } \ x \le 0,\, y \preceq_{\mathcal{C}} 0, \end{cases}
$$

and

$$
g_1(t,x,y) = \begin{cases} \tanh(\|x\|_{\mathcal{C}}) + \tanh y + 1, & \text{if } \ x \succeq_{\mathcal{C}} 0,\, x \ne 0,\, -\infty < y < \infty, \\ \tanh y + 1, & \text{if } \ x \preceq_{\mathcal{C}} 0,\, -\infty < y < \infty, \end{cases}
$$

for all $t \in I$.

Clearly, f_1 is continuous and bounded on $I \times \mathbb{R} \times \mathcal{C}$ with bound $M_{f_1} = 2$ and so the hypothesis (H_1) is satisfied. We show that f_1 satisfies the hypothesis (H_2). Let $x_1, y_2 \in \mathbb{R}$ and $x_2, y_2 \in \mathcal{C}$ be such that $x_1 \ge y_1$ and $x_2 \succeq_{\mathcal{C}} y_2 \succeq_{\mathcal{C}} 0$. Then $\|x\|_{\mathcal{C}} \ge \|y\|_{\mathcal{C}} > 0$ and therefore, we have

$$
\begin{aligned}
0 &\le f_1(t,x_1,x_2) - f_1(t,y_1,y_2) \\
&= \frac{1}{2}\left|\frac{x_1}{1+x_1} - \frac{y_1}{1+y_1}\right| + \frac{1}{2}\left|\frac{\|x_2\|_{\mathcal{C}}}{1+\|x_2\|_{\mathcal{C}}} - \frac{\|y_2\|_{\mathcal{C}}}{1+\|y_2\|_{\mathcal{C}}}\right| \\
&\le \frac{1}{2}\left[\frac{|x_1 - y_1|}{1+|x_1-y_1|} + \frac{\|x_2 - y_2\|_{\mathcal{C}}}{1+\|x_2-y_2\|_{\mathcal{C}}}\right] \\
&\le \frac{\max\{|x_1-y_1|\,,\, \|x_2-y_2\|_{\mathcal{C}}\}}{1+\max\{|x_1-y_1|\,,\, \|x_2-y_2\|_{\mathcal{C}}\}} \\
&= \psi\Big(\max\{x_1 - y_1\,,\, \|x_2-y_2\|_{\mathcal{C}}\}\Big)
\end{aligned}
$$

for all $t \in I$, where $\psi(r) = \dfrac{r}{1+r} < r$, $r > 0$. Similarly, other cases of the variables x, y of $f(t,x,y)$ are treated and we obtain the final estimate as

$$
0 \le f_1(t,x_1,x_2) - f_1(t,y_1,y_2) \le \psi\Big(\max\{x_1 - y_1|\,,\, \|x_2-y_2\|_{\mathcal{C}}\}\Big)
$$

for all comparable $x_1, y_2 \in \mathbb{R}$ and $x_2, y_2 \in \mathcal{C}$. Thus, the function f_1 satisfies the hypothesis (H_2).

Next, $k_1(t,x)$ is is nondcreasing in x for each $t \in [0,1]$ and so the hypothesis (H_3) is satisfied.. Again, the function g is bounded on $I \times \mathcal{C} \times \mathbb{R}$ with $M_{f_1} = 3$ and so the hypothesis (H_4) is satisfied. Again, it is easy to verify that the function $g_1(t,x,y)$ is nondecreasing in x as well as in y for each $t \in [0,1]$ and so the hypothesis (H_5) is satisfied. Finally, the function $u \in C(J,\mathbb{R})$ defined by

$$
u(t) = \begin{cases} \dfrac{3}{4}t, & \text{if } \ t \in [0,1], \\ \sin t, & \text{if } \ t \in [-1,0], \end{cases}
$$

is a lower solution of the neutral HFIDE (3.9) defined on J. Thus, the functions k_1, f_1 and g_1 satisfy all the hypotheses (H_1) through (H_6). Hence we apply Theorem 3.1 and conclude that the neutral HFIDE (3.9) has a solution x^* on J and the sequence $\{x_n\}$ of successive approximation defined by

$$x_0(t) = \begin{cases} \dfrac{3}{4}\,t, & \text{if } \ t \in [0,1], \\ \sin t, & \text{if } \ t \in [-1,0] \end{cases}$$

$$x_{n+1}(t) = \begin{cases} -\dfrac{1}{4}\,t + \displaystyle\int_0^t f_1(s, x_n(s), x_s^n)\,ds \\ \quad + \displaystyle\int_0^t (t-s)\,g_1\left(s, x_s^n, \int_0^s k_1(\tau, x_\tau^n)\,d\tau\right)ds, & \text{if } \ t \in [0,1], \\ \sin t, & \text{if } \ t \in [-1,0], \end{cases}$$

for $n = 0, 1, \ldots,$ converges monotonically to x^*.

Remark 3.2. The conclusion given in Example 3.1 also remains true if we replace the lower solution u with the upper solution v of the neutral HFIDE (3.9) defined by

$$v(t) = \begin{cases} t\,(3t+2)\,, & \text{if } \ t \in [0,1], \\ \sin t, & \text{if } \ t \in \left[-\frac{\pi}{2}, 0\right]. \end{cases}$$

Remark 3.3. We note that if the neutral HFIDE (1.3) has a lower solution u as well as an upper solution v such that $u \preceq v$, then under the given conditions of Theorem 3.1 it has corresponding solutions x_* and x^* and these solutions satisfy the inequality

$$u = x_0 \preceq x_1 \preceq \cdots \preceq x_n \preceq x_* \preceq y^* \preceq y_n \preceq \cdots \preceq y_1 \preceq y_0 = v.$$

Hence they are the minimal and maximal solutions of the neutral HFIDE (1.3) respectively in the vector segment $[u, v]$ of the Banach space $E = C(J, \mathbb{R})$, where the vector segment $[u, v]$ is a set in $C(J, \mathbb{R})$ defined by $[u, v] = \{x \in C(J, \mathbb{R}) \mid u \preceq x \preceq v\}$. This is because the order relation $\preceq$ defined by (3.2) is equivalent to the order relation defined by the order cone $\mathcal{K} = \{x \in C(J, \mathbb{R}) \mid x \succeq \theta\}$ which is a closed set in the Banach space $C(J, \mathbb{R})$. The existence of extremal solutions for the neutral HFIDE (1.3) may be obtained in the vector segment $[u, v]$ via generalized iteration method given Heikkilá and Lakshmikatham [25], but in that case we do not get the algorithm for the extremal solutions. Therefore our Dhage iteration method has some advantages over the iteration method presented in Heikkilá and Lakshmikatham [25].

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ON THE FISHER DISTRIBUTION OF PALEOMAGNETIC DIRECTIONS ON A SPHERE

By

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Abstract

The Fisher distribution is one of the most important distributions in statistics to deal with the paleomagnetic data; (cf. Fisher, R. A. (1953). Dispersion on a sphere. *Proc. R. Soc. Lond. A*, **217** (1130), 295-305). In this paper, we have considered several distributional properties of the Fisher distribution. Based on these distributional properties, we have established some new characterizations of the Fisher distribution by truncated first moment, order statistics and upper record values.

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1 Introduction

Paleomagnetism is the study of the record of the Earth's magnetic field in rocks, sediment, or archeological materials; (cf. `https://en.wikipedia.org/wiki/Paleomagnetism`). "The conceptual model for Earth's magnetic field is that of a dipole (i.e. bar magnet) positioned at Earth's center and aligned with the rotational axis of the Earth (see Figure 1). This allows us to predict the direction of the magnetic field at any location on Earth's surface using the fundamental equations of a dipole field. This equation gives a direct relation between magnetic inclination and geographic latitude at the point of observation"; (cf. `http://www.geo.mtu.edu/KeweenawGeoheritage/IRKeweenawRift/Paleomagnetism.html`).

The British statistician Sir R. A. Fisher introduced a probability density function to study certain statistical problems in paleomagnetism, which, in literature, is now known as the Fisher distribution, see, for example, Fisher [11], Mardia and Jupp [22], among others. "However, as pointed out by Mardia and Jupp [22], Fisher distributions first appeared in a paper on statistical mechanics by Langevin [21]. Further, according to Mardia and Jupp [22], the maximum likelihood estimation in the Fisher distributions and a corresponding characterization were considered by Arnold [2]. Also, Kuhn and Grün [20] found Fisher distributions as approximate solution to a problem associated with paths and chains of random segments in three dimensions. Significant advances in statistical applications were made by Fisher [11], who used these distributions to investigate certain statistical problems in

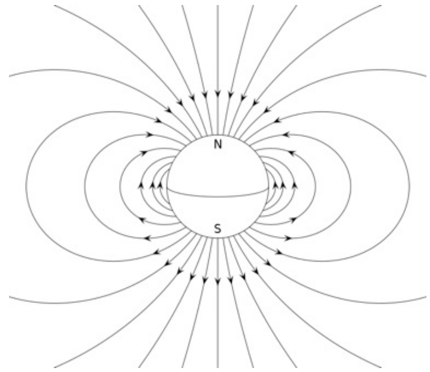

Figure 1.1: Earth's magnetic field (Source:http://www.geo.mtu.edu/KeweenawGeoheritage/IRKeweenawRift/Paleomagnetism.html)

paleomagnetism. Watson and Williams [26] studied the extension of the Fisher distribution to higher dimension." For further reading, the interested readers are also referred to some relevant literature, such as, Jammalamadaka and Sengupta [16], Jeffreys [17] and Jones and Pewsey [18], among others.

We say that a random variable X has the standard Fisher distribution, denoted as $X \sim FI(\kappa)$ (here we use FI in honor of Fisher) if its probability density function (*pdf*) is given by

$$f(\theta) = \frac{\kappa e^{\kappa \cos(\theta)} \sin(\theta)}{2 \sinh(\kappa)}, \quad 0 \leq \theta \leq \pi, \kappa \geq 0 \tag{1.1}$$

which defines the probability of finding a direction, where θ is the realization of X and denotes the angle from true mean direction (and is $= 0$ at true mean) and κ is the precision parameter and measures the concentration of the distribution about the true mean direction and is $= 0$ for a distribution of directions that is uniform over the sphere and approaches ∞ for directions concentrated at a point. Also, note that the distribution of directions is azimuthally symmetric about the true mean. It is easy to see that that the *pdf* in (1.1) integrates to 1. To describe the shapes of the Fisher distribution, $X \sim FI(\kappa)$, the plots of the *pdf* (1.1) for some values of the parameter κ are provided below in Figures 1.2 (a) and (b).

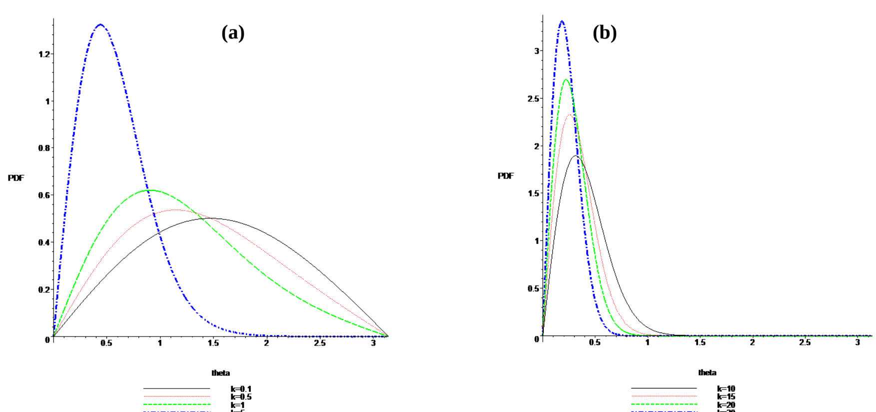

Figure 1.2: (a and b): Plots of the *pdf* of the standard Fisher distribution, $X \sim FI(\kappa)$.

