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SLOW ROTATION OF A SPHERICAL CAVITY ENCLOSING A CONCENTRIC POROUS SPHERE OF VARIABLE PERMEABILITY

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Abstract

In this paper we present an analytical investigation of slow motion generated by rotation of a spherical container about a concentric porous sphere of variable permeability (kr^{2-n},r being the polar distance). The exterior space is filled with viscous fluid. Stokes and Brinkman equations are employed to obtain the flow fields respectively in the region exterior to the porous sphere and inside it. The torque on the sphere is evaluated and wall correction factor are obtained. Graphs to show the effects of various parameters on torque, wall correction factor and velocity are presented and discussed. It is concluded that in general increasing index n the effects decrease.
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1 Introduction

The problem addressed in this paper is to obtain the solution for the slow rotation of a spherical cavity enclosing a concentric porous sphere of variable permeability and filled with viscous fluid. Here the permeability term (r^n/k) in the porous sphere equation has some power n of radial distance r instead of power *2* in the constant permeability case. The flow inside cavity wall and outside the porous sphere is governed by Stokes equation and the flow within the porous sphere by modified Brinkman equation. Boundary conditions of no slip at the surface of spherical cavity, matching conditions of continuity of velocity and tangential stress at interface at the porous sphere are applied; of course the velocity at the centre is zero. Our objective here is to determine the velocities field in the Stokes region and Brinkman region. The torque and the wall correction factor are evaluated and studied graphically. We also present graphs for velocity for different parameters.

The problem has many applications in physical, biomedical, environmental and chemical engineering sciences. Happle and Brenner [6] and Kim and Karrila [10] presented the history, references and competent information in this area. Considerable amount of work has been done on rotation of porous sphere, porous spherical shell, composite sphere having a solid core bounded by porous spherical shell and approximate sphere in approximate spherical container. To cite a few examples, we may mention the papers of Prasad,Kaur and Srinivasacharya[11], Srinivasacharya and Prasad [14], Keh and Lu [9], Keh and Chou

[8]. There are many papers concerns with flow past spherical particles involving constant permeability porous components e.g Padmavathi and Amaranath[13], Grosan,Postelnicu and Pop[4], Awasthi,Pandya and Datta[3], Awasthi and Pandya[2]. But scant literature is available on the sphere with variable permeability, a few are Verma and Datta [15], Kabeir, Hakiem, and Rashad[7], Hamdan and Kamel[5].

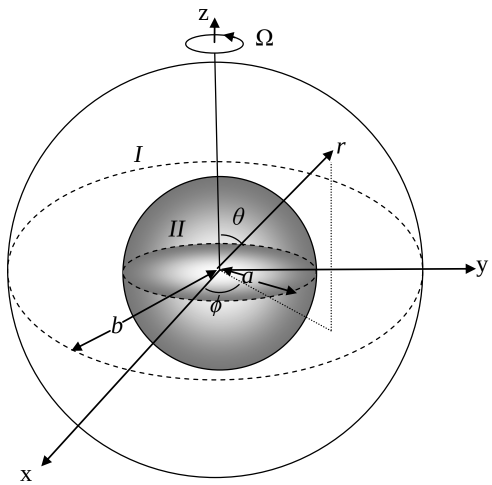

Figure 1: Figure for the rotation of spherical cavity enclosing concentric porous sphere of variable permeability

2 Formulation of the Problem

Consider the slow rotation of a spherical cavity of radius b with constant angular velocity Ω about the positive z direction and enclosing a concentric non -deformable porous sphere of variable permeability k'/r'^{2-n} and having radius a . The spherical cavity is filled with an incompressible viscous fluid of viscosity . The flow regions comprise of two parts, one part is the region inside spherical cavity and outside porous sphere denoted as Region I and other part is region inside porous sphere denoted by Region II. Because of axial symmetry only the non-zero azimuthal components $u'_{1\phi}$ and $u'_{2\phi}$ of velocities appear in region I and II respectively. In region $I(a \leqslant r' \leqslant b)$ the fluid flow is governed by Stokes equation while continuity equation is automatically satisfied. Similarly in region II the flow is governed by Brinkman equation. It is convenient to use the non-dimensional quantities

$$ar = r', a^2k = k', \Omega a u_{1\phi} = u'_{1\phi}, \Omega a u_{2\phi} = u'_{2\phi}. \tag{2.1}$$

The relevant equations are expressible, in spherical coordinate system (r, θ, ϕ) with the origin located at the centre of spherical cavity as (see Awasthi and Pandya[2])
Stokes:$(1 \leqslant r \leqslant b/a = \alpha)$

$$\frac{\partial}{\partial r}(r^2\frac{\partial u_{1\phi}}{\partial r}) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial u_{1\phi}}{\partial\theta}) - \frac{u_{1\phi}}{\sin^2\theta} = 0. \tag{2.2}$$

Brinkman:$(0 \leqslant r \leqslant 1)$

$$\frac{\partial}{\partial r}(r^2\frac{\partial u_{2\phi}}{\partial r}) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial u_{2\phi}}{\partial\theta}) - (\frac{1}{\sin^2\theta} + \frac{r^2}{k})u_{2\phi} = 0. \tag{2.3}$$

