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ON EINSTEIN $(\alpha, \beta)-$METRIC

By

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Abstract

The purpose of the present paper is to find the necessary and sufficient condition for the special $(\alpha, \beta)-$metric $L = \frac{b_1\alpha^2+b_2\alpha\beta+b_3\beta^2}{a_1\alpha+a_2\beta}$ to be an Einstein metric. Here α is a Riemannian metric and β is a constant Killing one-form.

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1 Introduction

A Finsler space is a manifold M equipped with a family of smoothly varying Minkowski norms, one on each tangent space. Riemannian metrics are examples of Finsler norms that arise from an inner product. A Finsler metric function $L(x, y)$ is called an $(\alpha, \beta)-$ metric if L is a positively homogeneous function of a Riemannian metric $\alpha(x, y) = (a_{ij}(x, y)y^iy^j)^{1/2}$ and a differential 1-form $\beta(x, y) = b_i(x)y^i$ of degree one. The especially interesting examples of $(\alpha, \beta)-$ metrics are Randers and Kropina metrics. Randers metric and its Ricci tensor are related by their histories in Physics. The well known Ricci tensor was introduced in 1904 by G. Ricci. Nine years later Ricci's work was used to formulate Einstein's gravitation theory [1].

Einstein metrics are defined in the next section but, loosely, we will say a Finsler metric F is Einstein if the average of its flag curvatures at a flag pole y is a function of position x alone, rather than the priori position x and flag pole y. C. Robles investigated Randers Einstein's metrics in her Ph.D. thesis in 2003. She obtained necessary and sufficient condition for Randers metric to be Einstein and by using Einstein navigation description; she proved the second Schur lemma [7].

The classification of projectively related Einstein Finsler metrics on compact manifold is investigated in [9, 10]. In [8] authors have considered the $(\alpha, \beta)-$metrics such as generalized Kropina, Matsumoto and $\frac{(\alpha+\beta)^2}{\alpha}$ metrics and have obtain the necessary and sufficient condition for them to be Einstein metrics when β is a constant Killing form. Also, they have proved under these conditions the above metrics are Riemannian or Ricci flat.

In this paper we consider the $(\alpha, \beta)-$metric $L = \frac{b_1\alpha^2+b_2\alpha\beta+b_3\beta^2}{a_1\alpha+a_2\beta}$, where a's and b's are constants. It is obvious that by homothetic change of α and β this kind of metric may be classified as follows:

$a_1 \neq 0$, $a_2 = 0$, we have the Randers metric $L = \alpha + \beta$ and

$$L = c_1\alpha + c_2\beta + \beta^2/\alpha \tag{1.1}$$

$a_1 = 0$, $a_2 \neq 0$, we have the Randers metric $L = \alpha + \beta$ and

$$L = c_1\alpha + c_2\beta + \alpha^2/\beta. \tag{1.2}$$

2 Preliminaries

Let M be an n-dimensional C^∞ manifold. Denote by T_xM the tangent space at $x \in M$, the tangent space at $x \in M$, and by $TM = \cup_{x\in M} T_X M$ the tangent space of M. Each element of TM has the form (x, y), where $x \in M$ and $y \in T_xM$. Let $TM_0 = TM\backslash\{0\}$. The natural projection $\pi : TM \to M$ is given by $\pi(x, y) = x$. The pull back tangent bundle π^*TM is a vector bundle over TM_0 whose fiber π_v^*TM at $v \in TM_0$ is just T_xM, where $\pi(v) = x$. Then

$$\pi^*TM = \{(x, y, v)|y \in T_xM_0,\ v \in T_xM\}.$$

A Finsler metric on a manifold M is a function $F : TM \to [0, \infty]$, having the following properties:

(1) F is C^∞ on TM_0;
(2) $F(x, \lambda y) = \lambda F(x, y), \quad \lambda > 0$;
(3) For any tangent vector $y \in T_xM$, the vertical Hessian of $\frac{F^2}{2}$ given by

$$g_{ij}(x, y) = \left[\frac{1}{2}F^2\right]_{y^iy^j},$$

is positive definite [2].