The effects of the parameter can easily be seen from the Figures 1.2 (a and b). Similar plots can be drawn for others values of the parameters. It is obvious from these figures that, as κ increases, the probability mass becomes more concentrated about $\theta = 0$.

The organization of this paper is given as follows: Some distributional properties of the Fisher distribution are given in Section 2. Section 3 discusses the characterizations of the Fisher distribution. Finally, some concluding remarks are given in section 4.

2 Distributional Properties

In this section, we will discuss some of the distributional properties of the Fisher distribution, $X \sim FI(\kappa)$.

2.1 Cumulative Distribution Function

The cumulative distribution function (*cdf*) of $X \sim FI(\kappa)$ is given by

$$F(\theta) = \frac{e^{\kappa} - e^{\kappa \cos(\theta)}}{2 \sinh(\kappa)}, \quad 0 \leq \theta \leq \pi, \kappa \geq 0. \tag{2.1}$$

Since

$$e^{\kappa \cos\theta} = (I_0(\kappa) + 2\sum_{j=1}^{\infty} I_j(\kappa)\cos(j\theta)),$$

where $I_0(\kappa)$ is the modified Bessel function of the first kind and order 0 and $I_j(\kappa)$ is the modified Bessel function of the first kind and order j given by

$$I_j(\kappa) = (\frac{\kappa}{2})^j \sum\nolimits_{m=0}^{\infty} (\frac{\kappa}{2})^{2m} \frac{1}{(m!)\Gamma(j+m+1)},$$

where $\Gamma(.)$ denotes the gamma function, see, for example, Gradshteyn and Ryzhik [15], we can also write the *pdf* (1.2), that is, $f(\theta)$ and the *cdf* (2.1), that is, $F(\theta)$, respectively as follows:

$$f(\theta) = \frac{\kappa \sin(\theta)}{2\sinh(\kappa)}(I_0(\kappa) + 2\sum_{j=1}^{\infty} I_j(\kappa)\cos(j\theta)), \quad 0 \leq \theta \leq \pi, \kappa \geq 0, \tag{2.2}$$

and

$$F(\theta) = \frac{e^{\kappa} - (I_0(\kappa) + 2\sum_{j=1}^{\infty} I_j(\kappa)\cos(j\theta))}{2\sinh(\kappa)}, \quad 0 \leq \theta \leq \pi, \kappa \geq 0. \tag{2.3}$$

The plots of the *cdf* (2.1) for some values of the parameter κ are provided below in Figures 2.1 (a and b).

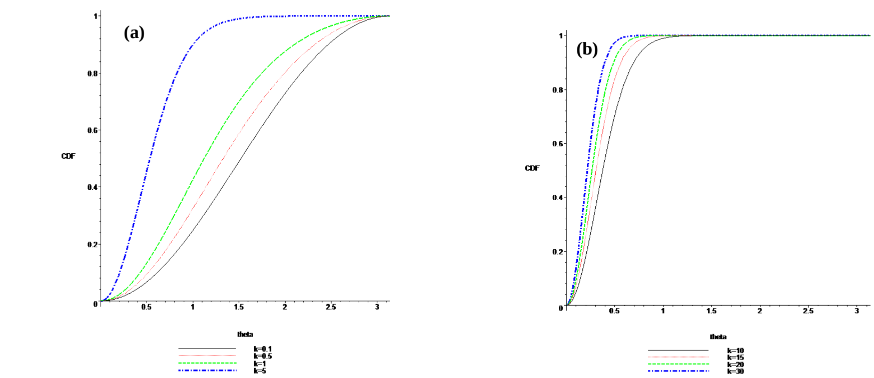

Figure 2.1: (a and b): Plots of the *cdf* standard Fisher distribution, $X \sim FI(\kappa)$.

2.2 Hazard Rate Function

Using the equations (1.1) and (2.1), the hazard rate (or failure rate) $h(\theta)$ of Fisher distribution, $X \sim FI(\kappa)$ is given by

$$h(\theta) = \frac{f(\theta)}{1 - F(\theta)} = \frac{\kappa e^{\kappa \cos(\theta)} \sin(\theta)}{2\sinh(\kappa) - [e^{\kappa} - e^{\kappa \cos(\theta)}]}, \quad 0 \leq \theta \leq \pi, \kappa \geq 0. \tag{2.4}$$

From (2.4), it is easily seen that its hazard rate, $h(\theta)$, has the following property: $h'(\theta) \geq 0$, that is, $f'(\theta)[1 - F(\theta)] + [f(\theta)]^2 \geq 0$, for all θ in $0 \leq \theta \leq \pi$ and for all $, \kappa \geq 0$, where $f'(\theta) = \frac{1}{2} \frac{\kappa e^{\kappa \cos(\theta)}}{\sinh(\kappa)} [\cos(\theta) - \kappa \sin(\theta)]$ and $1 - F(\theta) = 1 - (\frac{e^{\kappa} - e^{\kappa \cos(\theta)}}{2\sinh(\kappa)})$. Thus, $X \sim FI(\kappa)$ has an increasing failure rate (IFR). To describe the shapes of the hazard rate (or failure rate) of the Fisher distribution, $X \sim FI(\kappa)$, the plots of the hazard rate(2.4) for some values of the parameter κ are provided below in Figures 2.2 (a and b). The effects of the parameter can easily be seen from these graphs. It is obvious from the Figure 2.2 (a) that the hazard rate, $h(\theta)$ of $X \sim FI(\kappa)$, is concave up, that is, bathtub shaped, when $0 < \kappa < 1$. Also, we observe from the Figure 2.2 (b) that the hazard rate, $h(\theta)$ of $X \sim FI(\kappa)$, is concave down, that is, upside down bathtub shaped, when $\kappa \geq 10$. Similar plots can be drawn for others values of the parameters.

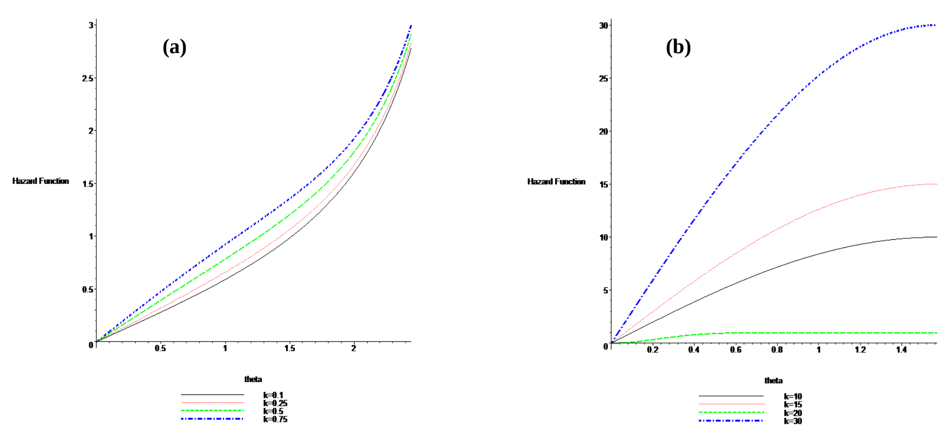

Figure 2.2: (a and b): Plots of the Hazard Function standard Fisher distribution, $X \sim FI(\kappa)$

2.3 Moment

Using the *pdf* in (2.2), we have the following expression for the first moment of the Fisher distribution, $X \sim FI(\kappa)$:

$$E(X) = \int_0^{\pi} \theta[\frac{\kappa \sin(\theta)}{2\sinh(\kappa)} + \frac{1}{I_0(\kappa)}\frac{\kappa}{\sinh(\kappa)}\sum_{j=1}^{\infty} I_j(\kappa)\sin(\theta)\cos(j\theta)]d\theta,$$

which, after integration and simplification, easily gives

$$E(X) = \frac{\pi\kappa}{2\sinh(\kappa)} + \frac{1}{I_0(\kappa)}\frac{\kappa}{\sinh(\kappa)}\sum_{j=1}^{\infty} I_j(\kappa)\frac{2j\sin(\pi j) - \pi(j^2-1)\cos(\pi j)}{(j^2-1)^2},$$

where $I_0(\kappa)$ is the modified Bessel function of the first kind and order 0 and $I_j(\kappa)$ is the modified Bessel function of the first kind and order j.

2.4 Shannon Entropy

An entropy provides an excellent tool to quantify the amount of information (or uncertainty) contained in a random observation regarding its parent distribution (population). A large value of entropy implies the greater uncertainty in the data. As proposed by Shannon [25], if X is a continuous random variable with *pdf* $f_X(x)$, defined over an interval Ω then Shannon's entropy of X, denoted by $H(f)$, is defined as

$$H(f) = E[-\ln\{f_X(x)\}] = -\int_{\Omega} f_X(x)\ln\{f_X(x)\}dx. \tag{2.5}$$

Now, using the *pdf* (1.1) of the Fisher distribution, $X \sim FI(\kappa)$, in Eq. (2.5) and integrating with respect to θ and simplifying, we obtain an explicit expression of Shannon entropy as follows:

$$\begin{aligned} H(f) &= -\int_0^{\pi} \frac{\kappa e^{\kappa\cos(\theta)}\sin(\theta)}{2\sinh(\kappa)}\ln[\frac{\kappa e^{\kappa\cos(\theta)}\sin(\theta)}{2\sinh(\kappa)}]d\theta \\ &= -\ln(\frac{\kappa}{2\sinh(\kappa)}) - [\kappa + \frac{(\kappa+1)e^{-\kappa}}{2\sinh(\kappa)}] - \int_0^{\pi}\frac{\kappa e^{\kappa\cos(\theta)}\sin(\theta)\ln[\sin(\theta)]}{2\sinh(\kappa)}d\theta. \end{aligned} \tag{2.6}$$

Now, by taking $\kappa\cos(\theta) = u$ in the integral in Eq. (2.6), recalling the definition of the definite integral of an even function and also the definition of the logarithmic series, we have

$$\begin{aligned} \int_0^{\pi}\frac{\kappa e^{\kappa\cos(\theta)}\sin(\theta)\ln[\sin(\theta)]}{2\sinh(\kappa)}d\theta &= \int_0^{\kappa}\frac{e^u\ln(1-\frac{u^2}{\kappa^2})}{2\sinh(\kappa)}du \\ &= \sum_{l=1}^{\infty}(-1)^{2l+1}\frac{1}{2\sinh(\kappa)}(\frac{1}{\kappa^2})^l\int_0^{\kappa} e^u u^{2l}du \\ &= \sum_{l=1}^{\infty}(-1)^{2l+1}\frac{1}{2\sinh(\kappa)}(\frac{1}{\kappa^2})^l B(1,2l+1)\kappa^{2l}{}_1F_1(2l;\ 1+2l;\kappa), \end{aligned} \tag{2.7}$$

which follows from Eq. 3.383.1, Page 318, of Gradshteyn and Ryzhik [15] and where $B(.,.)$ denotes the beta function and ${}_1F_1(.;.;.)$ denotes the degenerate hypergeometric function; see, for example, Gradshteyn and Ryzhik [15], among others. Thus, in view of (2.7), the expression (2.6) of Shannon entropy of $X \sim FI(\kappa)$ is given by

$$\begin{aligned} H(f) = &-\ln(\frac{\kappa}{2\sinh(\kappa)}) - [\kappa + \frac{(\kappa+1)e^{-\kappa}}{2\sinh(\kappa)}] \\ &- \sum_{l=1}^{\infty}(-1)^{2l+1}\frac{1}{2\sinh(\kappa)}(\frac{1}{\kappa^2})^l B(1,2l+1)\kappa^{2l}{}_1F_1(2l;\ 1+2l;\kappa). \end{aligned} \tag{2.8}$$

2.5 Percentile Points

Here we compute the percentage points of the standard Fisher distribution, $X \sim FI(\kappa)$, with the *pdf* (1.1) and the *cdf* (2.1). For any $0 < p < 1$, the 100 p^{th} percentile, (also called the quantile of order p), of $X \sim FI(\kappa)$, with *pdf* $f_X(\theta)$, is a number θ_p such that the area under $f_X(\theta)$ to the left of θ_p is p. That is, θ_p is any root of the equation given by

$$F(\theta_p) = \int_0^{\theta_p} f_X(u)du = p.$$

The percentage points θ_p associated with the *cdf* (2.1) of $X \sim FI(\kappa)$ are computed for some selected values of the parameters by using Maple software. These are provided in the Table 2.1 below:

Table 2.1: Percentage Points of the standard Fisher distribution, $X \sim FI(\kappa)$

Parameter *k*	**75%**	**80%**	**85%**	**90%**	**95%**	**99%**
0.1	2.05022	2.17331	2.30945	2.46708	2.66794	2.93089
0.2	2.00444	2.13044	2.27065	2.43402	2.64358	2.91964
0.5	1.86007	1.99238	2.14285	2.32248	2.55921	2.87979
1.0	1.61635	1.74851	1.90487	2.10130	2.37862	2.78781
1.5	1.40138	1.52266	1.67003	1.86317	2.15714	2.65419
2.0	1.23068	1.33807	1.46981	1.64587	1.92683	2.47416
2.5	1.10012	1.19516	1.31177	1.46810	1.72147	2.26223
3.0	0.99992	1.08512	1.18934	1.32861	1.55403	2.05094
3.5	0.92141	0.99893	1.09343	1.21914	1.42125	1.86614
4.0	0.85836	0.92982	1.01668	1.13172	1.31537	1.71441
4.5	0.80652	0.87310	0.95382	1.06036	1.22940	1.59147
5.0	0.76299	0.82556	0.90125	1.00088	1.15816	1.49084
5.5	0.72582	0.78500	0.85650	0.95038	1.09799	1.40707
6.0	0.69359	0.74989	0.81781	0.90683	1.04635	1.33607
6.5	0.66531	0.71911	0.78394	0.86878	1.00140	1.27495
7.0	0.64023	0.69183	0.75396	0.83516	0.96180	1.22161
7.5	0.61779	0.66744	0.72718	0.80517	0.92658	1.17452
8.0	0.59756	0.64546	0.70307	0.77820	0.89498	1.13255
8.5	0.57919	0.62553	0.68121	0.75378	0.86642	1.09481
9.0	0.56242	0.60733	0.66128	0.73152	0.84043	1.06064
9.5	0.54703	0.59064	0.64299	0.71113	0.81666	1.02950
10.0	0.53283	0.57525	0.62615	0.69236	0.79480	1.00097

3 Characterizations

Since characterization can be used to confirm whether the given continuous probability distribution satisfies the underlying requirements,many researchers have investigated the characterizations of probability distributions by different methods, see, for example, Galambos and Kotz [12], Glänzel [13], Glänzel et al. [14], Kotz and Shanbhag [19] and Nagaraja [23], among others. For recent developments on the characterizations of probability distributions by truncated moment method, the interested readers are referred to Ahsanullah [3], Ahsanullah and Shakil [4], Ahsanullah et al. [5, 6 and 7].It appears from the literature that not much attention has been paid to the characterizations of the Fisher distribution which can be usefully employed. However, as Mardia and Jupp [22] point out, “Arnold [2] gave a maximum likelihood characterization of the Fisher distribution, which was later proved by Breitenberger [9] by a simpler method. The general result was proved by Bingham and Mardia [8].” In this section, we present the characterizations of the Fisher distribution, $X \sim FI(\kappa)$ by truncated moments. For this, we will need some assumption and lemmas as provided below.

Assumption 3.1. *Suppose the random variable X is absolutely continuous with cumulative distribution function (cdf) $F(x)$ and probability density function (pdf) $f(x)$. We assume that $\gamma = \{x|F(x) > 0\}$and $\delta = \inf\{x|F(x) < 1\}$. We further assume that $E(X)$ exists.*

Lemma 3.1. *Under the assumption 3.1, if $E(X|X \leq x) = g(x)\tau(x)$, where $\tau(x) = \frac{f(x)}{F(x)}$ and $g(x)$ is a continuous differentiable function of x with the condition that $\int_\gamma^x \frac{u-g'(u)}{g(u)}du$ is finite for all x, $\gamma < x < \delta$, then $f(x) = ce^{\int_\gamma^x \frac{u-g'(u)}{g(u)}du} = ce^{\int_\gamma^x \frac{f'(u)}{f(u)}du}$, where $\frac{u-g'(u)}{g(u)} = \frac{f'(u)}{f(u)}$ and c is determined by the condition $\int_\gamma^\delta f(x)dx = 1$.*

Proof. For proof, see Ahsanullah and Shakil [4]. □

Lemma 3.2. *Under the assumption 3.1, if $E(X|X \geq x) = g(x)r(x)$, where $r(x) = \frac{f(x)}{1-F(x)}$ and $g(x)$ is a continuous differentiable function of x with the condition that $\int_x^\delta \frac{u+g'(u)}{g(u)}du$ is finite for all x, $\gamma < x < \delta$, then $f(x) = ce^{-\int_x^\delta \frac{u+g'(u)}{g(u)}du} = ce^{\int_x^\delta \frac{f'(u)}{f(u)}du}$, where $-\frac{u+g'(u)}{g(u)} = \frac{f'(u)}{f(u)}$ and c is determined by the condition $\int_\gamma^\delta f(x)dx = 1$.*

Proof. For proof, see Ahsanullah and Shakil [4]. □

3.1 Characterization by Truncated Moments

Following theorems contain the characterizations of the Fisher distribution by truncated moments. Without loss of generality, we will consider the e *pdf* $f(\theta)$ of the standard Fisher distribution, $X \sim FI(\kappa)$, as given in 2.2 above.

Theorem 3.1.1. *Suppose that X is absolutely continuous bounded random variable with cdf $F(x)$ such that $F(0) = 0$ and $F(\pi) = 1$, then $E(X|X \leq \theta) = g(\theta)\tau(\theta)$, where $\tau(\theta) = \frac{f(\theta)}{F(\theta)}$ and $g(\theta)$ is a continuous differentiable function of θ, $0 < \theta < \pi$, given by*

$$g(\theta) = \frac{e^{-k\cos\theta}}{\sin(\theta)}\{I_0(\kappa)(\sin(\theta) - \theta\cos(\theta))$$
$$+ \sum_{j=1}^{\infty} I_j(\kappa)[\frac{\sin((j+1)\theta) - (j+1)\theta\cos((j+1)\theta)}{(j+1)^2} - \frac{\sin((j-1)\theta) - (j-1)\theta\cos((j-1)\theta)}{(j-1)^2}]\},$$

if and only if

$$f(\theta) = \frac{\kappa e^{\kappa\cos(\theta)}\sin(\theta)}{2\sinh(\kappa)}, \quad 0 \leq \theta \leq \pi, \kappa \geq 0.$$

Proof. If

$$f(\theta) = \frac{\kappa e^{\kappa\cos(\theta)}\sin(\theta)}{2\sinh(\kappa)}, \quad 0 \le \theta \le \pi, \kappa \ge 0,$$

that is,

$$f(\theta) = \frac{\kappa\sin(\theta)}{2\sinh(\kappa)}(I_0(\kappa) + 2\sum_{j=1}^{\infty} I_j(\kappa)\cos(j\theta)), \quad 0 \le \theta \le \pi, \kappa \ge 0,$$

then it is easily seen, by direct integration and simplification, that

$$\begin{aligned} g(\theta) &= \frac{\int_0^\theta uf(u)du}{f(\theta)} = \frac{\int_0^\theta [I_0(\kappa)u\sin(u) + 2\sum_{j=1}^{\infty} I_j(\kappa)u\sin(u)\cos(ju)]du}{e^{k\cos\theta}\sin(\theta)} \\ &= \frac{e^{-k\cos\theta}}{\sin(\theta)}\{I_0(\kappa)(\sin(\theta) - \theta\cos(\theta)) \\ &+ \sum_{j=1}^{\infty} I_j(\kappa)[\frac{\sin((j+1)\theta) - (j+1)\theta\cos((j+1)\theta)}{(j+1)^2} - \frac{\sin((j-1)\theta) - (j-1)\theta\cos((j-1)\theta)}{(j-1)^2}]\} \end{aligned}$$

Suppose that

$$\begin{aligned} g(\theta) &= \frac{e^{-k\cos\theta}}{\sin(\theta)}\{I_0(\kappa)(\sin(\theta) - \theta\cos(\theta)) \\ &+ \sum_{j=1}^{\infty} I_j(\kappa)[\frac{\sin((j+1)\theta) - (j+1)\theta\cos((j+1)\theta)}{(j+1)^2} - \frac{\sin((j-1)\theta) - (j-1)\theta\cos((j-1)\theta)}{(j-1)^2}]\}. \end{aligned}$$

Then, differentiating both sides of the above equations with respect to θ and simplifying, it is easily seen that

$$g'(\theta) = \theta - [\cot(\theta) - k\sin(\theta)]g(\theta), \text{since } e^{\kappa\cos\theta} = (I_0(\kappa) + 2\sum_{j=1}^{\infty} I_j(\kappa)\cos(j\theta)).$$

Thus $\dfrac{\theta - g'(\theta)}{g(\theta)} = \cot(\theta) - k\sin(\theta)$,

from which, on using Lemma 3.1, we have

$$\frac{f'(\theta)}{f(\theta)} = \frac{\theta - g'(\theta)}{g(\theta)} = \cot(\theta) - k\sin(\theta),$$

or, $$d[\ln(f(\theta))] = \cot(\theta) - k\sin(\theta).$$

On integrating the above equation with respect to θ, we obtain $f(\theta) = c\sin(\theta)e^{k\cos\theta}$, where c is a constant to be determined.