Every Finsler metric F induces a spray $G = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y)\frac{\partial}{\partial y^i}$ which is defined by

$$G^i(x, y) = \frac{1}{4}g^{il}(x, y)\big\{2\frac{\partial g_{jl}}{\partial x^k}(x, y) - \frac{\partial g_{jk}}{\partial x^l}(x, y)\big\}\, y^jy^k,$$

where the matrix (g^{ij}) means the inverse of the matrix (g_{ij}) [11]. The coefficients G^i_j, G^i_{jk} of the berwald connection can be derived from the spray G^i as follows:

$$G^i_j = \frac{\partial G^i}{\partial y^j}, \quad G^i_{jk} = \frac{\partial G^i_j}{\partial y^k}.$$

The Riemannian curvature $K_y = K^i_k dx^k \times \frac{\partial}{\partial x^i}|_p : T_pM \to T_pM$ is defined by

$$K^i_k(y) = 2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^j \partial y^k}y^j + 2G^j\frac{\partial^2 G^i}{\partial y^i \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}. \tag{2.1}$$

When $F = (a_{ij}(x)y^iy^j)^{1/2}$ is Riemannian metric, $K^i_k = R^i_{jkl}(x)\dot{y}^jy^l$, where $R^i_{jkl}(x)$ denote the coefficients of the usual Riemannian curvature tensor. Thus the quantity K_y in Finsler geometry is still called the Riemannian curvature [12]. The Ricci scalar function is given by $\rho = \frac{1}{F^2}K^i_i$.

Therefore the Ricci scalar function is positive homogeneous of degree 0 in y [8]. This means $\rho(x, y)$ depends on the direction of the flag pole y, but not its length. The Ricci tensor of a Finsler metric F is defined by $\mathrm{Ric}_{ij}\big\{\frac{1}{2}K^m_m\big\}_{y^iy^j}$.

Ricci-flat manifold is manifold whose Ricci tensor vanishes. In Physics, Riemannian Ricci-flat manifolds are important, because they represent vacuum solutions to Einstein's equations.

A Finsler metric is said to be an Einstein metric if the Ricci scalar function is a function of x alone, or equivalently [3].

$$\mathrm{Ric}_{ij} = \rho(x)g_{ij}.$$

Ricci-flat manifolds are special cases of Einstein manifolds.

Let (M^n, L) be an n-dimensional Finsler space equipped with an $(\alpha, \beta)-$ metric L, where

$$\alpha(y) = (a_{ij}(x)y^i y^j)^{1/2}, \qquad \beta(y) = b_i(x)y^i.$$

M. Matsumoto [6] showed that G^i for $(\alpha, \beta)-$metric is given by

$$2G^i = \gamma^i_{00} + 2B^i, \tag{2.2}$$

where

$$B^i = \Big(\frac{E}{\alpha}\Big)y^i + \Big(\alpha\frac{L_\beta}{L_\alpha}\Big)s^i_0 - \Big(\alpha\frac{L_{\alpha\alpha}}{L_\alpha}\Big)C\{\Big(\frac{y^i}{\alpha}\Big) - \Big(\frac{\alpha}{\beta}\Big)b^i\},$$

$$E = \frac{\beta L_\beta}{L}C,$$

$$C = \frac{\alpha\beta(r_{00}L_\alpha - 2\alpha s_0 L_\beta)}{2(\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha})},$$