On integrating the above equation with respect to θ from 0 to π and using the condition

$$\int_0^\pi f(\theta)d\theta = 1, \text{we easily obtain } c = \frac{\kappa}{2\sinh(\kappa)}, \text{and thus}$$

$$f(\theta) = \frac{\kappa e^{\kappa\cos(\theta)}\sin(\theta)}{2\sinh(\kappa)}, \quad 0 \le \theta \le \pi, \kappa \ge 0,$$

which is the *pdf* of the standard Fisher distribution, $X \sim FI(\kappa)$. This completes the proof of the Theorem 3.1.1. □

Theorem 3.1.2. *Suppose that X is absolutely continuous bounded random variable with cdf $F(x)$ such that $F(0)=0$ and $F(\pi)=1$, then $E(X|X\geq\theta)=h(\theta)r(\theta)$, where $r(\theta)=\frac{f(\theta)}{1-F(\theta)}$ and $h(\theta)$ is a continuous differentiable function of θ, $0<\theta<\pi$, given by*

$$
\begin{aligned}
h(\theta) &= \frac{\int_\theta^\pi uf(u)du}{f(\theta)} = \frac{\int_\theta^\pi [I_0(\kappa)u\sin(u)+2\sum_{j=1}^\infty I_j(\kappa)u\sin(u)\cos(ju)]du}{e^{k\cos\theta}\sin(\theta)} \\
&= \frac{e^{-k\cos\theta}}{\sin(\theta)}\{I_0(\kappa)(\pi-\sin(\theta)+\theta\cos(\theta)) \\
&+ \sum_{j=1}^{\infty} I_j(\kappa)[\frac{(j+1)\theta\cos((j+1)\theta)+(j+1)\pi\cos(\pi j)-\sin((j+1)\theta)-\sin(\pi j)}{(j+1)^2} \\
&- \frac{(j-1)\theta\cos((j-1)\theta)+(j-1)\pi\cos(\pi j)-\sin((j-1)\theta)-\sin(\pi j)}{(j-1)^2}]\}
\end{aligned}
$$

if and only if

$$f(\theta)=\frac{\kappa e^{\kappa\cos(\theta)}\sin(\theta)}{2\sinh(\kappa)},\quad 0\leq\theta\leq\pi,\kappa\geq 0.$$

Proof. The proof is similar to the Theorem 3.1.1 and easily follows from Lemma 3.2. □

3.2 Characterizations by Order Statistics

Here, we will provide the characterizations based on order statistics, for which we first recall the following well-known results. Let$X_1,X_2,...,X_n$ be n independent copies of the random variable X having absolutely continuous distribution function $F(x)$ and *pdf* $f(x)$. Suppose that $X_{1,n}\leq X_{2,n}\leq...\leq X_{n,n}$ are the corresponding order statistics. It is known that $X_{j,n}|X_{k,n}=x$, for $1\leq k<j\leq n$, is distributed as the $(j-k)^{\text{th}}$ order statistics from $(n-k)$ independent observations from the random variable V having the pdf $f_V(v|x)$ where $f_V(v|x)=\frac{f(v)}{1-F(x)}$, $0\leq v<x$, see, for example, Arnold et al. [1], chapter 2, among others. Further, $X_{i.,n}|X_{k,n}=x, 1\leq i<k\leq n$, is distributed as i^{th} order statistics from k independent observations from the random variable W having the pdf $f_W(w|x)$ where $f_W(w|x)=\frac{f(w)}{F(x)}, w<x$. Let

$$S_{k-1}=\tfrac{1}{k-1}(X_{1,n}+X_{2,n}+...+X_{k-1,n}),$$

and

$$T_{k,n}=\tfrac{1}{n-k}(X_{k+1,n}+X_{k+2,n}+...+X_{n.n}).$$

In the following two theorems, we will provide the characterization of the standard Fisher distribution, $X\sim FI(\kappa)$, based on order statistics.

Theorem 3.2.1. *Suppose the random variable X satisfies the assumption 1 with $\gamma=0$ and $\delta=\pi$, then*

$$E(S_{k-1}|X_{k,n}=\theta)=g(\theta)\tau(\theta),\text{where }\tau(\theta)=\frac{f(\theta)}{F(\theta)}$$

and

$$g(\theta)=\frac{e^{-k\cos\theta}}{\sin(\theta)}\{I_0(\kappa)(\sin(\theta)-\theta\cos(\theta))$$

$$+\sum_{j=1}^{\infty} I_j(\kappa)[\frac{\sin((j+1)\theta)-(j+1)\theta\cos((j+1)\theta)}{(j+1)^2}-\frac{\sin((j-1)\theta)-(j-1)\theta\cos((j-1)\theta)}{(j-1)^2}]\},$$

if and only if

$$f(\theta)=\frac{\kappa e^{\kappa\cos(\theta)}\sin(\theta)}{2\sinh(\kappa)},\quad 0\le\theta\le\pi,\kappa\ge 0.$$

Proof. It is known, see David and Nagaraja [10], that

$$E(S_{k-1}|X_{k,n}=\theta)=E(X|X\le\theta).$$

Thus the result follows from Theorem 3.1.1. □

Theorem 3.2.2. *Suppose the random variable X satisfies the assumption* 1 *with $\gamma=0$ and $\delta=\pi$, then*

$$E(T_{k,n}|X_{k,n}=\theta)=h(\theta)r(\theta),\text{where } r(\theta)=\frac{f(\theta)}{1-F(\theta)} \text{ and}$$

$$\begin{aligned}h(\theta)&=\frac{\int_\theta^\pi uf(u)du}{f(\theta)}=\frac{\int_\theta^\pi[I_0(\kappa)u\sin(u)+2\sum_{j=1}^{\infty}I_j(\kappa)u\sin(u)\cos(ju)]du}{e^{k\cos\theta}\sin(\theta)}\\&=\frac{e^{-k\cos\theta}}{\sin(\theta)}\{I_0(\kappa)(\pi-\sin(\theta)+\theta\cos(\theta))\\&+\sum_{j=1}^{\infty}I_j(\kappa)[\frac{(j+1)\theta\cos((j+1)\theta)+(j+1)\pi\cos(\pi j)-\sin((j+1)\theta)-\sin(\pi j)}{(j+1)^2}\\&-\frac{(j-1)\theta\cos((j-1)\theta)+(j-1)\pi\cos(\pi j)-\sin((j-1)\theta)-\sin(\pi j)}{(j-1)^2}]\}\end{aligned}$$

if and only if $$f(\theta)=\frac{\kappa e^{\kappa\cos(\theta)}\sin(\theta)}{2\sinh(\kappa)},\quad 0\le\theta\le\pi,\kappa\ge 0.$$

Proof. It is known, see David and Nagaraja [10], that

$$E(T_{k,n}|X_{k,n}=\theta)=E(X|X\ge\theta).$$

Thus the result follows from Theorem 3.1.2. □

3.3 Characterization by Upper Record Values

Here, we will provide the characterizations based on upper record values, for which we first recall the following definitions. Suppose that $X_1, X_2, ...$ is a sequence of independent and identically distributed absolutely continuous random variables with distribution function $F(x)$ and *pdf* $f(x)$. Let $Y_n=\max(X_1,X_2,...,X_n)$ for $n\ge 1$. We say that X_j is an upper record value of $\{X_n, n\ge 1\}$ if $Y_j>Y_{j-1}, j>1$. The indices at which the upper records occur are given by the record times $\{U(n)>\min(j|j>U(n+1), X_j>X_{U(n-1)}, n>1)\}$ and $U(1)=1$. We will denote the n^{th} upper record value as $X(n)=X_{U(n)}$. In the following theorem, we will provide the standard Fisher distribution, $X\sim FI(\kappa)$, based on upper record values.

Theorem 3.3.1. *Suppose the random variable X satisfies the assumption* 1 *with $\gamma = 0$ and $\delta = \pi$, then $E(X(n+1)|X(n) = \theta) = h(\theta)r(\theta)$, where $r(\theta) = \frac{f(\theta)}{1-F(\theta)}$ and*

$$
\begin{aligned}
h(\theta) &= \frac{\int_\theta^\pi uf(u)du}{f(\theta)} = \frac{\int_\theta^\pi [I_0(\kappa)u\sin(u) + 2\sum_{j=1}^\infty I_j(\kappa)u\sin(u)\cos(ju)]du}{e^{k\cos\theta}\sin(\theta)} \\
&= \frac{e^{-k\cos\theta}}{\sin(\theta)}\{I_0(\kappa)(\pi - \sin(\theta) + \theta\cos(\theta)) \\
&+ \sum\nolimits_{j=1}^\infty I_j(\kappa)[\frac{(j+1)\theta\cos((j+1)\theta) + (j+1)\pi\cos(\pi j) - \sin((j+1)\theta) - \sin(\pi j)}{(j+1)^2} \\
&- \frac{(j-1)\theta\cos((j-1)\theta) + (j-1)\pi\cos(\pi j) - \sin((j-1)\theta) - \sin(\pi j)}{(j-1)^2}]\}
\end{aligned}
$$

if and only if $$f(\theta) = \frac{\kappa e^{\kappa\cos(\theta)}\sin(\theta)}{2\sinh(\kappa)}, \quad 0 \le \theta \le \pi, \kappa \ge 0.$$

Proof. It is known, see Nevzorov [24], that

$$E(X(n+1)|X(n) = \theta) = E(X|X \ge \theta).$$

Thus the result follows from Theorem 3.1.2. □

4 Concluding Remarks

Characterization of a probability distribution plays an essential role in probability and statistics and other applied sciences. It can be used to endorse whether the given probability distribution satisfies the underlying requirements before a particular probability distribution model is applied to fit the real world data. The Fisher distribution is one of the most important distributions in statistics to deal with the paleomagnetic data. In this paper, we have considered its several distributional properties. Based on these distributional properties, we have established some new characterizations of the Fisher distribution by truncated first moment, order statistics and upper record values. We believe that the findings of the paper will be useful for researchers in the fields of probability, statistics and other applied sciences.

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INVENTORY MODEL FOR SINGLE ITEM-MULTI SUPPLIER WITH SHORTAGES

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Abstract

This paper deals with the problem of developing an acquisition policy for solving the single-item multi-supplier problem with shortages and real world constraints. We evaluate the optimal quantity of item to be ordered to the different suppliers so as to optimize the total cost incurred. The optimal solution is obtained to indicate the best acquisition policy. In addition, the effects of shortages are also considered. By taking numerical example, the sensitivity analysis has been carried out to explore the effect of system parameters on optimal policy.

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1 Introduction

Traditionally there has always been some friction between organizations and their suppliers. This friction causes serious problems with loyalty and co-operation. Suppliers set rigid conditions, and as they have no guarantee of repeat business, they try to make as much profit from each sale as possible; organizations shop around to make sure they get best deal, and remind suppliers of the competition. The result is uncertainty about orders, constant changes in an organizations suppliers and customers, changing products, varying order sizes, different times between orders, uncertainty about repeat orders, changes in the costs, and so on.

Just-in-time production models have been developed in recent years in order to reduce the costs of diversified small-lot productions. Organizations using JIT rely on their suppliers being completely dependable and removing all uncertainty from supply. The only way of achieving this is through co-operation, recognizing that organizations and their suppliers both want a mutually beneficial trading arrangement. If they can agree conditions that satisfy both the organization and the supplier- with the feeling that they get best possible deal this is far better than working with unnecessary friction. The implication is that organizations should identify the best suppliers and always give order to that supplier. This reinforces the ideas behind strategic alliances and partnerships. With close co-operation, it can be difficult for suppliers to make the small, frequent delivery needed, and to co-ordinate their delivery with demands. Sometimes they respond by increasing their stock of finished goods to ensure the required pattern of delivery. The effect of this is simply to move stocks from an organizations raw material store to the suppliers finished goods store. This does not reduce overall costs and might even increase them. The aim of JIT is to eliminate stocks rather than move them to another point in the supply chain. And, again, the way to achieve this is through co-operation. Hax and Candea [10] indicated that unconstrained optimization of economic order quantity (EOQ) is often an unrealistic assumption. Burton [5] considered JIT and introduced repetitive sourcing strategy. The greatest flaw of the weights and criteria grading are open to subjectiveness. However, the clear advantage of these methods is that many important criteria that cannot be quantified are taken into consideration. For a summary of vendor selection criteria, we refer Weber et al. [19]. Benton [2] derived a heuristic procedure for treating PQD decisions under the conditions of multiple items, multiple suppliers and resource limitations.