$$b^i = a^{ir}b_r,$$

$$b^2 = b^r b_r \gamma^2 = b^2\alpha^2 - \beta^2,$$

$$r_{ij} = \frac{1}{2}(b_{i|j} + b_{j|i}),$$

$$s_{ij} = \frac{1}{2}(b_{i|j} - b_{j|i}),$$

$$s^i_j = a^{ih}s_{hj}, \quad s_j = b_i s^i_j.$$

The symbol "|" in the above formula stands for the h-covariant derivation with respect to the Riemannian connection in the space (M, α), and the matrix (a^{ij}) means the inverse of the matrix (a_{ij}). The functions γ^i_{jk} stands for the Chirstoffel symbols in the space (M, α), and the suffix '0' means transvecting by y^i. Putting

$$p = \beta(\beta L_\alpha L_\beta - \alpha L L_{\alpha\alpha})/2L(\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha}), \tag{2.3}$$

$$q = -\alpha\beta L_\beta(\beta L_\alpha L_\beta - \alpha L L_{\alpha\alpha})/L L_\alpha(\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha}), \tag{2.4}$$

$$r = \frac{\alpha L_\beta}{L_\alpha}, \tag{2.5}$$

$$\overline{s_0} = \alpha^3 L_{\alpha\alpha}/2(\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha}), \tag{2.6}$$

$$t = -\alpha^4 L_{\alpha\alpha}L_\beta/L_\alpha(\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha}), \tag{2.7}$$

we get

$$B^i = (pr_{00} + q\overline{s_0})y^i + rs^i_0 + (\overline{s_0}r_{00} + t\overline{s_0})\,b^i. \tag{2.8}$$

Substituting (2.8) in (2.2) we get G^i, and using this value of and G^i in (2.1), we obtain Berwald's formula in split and covariantized form:

$$K^i_k(y) = \overline{K}^i_k + \{2B^i_{|k} - y^j(B^i_{|j})_{y^k} - (B^i)_{y^j}(B^j)_{y^k} + 2B^j(B^i)_{y^jy^k}\}. \tag{2.9}$$

where $\overline{K}^i_i$ is the spray curvature of the Riemannian metric a_{ij}. The one form β is said to be Killing (closed) one form if $r_{ij} = 0 (s_{ij} = 0)$. ? is said to be constant killing form if it is Killing and has constant length with respect to α, equivalently $r_{ij} = s_{ij} = 0$.

Example. Let $L = \alpha + \beta$ be the family of Randers metrics on S^3 constructed in [4]. β satisfies that $r_{ij} = 0$ and $s_{ij} = 0$.

3 The $(\alpha, \beta)-$Metric (1.1)

From the $(\alpha, \beta)-$metric (1.1), we have

$$L_\alpha = \frac{c_1\alpha^2 - \beta^2}{\alpha^2}, \qquad L_\beta = \frac{c_2\alpha + 2\beta}{\alpha}, \tag{3.1}$$

Substituting (3.1) in (2.5) we get

$$r = \frac{\alpha^2(c_2\alpha + 2\beta)}{c_1\alpha^2 - \beta^2}, \tag{3.2}$$

Now we suppose that ? is a constant Killing form, then by substituting (3.2) in (2.8), we have

$$B^i = \frac{\alpha^2(c_2\alpha + 2\beta)}{c_1\alpha^2 - \beta^2}\, s^i_0, \tag{3.3}$$

From (3.3) we obtain

$$B^i_{|j} = \frac{\alpha^2(c_2\alpha + 2\beta)}{c_1\alpha^2 - \beta^2}\, s^i_{0|j} + \frac{2\alpha^3(c_1\alpha + c_2\beta) + 2\alpha^2\beta^2}{(c_1\alpha^2 - \beta^2)^2}\, b_{0|j}s^i_0, \tag{3.4}$$

$$B^i_{.j} = \Big[\frac{c_1c_2\alpha^3 - \beta^2(3c_2\alpha - 4\beta)}{(c_1\alpha^2 - \beta^2)^2} y_j + \frac{2\alpha^2(c_1\alpha^2 + \beta^2 + c_\beta\alpha\beta)}{(c_1\alpha^2 - \beta^2)^2} b_j\Big] s^i_0$$
$$+ \frac{\alpha^2(c_2\alpha + 2\beta)}{c_1\alpha^2 - \beta^2} s^i_j. \tag{3.5}$$

where $B^i_{.j} = B^i_{y^j}$. From (3.5) we get

$$B^i_{.j.i} = 0, \tag{3.6}$$

and

$$B^j_{.i}B^i_{.j} = \frac{2\{c_1c_2\alpha^3 - \beta^2(3c_2\alpha - 4\beta)\}\{\alpha^2(c_2\alpha + 2\beta\}}{(c_1\alpha^2 - \beta^2)^3} s^i_0 s_{0i} + \frac{\alpha^4(c_2\alpha + 2\beta)^2}{(c_1\alpha^2 - \beta^2)^2} s^{ij}s_{ij}, \tag{3.7}$$

Differentiating (3.4) by y^i and then transvecting this derivative by y^j, we get

$$y^j(B^i_{|j})_{,i} = 0. \tag{3.8}$$

Using (3.4), (3.6),(3.7) and (3.8) in Berwald's formula (2.9), we have

$$K_i^i = K_i^i + \frac{2\alpha^2(c_2\alpha + 2\beta)}{(c_1\alpha^2 - \beta^2)} s^i_{0|i} + \frac{2\alpha^4(c_2\alpha + 2\beta)^2}{(c_1\alpha^2 - \beta^2)^2} s^{ij} s_{ij}$$

$$+ \frac{2\{c_1c_2\alpha^3 - \beta^2(3c_2\alpha - 4\beta)\}\{\alpha^2(c_2\alpha + 2\beta\}}{(c_1\alpha^2 - \beta^2)^3} s_0^i s_{0i}, \tag{3.9}$$

As a result of (3.9), we can derive the spray curvature of the metric (1.1) with constant Killing vector.

4 Einstein Criterion for Metric (1.1)

In this section we assume that Ricci scalar of the $(\alpha, \beta)-$metric (1.1) which is function of x alone, i.e. L is Einstein. We have $L^2\mathrm{Ric}(x) = K_i^i$, so we can derive necessary and sufficient conditions for metric (1.1) to be Einstein. From (3.9), we have

$$0 = \overline{\mathrm{Ric}}_{00} + \frac{2\alpha^2(c_2\alpha + 2\beta)}{(c_1\alpha^2 - \beta^2)} s^i_{0|i} + \frac{2\alpha^4(c_2\alpha + 2\beta)^2}{(c_1\alpha^2 - \beta^2)^2} s^{ij} s_{ij}$$

$$+ \frac{2\{\alpha^2(c_2\alpha + 2\beta)\}\{c_1c_2\alpha^3 - \beta^2(3c_2\alpha - 4\beta)\}}{(c_2\alpha + 2\beta)^3} s_0^i s_{0i} - \left(c_1\alpha + c_2\beta + \frac{\beta^2}{\alpha}\right)^2 \mathrm{Ric}(x), \tag{4.1}$$

Multiplication of (4.1) by $\alpha^2(c_1\alpha^2 - \beta^2)^3$ removes y from the denominator in (4.1). We then obtain

$$\text{Rat part} + \text{Irrat part} = 0,$$

where Rat part and Irrat part are respectively polynomial in y of degree 10 and 9 as given bellow:

Where

$$\begin{aligned}
\text{Rat part} =& (c_1^3\alpha^8 - \alpha^2\beta^6 - 3c_1^2\alpha^6\beta^2 + 3c_1\alpha^4\beta^4)(Ric)?_00 + (4c_1^2\beta\alpha^8 + 4\alpha^4\beta^5 \\
&- 8c_1\alpha^6\beta^3)s^i_{0|i} + (2c_1c_2^2\alpha^{10} + 8c_1\alpha^8\beta^2 - 2c_2^2\alpha^2\beta^8 - 8\alpha^6\beta^4)s^{ij}s_{ij} \\
&- (c_1^5\alpha^{10} + c_1^3c_2^2\beta^2\alpha^8 + c_1^3\alpha^6\beta^4 + 2c_1^4\alpha^8\beta^2 - c_1^2\alpha^4\beta^6 - c_2^2\beta^8\alpha^2 \\
&- \beta^{10} + 2c_1\alpha^2\beta^8 - 3c_1^4\alpha^8\beta^2 - 3c_1^2c_2^2\alpha^6\beta^4 - 3c_1^2\alpha^4\beta^6 - 6c_1^3\alpha^6\beta^4 \\
&+ 3c_1^3\alpha^6\beta^4 + 3c_1c_2^2\alpha^4\beta^6 + 3c_1\alpha^2\beta^8 + 6c_1^2\alpha^4\beta^6)\mathrm{Ric}(x) \\
&+ (3c_1c_2^2\alpha^8 - 6c_2^2\alpha^6\beta^2 + 16\alpha^4\beta^4)s_0^i s_{0i},
\end{aligned}$$

and

$$\begin{aligned}
\text{Irrat part} =& (2c_1c_2\alpha^8 + 2c_2\alpha^4\beta^4 - 4c_1c_2\alpha^6\beta^2)s^i_{0|i} + (8c_1c_2\alpha^8\beta - 8c_2\alpha^6\beta^3)s^{ij}. \\
& s_{ij} + (8c_2\alpha^4\beta^3 + 4c_1c_2\beta\alpha^6 - 12c_2\alpha^4\beta^3)s_0^i s_{0i} + (2c_1^4c_2\beta\alpha^8 \\
&+ 2c_1^3c_2\alpha^6\beta^3 - 2c_1c_2\alpha^2\beta^7 - 2c_2\beta^9 - 6c_1^2\alpha^4\beta^5 - 6c_1^3c_2\alpha^6\beta^3 \\
&+ 6c_2c_1^2\alpha^4\beta^5 + 6c_1c_2\alpha^2\beta^7)\mathrm{Ric}(x).
\end{aligned}$$

Lemma 4.1. *In metric (1.1) with constant Killing form β is Einstein iff Rat part $= 0$ and Irrat part $= 0$.*

Notice that Rat part $= 0$ shows that α^2 divides either β^{10} or Ric (x). Since α^2 is an irreducible polynomial of degree two in y and β^{10} factors into ten linear terms, it follows that

α^2 divides Ric(x). Therefore Ric(x) = $c\alpha^2$, where c is a constant by the Riemannian Schur lemma. We can get some more information out of the two equations Rat part = 0 and Irrat part = 0 in Lemma 4.1. Replacing every Ric(x) by $c\alpha^2$ and dividing Rat part = 0 through by α^2, we get a relation of the form

$$0 = \alpha^2 M_1 + N_1, \tag{4.2}$$

where

$$\begin{aligned} M_1 =&(c_1^3\alpha^4 - 3c_1^2\alpha^2\beta^2 + 3c_1\beta^4)\overline{\mathrm{Ric}}_{00} + (4c_1^2\beta\alpha^4 + 4\beta^5 - 8c_1\alpha^2\beta^3)s^i_{0|i} \\ &+ (2c_1c_2^2\alpha^6 + 8c_1\alpha^4\beta^2 - 8\alpha^2\beta^6)s^{ij}s_{ij} - (c_1^5\alpha^6 + c_1^3c_2^2\beta^2\alpha^4 + c_1^3\alpha^2\beta^4 \\ &+ 2c_1^4\alpha^4\beta^2 - c_1^2\beta^6 - 3c_1^4\alpha^4\beta^2 - 3c_1^2c_2^2\alpha^2\beta^4 - 3c_1^2\beta^6 - 6c_1^3\alpha^2\beta^4 \\ &+ 3c_1^3\alpha^2\beta^4 + 3c_1c_2^2\beta^6 + 6c_1^2\beta^6)c + (3c_1c_2^2\alpha^2 - 6c_2^2\beta^2)s^i_0 s_{0i}, \end{aligned}$$

and

$$N_1 = -\beta^6\overline{\mathrm{Ric}}_{00} - 2c_2^2\beta^8 s^{ij}s_{ij} + (-c_2^2\beta^8 - \beta^{10} + 2c_1\beta^2 + 3c_1\beta^8)c + 16\beta^4 s^i_0 s_{0i}.$$