Dickson [7] analyzed the supplier selection problem in supply chain management. Activity based costing approach was developed by Roodhooft and Konings [16]. They gave various aspects of vendor selection considering the total cost of ownership. Ghodsyapour and O'Brien [9] applied integrated analytical hierarchical process and linear programming to decision support system for supplier selection. Rosenblatt et al. [17] considered the problem of developing an acquisition policy. Specifically, given a set of potential suppliers, from whom should the firm buy the product, in what quantities, and how often? They provided an algorithm for multi-supplier single item system.

Yang and Wee [22] developed a production-inventory system model of a deteriorating item, taking into account the view of both the vendor and the multi-buyers. Minner [15] reviewed inventory models with multi supply options and discussed their contribution to supply chain management. He also discussed strategic aspects of supplier competition and the role of operational flexibility in global sourcing and outlined inventory models, which use several suppliers in order to avoid or reduce the effects of shortage situations. Li [11] discussed the multi-buyer joint replenishment problem (MJRP) of coordinating the replenishment of a group of items that are jointly ordered from a single supplier. Xu [20] considered a class of multi-period dynamic supply contracts in which a buyer orders a product from a supplier in each period and the supplier allows the buyer to cancel a portion of an outstanding order with penalty during a planning horizon. He used simulation to show how the supplier chooses cancellation costs that minimize his expected cost during the planning horizon.

Bretthauer et al. [3] presented a model and solution methodology for production and inventory management problems that involve multiple resource constraints. Their model formulation was general, allowing organizations to handle a variety of multi-item decisions such as determining order quantities, production batch sizes, number of production runs, or cycle times. They presented efficient algorithms for solving both continuous and integer variable versions of the resource constrained production and inventory management model. Chan et al. [6] addressed the supplier-scheduling problem by considering the deliveries scheduling issue, once the optimal replenishment cycles are determined. They considered four large integer programming problems according to four different objectives in cost and resource minimization and solved them by converting them into network flow problems.

Lebacque et al. [12] suggested the methods which are aimed at maintaining the production rate of each type of part as smooth as possible and therefore holding small inventory and shortage costs. Burke et al. [4] analyzed single period, single product sourcing decisions under demand uncertainty. Their approach includes the product prices, supplier costs, supplier capacities, historical supplier reliabilities and firm specific inventory costs. Rong et al. [8] presented an EOQ model with two-warehouses for the perishable goods with fuzzy lead time and partially/ fully backlogged shortage. Again Madhavilata et al. [14] introduced two levels of storage for inventory of single item in their research work. Min et al. [13] developed inventory model in which items are deteriorating exponentially and shortages are allowed. Agrawal et al. [1] also considered an inventory system with two warehouses where demand rate is ramp type and deterioration rate is constant. Guchhait et al. [8] developed a model for inventory system with time dependent deteriorating items to determine the profit maximization. Xu.et al. [21] proposed an inventory system periodic review base stock with partial backlogging.

In this investigation we extend the model of Rosenblatt et al. [17]. In their model, shortages are not allowed, so we add the concept of shortages which are satisfied at the arrival epoch of the next replenishment. Since demand for an item may repeat either very frequently or after a long time, so the organization may face the problem of (i) how much to restock at each replenishment cycle, and (ii) how many intervals should a replenishment cycle be. To deal with this problem, this chapter presents an inventory model having real-world constraints with back order. The rest of the chapter is organized as follows. In the next section we describe our model. An optimal policy is given in section 3. Section 4 provides an acquisition policy from which we find that from which supplier we buy the product and how much quantity to buy. Section 5 is devoted to numerical illustrations. Finally section 6 presents conclusions and some recommendations for future research work.

2 The Model

A deterministic inventory model is developed for a single item, multi supplier, where each supplier has a limit on the average capacity of the items, which he can deliver per unit of time. Each of the suppliers has its own cost parameters and a finite long-run average capacity. The total cost is comprised of the sum of the total periodic purchasing, ordering, and inventory carrying, shortage and supplier management costs.

Ordering cost is a cost associated with ordering of raw material for production purposes. Advertisements, consumption of stationery and postage, telephone charges, telegrams, rent for space used by the purchasing department, travelling expenditures incurred, etc. constitute the ordering cost. The cost of purchasing total units of an item is known as total periodic purchasing cost. The carrying cost is associated with carrying (or holding) inventory. This cost generally includes the costs such as rent for space used for storage, interest on the money locked-up, insurance of stored equipment, production, taxes, depreciation of equipment and furniture used etc.. The penalty cost for running out of stock (i.e., when an item cannot be supplied on the customers demand) is known as shortage cost. This cost includes the loss of potential profit through sales of items and loss of goodwill, in terms of permanent loss of goodwill, in terms of permanent loss of customers and its associated lost profit in future sales.

The supplier management costs are fixed periodic costs incurred for every supplier who supplies a nonzero amount, independent of the supply volume and the number of orders. In general, any cost associated with maintaining a relationship with a supplier will constitute a part of supplier management costs.

The following notations are used for mathematical formulation of the inventory model:

M : Total number of suppliers.
b_i : Periodic capacity of item for i^{th} supplier.

d_i : Average periodic quantity ordered from supplier i.
C_i : The cost per item procured from i^{th} supplier.
D : The demand for item per period.
C_{mi} : The periodic supplier management cost incurred when using i^{th} supplier.
C_{hi} : The periodic holding cost associated with holding one unit from i^{th} supplier.
C_{si} : The fixed ordering cost associated with each order from i^{th} supplier.
Q_i : The order quantity of the item from i^{th} supplier.
q_i : Amount to be back-ordered.
y_i : An integer variable, set equal to 1 if supplier i is used and 0 otherwise.
$\frac{d_i}{Q_i}$: The number of orders from supplier i.
τ : Cycle time.
α_i : Number of orders placed per cycle by i^{th} supplier.
K : Maximum limit on inventory.

For all the above notations, $i = 1, 2, ..., M$. In our model, we allow the shortages, therefore the demand of the item is not fully met, it is greater than the ordered item. Hence,

$$\sum_{i=1}^{M} d_i \leq D. \tag{2.1}$$

Individual cost for the item from different suppliers is now evaluated before they are grouped together.

The total periodic purchasing cost from supplier $i = C_i d_i$.

The total ordering cost associated with supplier $i = C_{si} \times \frac{d_i}{Q_i}$.

The periodic holding cost for supplier i for the item $C_{hi} \left(\frac{Q_i - q_i}{2D}\right)^2 \times \frac{d_i}{Q_i}$.

The shortage cost for i^{th} supplier for the item $= S_i \left(\frac{q_i^2}{2D}\right) \times \frac{d_i}{Q_i}$.

The periodic supplier management cost associated with i^{th} supplier $= C_{mi} Y_i$.

The total average cost per period can be expressed as follows:

TC = Purchasing cost + Ordering cost + Holding cost + Shortage cost +Management cost.

Therefore,

$$TC = \sum_{i=1}^{M} \left[C_i d_i + C_{si} \frac{d_i}{Q_i} + C_{hi} \left(\frac{(Q_i - q_i)^2}{2D} \right) \frac{d_i}{Q_i} + S_i \left(\frac{q_i^2}{2D} \right) \frac{d_i}{Q_i} + C_{mi} Y_i \right]. \tag{2.2}$$

3 Cost Optimization Problem

The purpose of this study is to minimize the total cost by simultaneously determining the values of Q_i and q_i optimally. According to equation (2.2), the total average cost function is a function of two variables Q_i and q_i. Our problem is to determine the values of Q_i and q_i which minimize the total cost. During the first part of the cycle all demand is met from the stock, so the amount sent to customers is $(Q_i - q_i)$. During the second part of the cycle all demand is back-ordered, so the shortage, q_i equals the unmet demand. After some manipulation the optimal order quantity (Q_i^*) and optimal amount to be back-ordered (q_i^*) is given by:

$$Q_i = \sqrt{\frac{2DC_{si}(S_i + C_{hi}}{C_{hi} S_i}} \tag{3.1}$$

and

$$q_i = \sqrt{\frac{2DC_{hi}DC_{si}}{(S_i + C_{hi})S_i}}. \tag{3.2}$$

4 The Acquisition Policy

The acquisition policy is maintained, which combines both the vendors selection and inventory management into the model. In this policy a firm can determine from which suppliers to buy, and what quantity to buy periodically.

The supplier can be selected on the basis of suppliers effective variable cost per unit item, which can be obtained as

$$f_i = C_i + \frac{2C_{hi} + S_i}{(C_{hi} + S_i)}\sqrt{\frac{C_{hi}S_iC_{si}}{2D\,(C_{hi} + S_i)}} - \frac{C_{hi}\,(C_{hi} + 2S_i)}{(C_{hi} + S_i)}\sqrt{\frac{C_{hi}C_{si}}{2DS_i\,(C_{hi} + S_i)}} + \sqrt{\frac{C_{hi}C_{si}\,(C_{hi} + S_i)}{2DS_i}} \tag{4.1}$$

A supplier (who is selected to supply units) will supply less than its capacity, therefore, any supplier who has a higher effective variable cost per unit item then this supplier will not be used. The objective function yields the following form:

$$\text{Min } TC = \sum_{i=1}^{M}(f_i d_i + C_{mi}y_i). \tag{4.2}$$

In association to execute a cyclic schedule one has to determine the cycle length, τ. Now the number of orders placed with supplier i for item per cycle is determined as:

$$\alpha_i = \frac{d_i\tau}{Q_i^*}, \tag{4.3}$$

where Q_i^* is obtained from equation (3.1). But α_i might not be an integer, hence, we round off α_i to the respective geometrically close integer. Since the allocation from the various suppliers cannot change, we set the actual order size $Q_i' = Q_i^*\,(\alpha_i/\alpha_i')$ so as to maintain $\alpha_i'Q_i' = d_i\tau_i$. Hence, $\alpha_i'Q_i'$ will be the actual amount of item supplied by i^{th} supplier during a cycle.

5 Numerical Illustrations

Table 5.1: Parameters and cost elements

Supplier	Cost per item (C_i)	Fixed ordering cost (C_{si})	Holding cost (C_{hi})	Periodic supplier management cost (C_{mi})	Shortage cost (S_i)	Periodic demand (d_i)
Supplier (1)	2.5	0.2	0.21	20	0.5	60
Supplier (2)	2.7	1.7	0.19	15	0.7	40

A single item multi supplier system with shortages and real world constraints is considered. The parameters and cost elements of suppliers are shown in Table 5.1. A commercial non-linear programming solver OPTIMIZATION TOOL in MATLAB 6.5 is used for solving this problem to obtain the optimal solution as $(Q_1, Q_2, q_1, q_2, TC) = (16.94, 47.70, 4.86, 10.18, 34409110)$.

From Table 5.2, we see that as the value of shortage cost (S_i) increases, the optimal order quantity (Q_i), optimal quantity to be back-ordered (q_i) and total cost (TC) decrease, which is quite obvious. Table 5.3 shows the effect of holding cost (C_{hi}) on optimal order size (Q_i), optimal amount to be back-ordered (q_i) and total cost (TC). It is clear from Table 5.3 that as the value of shortage cost (S_i) increases, the optimal order quantity (Q_i), optimal quantity to be back-ordered (q_i) and total cost (TC) decrease.