From (4.2) we conculude that α^2 divides N_1 and so $\beta = 0$. Then the metric (1.1) is actually Riemannian. In fact we proved the following theorem:

Theorem 4.1. *Let M^n be a special Finsler space with metric (1.1), where β is a constant Killing form. Then metric (1.1) is Einstein iff it is a Riemannian Einstein metric.*

5 The $(\alpha, \beta)-$Metric (1.2)

From the metric (1.2), we get

$$L_\alpha = \frac{c_1\beta + 2\alpha}{\beta}, \qquad L_\beta = \frac{c_2\beta^2 - \alpha^2}{\beta^2}, \tag{5.1}$$

Substituting (5.1) in (2.5), we get

$$r = \frac{\alpha(c_2\beta^2 - \alpha^2)}{\beta(c_1\beta + 2\alpha)}, \tag{5.2}$$

Now we suppose that β is a constant Killing form. Then by substituting (5.2) in (2.8), we have

$$B^i = \frac{\alpha(c_2\beta^2 - \alpha^2)}{\beta(c_1\beta + 2\alpha)}, \tag{5.3}$$

From (5.3), we get

$$B^i_{|j} = \frac{\alpha(c_2\beta^2 - \alpha^2)}{\beta(c_1\beta + 2\alpha)} s^i_{0|j} + \frac{2\alpha^2\beta(2c_2\beta + c_1\alpha)}{\beta^2(c_1\beta + 2\alpha)^2} s^i_0 b_{0|j}, \tag{5.4}$$

$$\begin{aligned} B^i_{.j} = \Big[\Big\{ &\frac{(c_2\beta^2 - 3\alpha^2)(c_1\beta + 2\alpha) - 2\alpha(c_2\beta^2 - \alpha^2)}{\alpha\beta(c_1\beta + 2\alpha)^2} y_j \\ &+ \frac{2c_2\alpha\beta^2(c_1\beta + 2\alpha) - (2c_1\alpha\beta + 2\alpha^2)(c_2\beta^2 - \alpha^2)}{\beta^2(c_1\beta + 2\alpha)^2} b_j \Big\}\Big] s^i_0 + \frac{\alpha(c_1\beta^2 - \alpha^2)}{c_1\beta^2 + 2\alpha\beta} s^i_j, \end{aligned} \tag{5.5}$$

where $B^i_{.j} = B^i_{y^j}$.

From (5.5), we get

$$B^j B^i_{.j.i} = 0, \tag{5.6}$$

and

$$B^j_{.i} B^i_{.j} = \frac{2\{(c_2\beta^2 - 3\alpha^2)(c_1\beta + 2\alpha) - 2\alpha(c_2\beta^2 - \alpha^2)\}\alpha(c_2\beta^2 - \alpha^2)}{\alpha\beta^2(c_1\beta + 2\alpha)^3} s^i_0 s_{0i} + \frac{\alpha^2(c_1\beta^2 - \alpha^2)^2}{(c_1\beta^2 + 2\alpha\beta)^2} s^{ij} s_{ij}, \tag{5.7}$$

Differentiation (5.5) by y^i and then transvecting this derivative by y^j, we get

$$y^j (B^i_{|j})_{,i} = 0, \tag{5.8}$$

Using (5.4), (5.6), (3.6), (3.7) and (3.8) in Berwald's formula (2.9), we have

$$\begin{aligned} K^i_i =& \overline{K}^i_i + \frac{2\alpha(c_1\beta^2 - \alpha^2)}{(c_1\beta^2 + 2\alpha\beta)} s_(0|i)^i + \frac{2\alpha^2(c_2\beta^2 - \alpha^2)^2}{(c_1\beta^2 + 2\alpha\beta)^2} s^{ij} s_{ij} \\ &+ \frac{2\{(c_2\beta^2 - 3\alpha^2)(c_1\beta + 2\alpha) - 2\alpha(c_2\beta^2 - \alpha^2)\}\alpha(c_2\beta^2 - \alpha^2)}{\alpha\beta^2(c_1\beta + 2\alpha)^3} s^i_0 s_{0i}. \end{aligned} \tag{5.9}$$

As a result of (5.9), we can derive the spray curvature of the metric (1.2) with constant Killing vector.

6 Einstein Criterion for Metric (1.2)

In this section we assume that Ricci scalar of the $(\alpha, \beta)-$metric (1.2) is a functions of x alone, i.e. L is Einstein. We have $L^2\,\mathrm{Ric}(x) = K^i_i$, so we can derive necessary and sufficient conditions for metric (1.2) to be Einstein. From (5.9), we have

$$\begin{aligned} 0 =& \overline{\mathrm{Ric}}_{00} + \frac{2\alpha(c_1\beta^2 - \alpha^2)}{(c_1\beta^2 + \alpha\beta)} s^i_{0|i} + \frac{2\alpha^2(c_2\beta^2 - \alpha^2)^2}{(c_1\beta^2 + 2\alpha\beta)^2} s^{ij} s_{ij} \\ &+ \frac{2\{(c_2\beta^2 -?3\alpha^2)(c_1\beta + 2\alpha) - 2\alpha(c_2\beta^2 - \alpha^2)\}(c_2\beta^2 - \alpha^2)}{\beta^2(c_1\beta + 2\alpha)^3} s^i_0 s_{0i} \\ &- \left(c_1\alpha + c_2\beta + \frac{\alpha^2}{\beta}\right)^2 \mathrm{Ric}(x), \end{aligned} \tag{6.1}$$

Multiplication (6.1) by $\beta^2(c_1\beta + 2\alpha)^3$ removes y from the denominator in (6.1). We then obtain

$$\text{Rat part} + \text{Irrat part} = 0, \tag{6.2}$$

where Rat part and Irrat part are respectively polynomial in y of degree 7 and 6 as given below:

$$\begin{aligned} \text{Rar part} =& (c_1\beta^5 - 12c_1\beta^3\alpha^2)\overline{\mathrm{Ric}}_{00} + (8c_1^2\alpha^2\beta^4 - 8c_1\alpha^4\beta^2)s^i_{0|i} + (2c_1c_2^2\alpha^2\beta^5 \\ &+ 2c_1\alpha^6\beta - 4c_1c_2\alpha^4\beta^3)s^{ij}s_{ij} - (c_1^5\alpha^2\beta^5 + c_1^3c_2^2\beta^7 + c_1^3\alpha^4\beta^3 \\ &+ 2c_1^3c_2\alpha^2\beta^5 + 16c_1c_2\alpha^4\beta^3 + 16c_1\beta\alpha^6 + 12c_1^3c_2\alpha^2\beta^5 + 12c_1^3\alpha^4\beta^3 \\ &+ 6c_1^3\alpha^4\beta^3 + 6c_1c_2^2\alpha^2\beta^5 + 6c_1\alpha^6\beta + 12c_1c_2\alpha^4\beta^3)\mathrm{Ric}(x) \\ &+ (2c_1c_2^2\beta^5 - 6c_1c_2\alpha^2\beta^3 - 2c_1c_2\alpha^2\beta^3 + 6c_1\beta\alpha^4)s^i_0 s_{0i}, \end{aligned}$$