Table 5.4 displays the effect of ordering cost (C_{si}) on optimal order quantity (Q_i), optimal quantity to be back-ordered (q_i) and total cost (TC). An optimal order quantity, optimal quantity to be back-ordered and total cost increase as the value of ordering cost (C_{si}) increases. It is observed from Table 5.5 that as annual demand (D) increases, the optimal order quantity (Q_i), optimal quantity to be back-ordered (q_i) and total cost (TC) of the system increase.

Table 5.6 examines the effect of cost per item (C_i) on the total cost of the system. The total cost increases with the increase in the cost per item. From Table 5.7 we find that as periodic supplier management cost increases, the total cost of the system increases.

Table 5.2: Effect of shortage cost on optimal order size, optimal amount to be back-ordered and total cost

S_1	S_2	Q_1	q_1	Q_2	q_2	TC
0.50	0.70	16.45	4.86	47.70	10.18	34409110
0.70	0.90	15.74	3.63	46.55	8.11	33545150
0.90	1.10	15.33	2.90	45.81	6.75	32990410
1.10	1.30	15.06	2.41	45.29	5.77	32603550
1.30	1.50	14.87	2.07	44.90	5.05	32318200
1.50	1.70	14.74	1.81	44.60	4.48	32098980
1.70	1.90	14.63	1.61	44.37	4.03	31925250
1.90	2.10	14.54	1.45	44.17	3.67	31784180
2.10	2.30	14.47	1.32	44.01	3.36	31667330
2.30	2.50	14.42	1.21	43.88	3.10	31568960

Table 5.3: Effect of holding cost on optimal order size, optimal amount to be back-ordered and total cost

C_{h1}	C_{h2}	Q_1	q_1	Q_2	q_2	TC
0.21	0.19	16.45	4.86	47.70	10.18	34409110
0.31	0.29	14.46	5.53	40.72	11.93	29424930
0.41	0.39	13.33	6.00	36.84	13.18	26653510
0.51	0.49	12.59	6.36	34.35	14.14	24865460
0.61	0.59	12.07	6.63	32.59	14.90	23608060
0.71	0.69	11.68	6.85	31.28	15.53	22672180
0.81	0.79	11.37	7.03	30.27	16.05	21946850
0.91	0.89	11.13	7.19	29.46	16.49	21367340
1.01	0.99	10.94	7.32	28.79	16.87	20893220
1.11	1.09	10.77	7.43	28.24	17.20	20497860

Table 5.4: Effect of ordering cost on optimal order size, optimal amount to be back-ordered and total cost

C_{s1}	C_{s2}	Q_1	q_1	Q_2	q_2	TC
0.20	1.70	16.45	4.86	47.70	10.18	34409110
0.30	1.80	20.14	5.96	49.08	10.48	38964810
0.40	1.90	23.26	6.88	50.43	10.77	43906640
0.50	2.00	26.00	7.69	51.74	11.04	49190800
0.60	2.10	28.49	8.43	53.01	11.32	54787200
0.70	2.20	30.77	9.10	54.26	11.58	60673360
0.80	2.30	32.89	9.73	55.48	11.84	66831580
0.90	2.40	34.89	10.32	56.67	12.10	73247370
1.00	2.50	36.77	10.88	57.84	12.35	79908590
1.10	2.60	38.57	11.41	58.99	12.59	86804810

Table 5.5: Effect of annual demand on optimal order size, optimal amount to be back-ordered and total cost

D	Q_1	q_1	Q_2	q_2	TC
100.00	16.45	4.86	47.70	10.18	34409110
110.00	17.25	5.10	50.03	10.68	43667100
120.00	18.02	5.33	52.25	11.15	54278160
130.00	18.75	5.55	54.39	11.61	66302510
140.00	19.46	5.76	56.44	12.05	79797880
150.00	20.14	5.96	58.42	12.47	94819830
160.00	20.80	6.15	60.33	12.88	111422000
170.00	21.44	6.34	62.19	13.28	129656100
180.00	22.06	6.53	63.99	13.66	149572500
190.00	22.67	6.71	65.75	14.04	171220000

Table 5.6: Effect of cost per item on total cost

C_1	C_2	TC
2.50	2.70	34409110
3.00	3.20	34409160
3.50	3.70	34409210
4.00	4.20	34409260
4.50	4.70	34409310
5.00	5.20	34409360
5.50	5.70	34409410
6.00	6.20	34409460
6.50	6.70	34409510
7.00	7.20	34409560

Table 5.7: Effect of periodic supplier management cost on total cost

C_{m1}	C_{m2}	TC
20	15	34409110
25	20	34409120
30	25	34409130
35	30	34409140
40	35	34409150
45	40	34409160
50	45	34409170
55	50	34409180
60	55	34409190
65	60	34409200

Conclusions

A single item multi supplier EOQ model with real world constraints has been investigated. We have dealt with an inventory problem wherein the shortage occurs. Our study will be helpful to determine an acquisition policy to determine from which supplier to buy, and what quantity to buy from each supplier periodically. The model formulated with shortages under more realistic assumptions provides insight for many big firms and malls.

For future research on this problem, it would be of interest to add the effect of more realistic demand rate in the model. The incorporation of some more realistic factors, such as quantity discounts, inflation etc., can further enrich our inventory model

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NEW PATHWAY FRACTIONAL INTEGRAL OPERATOR INVOLVING THE PRODUCT OF TWO I-FUNCTIONS

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Abstract

The object of the present paper is to study pathway fractional integral operator associated with the pathway model and pathway probability density for certain product of special functions with general argument. We establish new results on applying the Saigo- operators to the product of two I-functions.

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1 Introduction

The fractional integral operator involves various special functions, which has found significant importance and applications in various subfields of applicable mathematical analysis. For last four decades, a number of workers like Mathai and Houbold [12,13], Love [10], Saigo [17], Chaurasia and Gill [1], Chaurasia and Kumar [2], Gorenflo, Kilbas and Rogosin [5], Kilbas Srivastava and Trujillo [6], Kilbas and Saigo [7], Kilbas, Saigo and Saxena [8], Kiryakova [9], Mathai and Saxena [14], Samko, Kilbas and Merichev [18] etc. have studied in depth, the properties, applications and different extensions of various hypergeometric operators of fractional integration. The Pathway Fractional integral operator introduced by Nair [15] is defined in the following manner:

$$(P_{0_+}^{(\eta,\alpha)} f)(x) = x^{\eta} \int_0^{[\frac{x}{a(1-\alpha)}]} \left[1 - \frac{a(1-\alpha)}{x}\right]^{\frac{\eta}{1-\alpha}} f(t)dt, \tag{1.1}$$

where $f(x) \in L(a,b)$; $\eta \in \mathbb{C}$; $\mathrm{Re}(\eta) > 0$; $a > 0$ and ‘pathway parameter $\alpha < 1$. The pathway model introduced by Mathai [11] and studied further by Mathai and Haubold [12, 13]. For real scalar α, the pathway model for scalar random variables is represented by the following probability density function (p.d.f.):

$$f(x) = c|x|^{\gamma-1}[1 - a(1-\alpha)|x|^{\delta}]^{\frac{\beta}{1-\alpha}}, \tag{1.2}$$

$-\infty < x < \infty$, $\delta > 0$, $\beta \geq 0$, $1 - a(1-\alpha)|x|^{\delta} > 0$, $\gamma > 0$, where c is the normalization constant and α is called the pathway parameter. For real α, the normalization constant is as follows:

$$c = \frac{1}{2} \frac{\delta[a(1-\alpha)]^{\frac{\gamma}{\delta}} \Gamma(\frac{\gamma}{\delta} + \frac{\beta}{1-\alpha} + 1)}{\Gamma(\frac{\gamma}{\delta})\Gamma(\frac{\beta}{1-\alpha} + 1)}, \text{ for } \alpha < 1 \tag{1.3}$$

$$= \frac{1}{2}\frac{\delta[a(1-\alpha)]^{\frac{\gamma}{\delta}}\Gamma(\frac{\beta}{\alpha-1})}{\Gamma(\frac{\gamma}{\delta})\Gamma(\frac{\beta}{\alpha-1}-\frac{\gamma}{\delta})},\ \text{for}\ \frac{1}{\alpha-1}-\frac{\gamma}{\delta}>0, \alpha>0, \tag{1.4}$$

$$= \frac{1}{2}\frac{\delta[a\beta]^{\frac{\gamma}{\delta}}}{\Gamma(\frac{\gamma}{\delta})},\ \text{for}\ \alpha\to 1, \tag{1.5}$$

for $\alpha < 1$, it is a finite range density with $1 - a(1-\alpha)|x|^{\delta} > 0$ and (1.3) remains in the extended generalized type-1 beta family. The pathway density in (1.3) for $\alpha < 1$, included the extended type-1 beta density, the triangular density, the uniform density, etc.

For $\alpha > 1$, writing $1-\alpha = -(\alpha - 1)$, we have

$$f(x) = c|x|^{\gamma-1}[1 - a(\alpha-1)|x|^{\delta}]^{-\frac{\beta}{\alpha-1}}, \tag{1.6}$$

$-\infty < x < \infty$, $\delta > 0$, $\beta \geq 0$, $\alpha > 1$, which is extended generalized type-2 beta model for real x. it includes the type-2 beta density, the F-density, the student-t density, the Cauchy density and many more.

In this paper, we consider only the case of pathway parameter $\alpha < 1$. For $\alpha \to 1$ both (1.2) and (1.6) take the exponential form, since

$$\lim_{\alpha\to 1} c|x|^{\gamma-1}[1-a(\alpha-1)|x|^{\delta}]^{\frac{\beta}{1-\alpha}} = \lim_{x\to 1} c|x|^{\gamma-1}[1-a(1-\alpha)|x|^{\delta}]^{-\frac{\beta}{\alpha-1}}$$
$$= c|x|^{\gamma-1}e^{-a\eta|x|^{\delta}}. \tag{1.7}$$

This includes the generalized Gamma, Weibull, Laplace Maxwell-Boltzmann and other related densities.

2 Pathway Integral Operator Involving The Product of Two I-Functions

The I-function introduced by Saxena [19] is defined and represented in the following way:

$$I^{m,n}_{p_i,q_i:r}\left[z|\begin{pmatrix}(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,q_i}\end{pmatrix}\right] = \frac{1}{2\pi i}\int_L \varphi(\xi)z^{\xi}d\xi, \tag{2.1}$$

where

$$\varphi(\xi) = \frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j\xi)}{\sum_{i=1}^{r}\{\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}+\beta_{ji}\xi)\prod_{j=n+1}^{p_i}\Gamma(a_{ji}-\alpha_{ji}\xi)\}}, \tag{2.2}$$

$m, n, p_i(i = 1,\dots,r)$; and $q_i(i = 1,\dots,r)$ are integers satisfying $0 \leq n \leq p_i, 1 \leq m \leq q_i(i = 1,\dots,r)$; $r$is finite $\alpha_j, \beta_j, \alpha_{ji}, \beta_{ji}$ are real and positive numbers ;a_j, b_j, a_{ji}, b_{ji} are complex numbers such that

$$\alpha_j(\beta_h + \nu) \neq \beta_h(\alpha_j - \lambda - 1)$$

for $\lambda, \nu = 0, 1, 2, \dots; h = 1, 2, \dots, m; i = 1, 2, \dots, r$. L is a contour running from $\sigma - i\infty to \sigma + i\infty$($\sigma$ is real), in the complex ξ - plane such that the points $\xi = \frac{(\alpha_j-\lambda-1)}{\alpha_j}, j = 1, 2, \dots n; \lambda = 0, 1, 2, \dots$.and $\xi = \frac{(b_j+\lambda)}{\beta_j}, j = 1, 2, \dots m; \lambda = 0, 1, 2,...$ lie to the left hand and right hand sides of L respectively. For the convergence conditions, existence of various contours L and other properties, we can refer Saxena [19, p. 35]. The importance of I-function lies in the fact that almost all the elementary and special functions in the literature follow as its special cases viz. Erdélyi [3], Fox [4] and Wright ([21], [22]). etc. These special functions appear in various problems arising in theoretical and applied branches of mathematics, statistics, physics, engineering and other areas. The series representation is given in (2.3) is as follows:

$$I^{m,n}_{p_i,q_i:r}\left[z|\begin{pmatrix}(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,q_i}\end{pmatrix}\right] = \sum_{h=1}^{N}\sum_{k=0}^{\infty}\frac{(-1)^k\chi(\xi)}{k!\alpha_h}(\frac{1}{z})^{\xi} \tag{2.3}$$

and

$$\chi(\xi)=\frac{\prod_{i=1}^{m}\Gamma(b_j-\beta_j\{\frac{a_h-k-1}{\alpha_h}\})\prod_{i=1}^{n}\Gamma(1-a_j+\alpha_j\{\frac{a_h-k-1}{\alpha_h}\})z^{\frac{a_h-k-1}{\alpha_h}}}{\sum_{i=1}^{r}\{\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}+\beta_{ji}\{\frac{a_h-k-1}{\alpha_h}\})\prod_{j=n+1}^{p_i}\Gamma(a_{ji}-\alpha_{ji}\{\frac{a_h-k-1}{\alpha_h}\})\}}.$$

Theorem 2.1. *Let* $\eta,\ \rho\varepsilon\mathbb{C}$, $\mathrm{Re}(\beta)>0$, $\mathrm{Re}(\delta)>0$, $\mathrm{Re}(1+\frac{h}{1-\alpha})>0$, $\mathrm{Re}(\rho)>0$ *and* $\alpha<1$, $b\in\mathrm{Re}$, $c\in\mathrm{Re}$. *Then for the pathway fractional integral* $P_{0_+}^{(\eta,\alpha)}$ *the following formula hold for the image of product of two* I*-functions:*

$$\left(P_{0_+}^{(\eta,\alpha)}t^{\rho-1}I^{M,N}_{P_l,Q_l:R}\left[ct^{\delta}|\begin{pmatrix}(e_j,E_j)_{1,N};(e_{jl},E_{jl})_{N+1,,P_l}\\(f_j,F_j)_{1,M};(f_{jl},F_{jl})_{M+1,Q_l}\end{pmatrix}\right]I^{m,n}_{p_i,q_i:r}\left[bt^{\beta}|\begin{pmatrix}(a_j,\alpha_j)_{1,N};(a_{ji},\alpha_{ji})_{N+1,,P_i}\\(b_j,\beta_j)_{1,M};(b_{ji},\beta_{ji})_{M+1,Q_i}\end{pmatrix}\right]\right)$$
$$=\frac{x^{\eta+\rho}\Gamma(1+\frac{\eta}{1-\alpha})}{[a(1-\alpha)]^{\rho}}I^{M,N}_{P_I,Q_I:R}\left[\frac{cx^{\delta}}{[(a(1-\alpha))]^{\delta}}|\begin{pmatrix}(e_j,E_j)_{1,N};(e_{jl},E_{jl})_{N+1,,P_l}\\(f_j,F_j)_{1,M};(f_{jl},F_{jl})_{M+1,Q_l}\end{pmatrix}\right]$$
$$\times I^{m,n+1}_{p_i+1,q_i+1:r}\left[\frac{bx^{\beta}}{[a(1-\alpha)]^{\beta}}|\begin{pmatrix}(1-\rho+\delta\xi,\beta),(a_1,\alpha_1),\ldots,(a_{p_i},\alpha_{p_i})\\(b_1,\beta_1),\ldots\ldots,(b_{q_i},\beta_{q_i}),(-\rho+\delta\xi-\frac{\eta}{1-\alpha},\beta)\end{pmatrix}\right]. \quad (2.4)$$

Proof. Let

$$\Delta=\left(P_{0_+}^{(\eta,\alpha)}t^{\rho-1}I^{M,N}_{P_l,Q_l:R}\left[ct^{\delta}|\begin{pmatrix}(e_j,E_j)_{1,N};(e_{jl},E_{jl})_{N+1,,P_l}\\(f_j,F_j)_{1,M};(f_{jl},F_{jl})_{M+1,Q_l}\end{pmatrix}\right]\right)$$
$$\times\left(I^{m,n}_{p_i,q_i:r}\left[bt^{\beta}|\begin{pmatrix}(a_j,\alpha_j)_{1,N};(a_{ji},\alpha_{ji})_{N+1,,P_i}\\(b_j,\beta_j)_{1,M};(b_{ji},\beta_{ji})_{M+1,Q_i}\end{pmatrix}\right]\right).$$

Now using the definitions (1.1), (2.1) and (2.3), we have

$$\Delta=x^{\eta}\left\{\int_0^{[\frac{x}{a(1-\alpha)}]}\left[1-\frac{a(1-\alpha)t}{x}\right]^{\frac{\eta}{1-\alpha}}t^{\rho-1}\sum_{h=1}^{N}\sum_{\nu=0}^{\infty}\frac{(-1)^{\upsilon}X(\xi)}{\upsilon!E_h}(\frac{1}{ct^{\delta}})^{\xi}\frac{1}{2\pi i}\int_L\varphi(s)(bt^{\beta})^{-s}ds\right\}dt.$$

Interchanging the order of integration, using a known result [15] and evaluating the integral with the help of beta function formula, we obtain

$$\Delta=\frac{x^{\eta+\rho}\Gamma(1+\frac{\eta}{1-\alpha})}{[a(1-\alpha)]^{\rho}}\sum_{h=1}^{N}\sum_{\upsilon=0}^{\infty}\frac{(-1)^{\upsilon}X(\xi)}{\upsilon!E_h}\left[\frac{cx^{\delta}}{[a(1-\alpha)]^{\delta}}\right]^{-\xi}$$
$$\times\frac{1}{2\pi i}\int_L\varphi(s)\frac{\Gamma(\rho-\delta\xi-\beta s)}{\Gamma(1+\rho+\frac{\eta}{1-\alpha}-\delta\xi-\beta s)}\left(\frac{bx^{\beta}}{[a(1-\alpha)]^{\beta}}\right)ds$$
$$=\frac{x^{\eta+\rho}\Gamma(1+\frac{\eta}{1-\alpha})}{[a(1-\alpha)]^{\rho}}I^{M,N}_{P_l,Q_I:R}\left[\frac{cx^{\delta}}{[(a(1-\alpha))]^{\delta}}|\begin{pmatrix}(e_j,E_j)_{1,N};(e_{jl},E_{jl})_{N+1,,P_l}\\(f_j,F_j)_{1,M};(f_{jl},F_{jl})_{M+1,Q_l}\end{pmatrix}\right]$$
$$\times I^{m,n+1}_{p_i+1,q_i+1:r}\left[\frac{bx^{\beta}}{[a(1-\alpha)]^{\beta}}|\begin{pmatrix}(1-\rho+\delta\xi,\beta),(a_1,\alpha_1),\ldots,(a_{p_i},\alpha_{p_i})\\(b_1,\beta_1),\ldots\ldots,(b_{q_i},\beta_{q_i}),(-\rho+\delta\xi-\frac{\eta}{1-\alpha},\beta)\end{pmatrix}\right].$$

This completes the proof of Theorem 1. □

3 Pathway Integral Operator Involving the Product of an *I*-Function and Mittag-Leffler Functions

The Swedish mathematician Mittag-Leffler introduced the function $E_{\beta}(z)$ [5] defined as:

$$E_{\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\beta k+1)}, \quad (3.1)$$

where z is a complex variable and $\Gamma(*)$ is a gamma function $\beta \geq 0$. The Mittag-Leffler function is the direct generalization of the exponential function to which it reduces for $\beta = 1$. For $0 < \beta < 1$, it interpolates between the pure exponential and a hypergeometric function $\frac{1}{1-z}$. Mittag-Leffler function naturally occurs as the solution of the fractional order differential and integral equations.

Wiman [20] studied the generalization of $E_\beta(z)$, that is given by

$$E_{\beta,\rho}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta k + \rho)}, \beta, \rho \in \mathbb{C}, \mathrm{Re}(\rho) > 0, \mathrm{Re}(\beta) > 0, \tag{3.2}$$

which is known as Wiman's function.

Prabhakar [16] investigated the function $E^{\gamma}_{\beta,\rho}(z)$ as

$$E^{\gamma}_{\beta,\rho}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\beta k + \rho)} \frac{(z)^k}{\Gamma(\beta k + \rho)}, \tag{3.3}$$

where $\rho, \beta, \gamma \in \mathbb{C}, \mathrm{Re}(\beta) > 0, \mathrm{Re}(\gamma) > 0, \mathrm{Re}(\rho) > 0$.

The Mittag-Leffler type functions belong to I-functions family, since they can be represented in terms of the I-function:

$$E_\beta(z) = I^{1,1}_{1,2:1} \left[-z | \begin{pmatrix} (0,1) \\ (0,1),(0,\beta) \end{pmatrix}, \beta \right], \beta \in \mathbb{C}, \mathrm{Re}(\beta) > 0, \tag{3.4}$$

$$E_{\beta,\rho}(z) = I^{1,1}_{1,2:1} \left[-z | \begin{pmatrix} (0,1) \\ (0,1),(1-\rho,\beta) \end{pmatrix} \right], \beta, \rho \in \mathbb{C}, \mathrm{Re}(\beta) > 0, \tag{3.5}$$

$$E^{\gamma}_{\beta,\rho}(z) = \frac{1}{\Gamma(\gamma)} I^{1,1}_{1,2:1} \left[-z | \begin{pmatrix} (1-\gamma,1) \\ (0,1),(1-\rho,\beta) \end{pmatrix} \right] = \frac{1}{2\pi i \Gamma\gamma} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(s)\Gamma(\gamma-s)}{\Gamma(\beta-\alpha s)} (-z)^{-s} ds, \tag{3.6}$$

where $\gamma, \beta, \rho \in \mathbb{C}, \mathrm{Re}(\beta) > 0$.

Theorem 3.1. *Let $\eta, \gamma, \rho \varepsilon \mathbb{C}$, $\mathrm{Re}(\eta) > 0$, $\mathrm{Re}(\gamma) > 0$, $\mathrm{Re}(\rho) > 0$, $\mathrm{Re}(1 + \frac{\eta}{1-\alpha}) > \max[0, -\mathrm{Re}(\rho)]$, $b \in \mathrm{Re}$, $c \in \mathrm{Re}$, $\beta > 0$, $\delta > 0$ and $\alpha < 1$. Then the image of the product of an I-function and Mittag-Leffler function under the pathway integral operator*

$$\begin{aligned} &\left(P^{(\eta,\alpha)}_{0_+} t^{\rho-1} I^{M,N}_{P_l,Q_l:R} \left[ct^\delta | \begin{pmatrix} (e_j,E_j)_{1,N}; (e_{jl},E_{jl})_{N+1,P_l} \\ (f_j,F_j)_{1,M}; (f_{jl},F_{jl})_{M+1,Q_l} \end{pmatrix} \right] E^{\gamma}_{\beta,\rho}(bt^\delta) \right) \\ &\quad = \frac{1}{\Gamma\gamma} \frac{x^{\eta+\rho}\Gamma(1+\frac{\eta}{1-\alpha})}{[a(1-\alpha)]^\rho} I^{M,N}_{P_i,Q_i:R} \left[\frac{cx^\delta}{[a(1-\alpha)]^\delta} | \begin{pmatrix} (e_j,E_j)_{1,N}; (e_{ji},E_{ji})_{N+1,P_i} \\ (f_j,F_j)_{1,M}; (f_{ji},F_{ji})_{M+1,Q_i} \end{pmatrix} \right] \\ &\qquad \times {}_2\psi_2 \left[\begin{pmatrix} (\rho-\delta\xi,\beta),(\gamma,1) \\ (\rho,\beta),(1+\rho+\frac{\eta}{1-\alpha}-\delta\xi,\beta) \end{pmatrix} | \frac{bx^\beta}{[a(1-\alpha)]^\beta} \right], \end{aligned} \tag{3.7}$$

where ${}_2\psi_2(x)$ is the Wright's generalized hypergeometric function.