and

$$
\begin{aligned}
\text{Irrat part} =&(8\alpha^2\beta^2 + 6c_1^2\beta^4)\overline{\text{Ric}}_{00} + (2c_1^3\beta^5 + 8c_1\alpha^2\beta^3 - 2c_1^2\alpha^2\beta^3 \\
&- 8\beta\alpha^4)s^i_{0|i} + (4c_2^2\alpha^2\beta^3 + 4\alpha^6 - 8c_2\beta^2\alpha^4)s^{ij}s_{ij} + (4c_2^2\beta^4 \\
&- 2c_2\alpha^2\beta^2 - 4c_2^2\beta^4 + 2\alpha^4)s^i_0 s_{0i} - (2c_1^4c_2\beta^6 + 2c_1^4\alpha^2\beta^4 \\
&+ 8c_1^2\alpha^4\beta^2 + c_2^2\alpha^2\beta^4 + 8\alpha^6 + 16c_2\beta^2\alpha^4 + 6c_1^4\alpha^2\beta^4 \\
&+ c_1^2c_2^2\beta^6 + 6c_1^2\alpha^4\beta^2 + 12c_1^2c_2\alpha^2\beta^4 + 12c_1^2\alpha^4\beta^2)\text{Ric}(x).
\end{aligned}
$$

Lemma 6.1. *The metric (1.2) with constant Killing form β is Einstein iff Rat part = 0 and Irrat part = 0.*

Notice that Rat part = 0 shows that α^2 divides β^5 or Ric$_0$0, β^7 or Ric(x) and β^5 or $s^i_0 s_{0i}$. Since α^2 is an irreducible polynomial of degree two in y and β^5, β^7 factors into five and seven linear terms, it follows that α^2 divides $\overline{\text{Ric}}_{00}$, Ric$(x)$ and $s^i_0 s_{0i}$. Therefore $\overline{\text{Ric}}_{00} = c\alpha^2$, Ric$(x) = c'\alpha^2$ and $s^i_0 s_{0i} = c''\alpha^2$ where c, c' and c'' are constant by the Riemannian Schure. We can get some more information out of two equations Rat part = 0 and Irrat part = 0 in lemma (6.1). Replacing $\overline{\text{Ric}}_{00} = c\alpha^2$, Ric $(x) = c'\alpha^2$ and $s^i_0 s_{0i} = c''\alpha^2$ and then dividing Rat part = 0, through out by α^2, we get a relation of the form

$$
\alpha^2 G_1 + G_(2) = 0, \tag{6.3}
$$

where

$$
\begin{aligned}
G_1 =&(-12c_1\beta^3)c + (-8c_1\beta^2)s^i_{0|i} + (2c_1\alpha^2\beta - 4c_1c_2\beta^3)s^{ij}s_{ij} - (c_1^3\alpha^4\beta^3 \\
&+ 16c_1c_2\beta^3 + 16c_1\beta\alpha^2 + 12c_1^3\beta^3 + 6c_1^3\beta^3 + 6c_1\alpha^2\beta + 12c_1c_2\beta^3)c' \\
&+ 6c_1\beta\alpha^4c'',
\end{aligned}
$$

and

$$
\begin{aligned}
G_2 =&c_1\beta^5c + 8c_1^2\beta^4s^i_{0|i} + 2c_1c_2^2\beta^5s^{ij}s_{ij} - (c_1^5\beta^5 + c_1^3c_2^2\beta^7 + 2c_1^3c_2\beta^5 + 12c_1^3c_2\beta^5 \\
&+ 6c_1c_2^2\beta^5)\text{Ric}(x) + (2c_1c_2^2\beta^5 - 6c_1c_2\beta^3 - 2c_1c_2\beta^3)c''.
\end{aligned}
$$

From (6.3) we conclude that α^2 divides G_2 and so $\beta = 0$. Then the metric (1.2) is actually Riemannian. In fact we proved the following theorem:

Theorem 6.1. *Let M^n be a special Finsler space with metric (1.2), where β is a constant Killing form. Then metric (1.2) is Einstein iff it is a Riemannian Einstein metric.*

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