Proof. This result can be derived from Theorem 1, by putting $m = 1$,$n = 1$, $p = 1$, $q =$, $b_1 = 0$, $\beta_1 = 1$, $b_2 = 1-\rho$,$\beta_2 = \beta$, $a_1 = 1-\gamma$, $\alpha_1 = 1$, $b = -b$, and $r = 1$, then (2.4) reduced to

$$\left(P^{(\eta,\alpha)}_{0_+} t^{\rho-1} I^{M,N}_{P_l,Q_l:R} \left[ct^\delta | \begin{pmatrix} (e_j,E_j)_{1,N}; (e_{jl},E_{jl})_{N+1,P_l} \\ (f_j,F_j)_{1,M}; (f_{jl},F_{jl})_{M+1,Q_l} \end{pmatrix} \right] I^{1,1}_{1,2:1} \left[-bt^\delta | \begin{pmatrix} (1-\gamma,\alpha) \\ (0,1),(1-\rho,\beta) \end{pmatrix} \right] \right)$$

$$= \frac{x^{\eta+\rho}\Gamma(1+\frac{\eta}{1-\alpha})}{[a(1-\alpha)]^{\rho}} I^{M,N}_{P_l,Q_l:R}\left[\frac{cx^{\delta}}{[a(1-\alpha)]^{\delta}}\Big|\begin{pmatrix}(e_j,E_j)_{1,N};(e_{jl},E_{jl})_{N+1,P_l}\\(f_j,F_j)_{1,M};(f_{jl},F_{jl})_{M+1,Q_l}\end{pmatrix}\right]$$
$$\times I^{1,2}_{2,3:1}\left[-\frac{bx^{\beta}}{[a(1-\alpha)]^{\beta}}\Big|\begin{pmatrix}(1-\rho+\delta\xi,\beta),(1-\gamma,1)\\(0,1),(1-\rho,\beta),(-\rho+\delta\xi-\frac{\eta}{1-\alpha},\beta)\end{pmatrix}\right]. \quad (3.8)$$

Equation (3.8) can be written in terms of Wright's generalized hypergeometric function as follows:

$$\left(P^{(\eta,\alpha)}_{0_+} t^{\rho-1} I^{M,N}_{P_l,Q_l:R}\left[ct^{\delta}\Big|\begin{pmatrix}(e_j,E_j)_{1,N};(e_{jl},E_{jl})_{N+1,P_l}\\(f_j,F_j)_{1,M};(f_{jl},F_{jl})_{M+1,Q_l}\end{pmatrix}\right] E^{\gamma}_{\beta,\rho}(bt^{\delta})\right)$$
$$= \frac{1}{\Gamma\gamma}\frac{x^{\eta+\rho}\Gamma(1+\frac{\eta}{1-\alpha})}{[a(1-\alpha)]^{\rho}} I^{M,N}_{P_i,Q_i:R}\left[\frac{cx^{\delta}}{[a(1-\alpha)]^{\delta}}\Big|\begin{pmatrix}(e_j,E_j)_{1,N};(e_{ji},E_{ji})_{N+1,P_i}\\(f_j,F_j)_{1,M};(f_{ji},F_{ji})_{M+1,Q_i}\end{pmatrix}\right]$$
$$\times {}_2\psi_2\left[\begin{pmatrix}(\rho-\delta\xi,\beta),(\gamma,1)\\(\rho,\beta),(1+\rho+\frac{\eta}{1-\alpha}-\delta\xi,\beta)\end{pmatrix}\Big|\frac{bx^{\beta}}{[a(1-\alpha)]^{\beta}}\right]. \quad (3.9)$$

This completes the proof of Theorem 2. □

4 Pathway Integral Operator Involving the Product of an I-Function and Bessel Functions

The Bessel function of the first kind $J_\nu(x)$ is defined for complex $x \in \mathbb{C}$, $\neq 0$ and $\nu \in \mathbb{C}$, $\mathrm{Re}(\nu) > -1$ by

$$J_\nu(x) = (\frac{x}{2})^{\nu}\sum_{k=0}^{\infty}\frac{(-1)^k(\frac{x}{2})^k}{\Gamma(\nu+k+1)k!}, \quad (4.1)$$

and its I-function representation is given by

$$J_\nu(x) = (\frac{x}{2})^{\nu} I^{1,0}_{0,2;1}\left[\frac{x^2}{4}\Bigg|\begin{pmatrix}\\(0,1),(-\nu,1)\end{pmatrix}\right], \nu\in\mathbb{C}, \mathrm{Re}(\nu)>0. \quad (4.2)$$

Here we find the pathway integral image in the following Theorem:

Theorem 4.1. *Let $\gamma,\eta,\nu \in \mathbb{C}$, $\mathrm{Re}(\eta) > 0$, $\mathrm{Re}(1+\frac{\eta}{1-\alpha}) > 0$, $\mathrm{Re}(\gamma+\nu) > 0$, $c \in \mathrm{Re}$, $\delta > 0$ and $\alpha < 1$. Let $P^{(\eta,\alpha)}_{0_+}$ be the pathway fractional integral. Then there holds the image*

$$\left(P^{(\eta,\alpha)}_{0_+}(\frac{t}{2})^{\gamma-1} I^{M,N}_{P_i,Q_i:R}\left[c(\frac{t}{2})^{\delta}\Big|\begin{pmatrix}(e_j,E_j)_{1,N},(e_{ji},E_{ji})_{N+1,P_i}\\(f_j,F_j)_{1,M},(f_{ji},F_{ji})_{M+1,Q_i}\end{pmatrix}\right] J_\nu(t)\right)$$
$$= \frac{x^{\eta+\gamma+\nu}\Gamma(1+\frac{\eta}{1-\alpha})}{[a(1-\alpha)]^{\gamma+\nu}x^2} I^{M,N}_{P_i,Q_i:R}\left[\frac{cx^{\delta}}{[a(1-\alpha)]^{\delta}}\Big|\begin{pmatrix}(e_j,E_j)_{1,N};(e_{ji},E_{ji})_{N+1,P_i}\\(f_j,F_j)_{1,M};(f_{ji},F_{ji})_{M+1,Q_i}\end{pmatrix}\right]$$
$$\times {}_1\psi_2\left[\begin{pmatrix}(\gamma+\nu-\delta\xi)\\(\nu+1,1),(1+\nu,\gamma+\frac{\eta}{1-\alpha}-\delta\xi)\end{pmatrix}\Big|-(\frac{x^2}{4a^2(1-\alpha)^2})\right]. \quad (4.3)$$

where ${}_1\psi_2(x)$ denotes the Wright's generalized hypergeometric function.

Proof. By putting $m=1$, $n=0$, $p=0$, $q=22$, $b_1=0$, $\beta_1=1$, $b_2=-\nu$, $\beta_2=1$, $\rho=\gamma+\nu$, $\beta=2$, $b=1$, $r=1$ and replacing t by $t/2$ in (2.4), we get

$$\left(P^{(\eta,\alpha)}_{0_+}(\frac{t}{2})^{\nu+\gamma-1} I^{M,N}_{P_l,Q_l:R}\left[c(\frac{t}{2})^{\delta}\Big|\begin{pmatrix}(e_j,E_j)_{1,N};(e_{jl},E_{jl})_{N+1,P_l}\\(f_j,F_j)_{1,M};(f_{jl},F_{jl})_{M+1,Q_l}\end{pmatrix}\right] I^{1,0}_{0,2:1}\left[\frac{t^2}{4}\Big|\begin{pmatrix}\\(0,1),(-\nu,1)\end{pmatrix}\right]\right)$$

$$= \frac{x^{\eta+\gamma+\nu}\Gamma(1+\frac{\eta}{1-\alpha})}{[a(1-\alpha)]^{\gamma+\nu}x^2} I^{M,N}_{P_l,Q_l:R}\left[\frac{cx^\delta}{[a(1-\alpha)]^\delta}|\begin{pmatrix}(e_j,E_j)_{1,N};(e_{jl},E_{jl})_{N+1,P_l}\\(f_j,F_j)_{1,M};(f_{jl},F_{jl})_{M+1,Q_l}\end{pmatrix}\right]$$

$$\times I^{1,1}_{1,3:1}\left[\frac{x^2}{4a^2(1-\alpha)^2}|\begin{pmatrix}(1-\gamma-\nu+\delta\xi,2)\\(0,1),(-\nu,1),(-\nu-\gamma+\delta\xi-\frac{\eta}{1-\alpha},2)\end{pmatrix}\right]. \quad (4.4)$$

Equation (4.2) can be represented in the form of Wright's generalized hypergeometric function as follows:

$$\left(P^{(\eta,\alpha)}_{0_+}(\frac{t}{2})^{\gamma-1}I^{M,N}_{P_l,Q_l:R}[c(\frac{t}{2})^\delta|\begin{pmatrix}(e_j,E_j)_{1,N};(e_{jl},E_{jl})_{N+1,P_l}\\(f_j,F_j)_{1,M};(f_{jl},F_{jl})_{M+1,Q_l}\end{pmatrix}]J_\nu(t)\right)$$

$$= \frac{x^{\eta+\gamma+\nu}\Gamma(1+\frac{\eta}{1-\alpha})}{[a(1-\alpha)]^{\gamma+\nu}x^2} I^{M,N}_{P_l,Q_l:R}\left[\frac{cx^\delta}{[a(1-\alpha)]^\delta}|\begin{pmatrix}(e_j,E_j)_{1,N};(e_{jl},E_{jl})_{N+1,P_l}\\(f_j,F_j)_{1,M};(f_{jl},F_{jl})_{M+1,Q_l}\end{pmatrix}\right]$$

$$\times {}_1\psi_2\left[\begin{pmatrix}(\gamma+\nu-\delta\xi,)\\(\nu+1,1),(1+\nu,\gamma+\frac{\eta}{1-\alpha}-\delta\xi,)\end{pmatrix}| - \frac{x^2}{4a^2(1-\alpha)^2}\right]$$

This completes the proof of Theorem 3. □

The importance of our results lies in their manifold generality. In view of the generality of I-function, on specializing the various parameters, we can obtain from our main result, several results containing remarkably wide variety of useful functions and their various special cases.

Special Cases

1. On choosing $r = R = 1$ in (2.4), I-function will change into H- function and we obtain the result of Chaurasia and Gill [1].
2. On putting $r = R = 1$ in (3.7), the result so obtain is in the form of Chaurasia and Gill [1].
3. While choosing $r = R = 1$ in (4.2), I-function will change into H-function and we obtain the result of Chaurasia and Gill [1].

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