3 Boundary Conditions

These equations are to be solved under following boundary conditions (see Awasthi and Pandya[10])
at $r = 0$

$$u_{2\phi} = 0 \text{ (no singularity condition)} \tag{3.1}$$

at the interface $r = 1$,

$$u_{2\phi} = u_{1\phi} \text{ (continuity of velocity)} \tag{3.2}$$

$$\tau_{2r\phi} = \tau_{1r\phi} \text{ (continuity of stress components)} \tag{3.3}$$

or

$$\frac{\partial}{\partial r}(u_{2\phi}) = \frac{\partial}{\partial r}(u_{1\phi}) \text{ using Equ.(5)} \tag{3.4}$$

at $r = b/a = 1/\alpha$,

$$u_{1\phi} = (1/\alpha)\sin\theta \text{ (no slip condition).} \tag{3.5}$$

4 Solution of the Problem

The boundary condition (8) implies that solution of Equations (2) and (3) are of the form $u_{1\phi} = f_1(r)\sin\theta$ and $u_{2\phi} = f_2(r)\sin\theta$ respectively and then Equations (2) and (3) provide

$$r^2 f_1''(r) + 2r f_1'(r) - 2f_1(r) = 0, \tag{4.1}$$

$$r^2 f_2''(r) + 2r f_2'(r) - (2 + \frac{r^2}{k}) f_2(r) = 0. \tag{4.2}$$

Now to incorporate the effect of variable permeability we replace r^2/k by r^n/k and then the Brinkman equation (10) provides the modified Brinkman equation

$$r^2 f_2''(r) + 2r f_2'(r) - (2 + \frac{r^n}{k}) f_2(r) = 0. \tag{4.3}$$

It may be noted that permeability here is kr^{2-n} instead of k. Here $n \geqslant 0$ and $0 \leqslant r \leqslant 1$. When $0 < n < 2$ then permeability increases with increase in r and when $n > 2$ then permeability decreases with increase in r. When $n = 2$ it remains uniform.
The boundary conditions (4), (5), (7) and (8) now reduce to

$$f_2(0) = 0, \tag{4.4}$$

$$f_1(1) = f_2(1), \tag{4.5}$$

$$f_1'(1) = f_2'(1), \tag{4.6}$$

$$f_1(1/\alpha) = 1/\alpha. \tag{4.7}$$

The general solution of equation (9) is given by

$$f_1(r) = c_1 r + \frac{c_2}{r^2}. \tag{4.8}$$

It may be observed that the differential equation (11) of the second order posses two kinds of solutions, one for $n > 0$ and the other for $n = 0$. For $n < 0$, equation (11) shows that both of the two linearly independent solutions are singular at $r = 0$, hence we discuss cases for $n \geqslant 0$.

Solution for $n > 0$

The solution of modified Brinkman equation (11) satisfying boundary condition (12) is (see Abramowitz and Stegun[1])

$$f_2(r) = r^{-\frac{1}{2}} c_3 I_{\frac{3}{n}}(\frac{2\sqrt{r^n}}{n\sqrt{k}}), \tag{4.9}$$

where $I_{\frac{3}{n}}(z)$ is modified Bessel function of first kind. For $n = 2$ we get the solution

$$f_2(r) = \frac{c_3}{k^{1/4}}\sqrt{\frac{2}{\pi}}(\frac{\sqrt{k}}{r}\cosh\frac{r}{\sqrt{k}} - \frac{k}{r^2}\sinh\frac{r}{\sqrt{k}}), \tag{4.10}$$

for a porous sphere of uniform permeability k as obtaind [8,9] .
Constants c_1, c_2 and c_3 are obtained by applying the boundary conditions (13, 14, 15), thus, we get the following system of equations

$$c_1 + c_2 = c_3 I_{\frac{3}{n}}(\frac{2}{n\sqrt{k}}), \tag{4.11}$$

$$c_1 - 2c_2 = c_3(I_{\frac{3}{n}}(\frac{2}{n\sqrt{k}}) + \frac{1}{\sqrt{k}} I_{\frac{3}{n}+1}(\frac{2}{n\sqrt{k}})), \tag{4.12}$$

$$c_1 + \alpha^3 c_2 = 1. \tag{4.13}$$

The constants involved in above system of equations are

$$c_1 = \frac{I_{\frac{3}{n}-1}(\frac{2}{n\sqrt{k}}) + 3\sqrt{k} I_{\frac{3}{n}}(\frac{2}{n\sqrt{k}}) + I_{\frac{3}{n}+1}(\frac{2}{n\sqrt{k}})}{(1-\alpha^3)(I_{\frac{3}{n}+1}(\frac{2}{n\sqrt{k}}) + I_{\frac{3}{n}-1}(\frac{2}{n\sqrt{k}})) + 3(1+\alpha^3)\sqrt{k} I_{\frac{3}{n}}(\frac{2}{n\sqrt{k}})}, \tag{4.14}$$

$$c_2 = \frac{I_{\frac{3}{n}-1}(\frac{2}{n\sqrt{k}}) + 3\sqrt{k} I_{\frac{3}{n}}(\frac{2}{n\sqrt{k}}) + I_{\frac{3}{n}+1}(\frac{2}{n\sqrt{k}})}{(1-\alpha^3)(I_{\frac{3}{n}+1}(\frac{2}{n\sqrt{k}}) + I_{\frac{3}{n}-1}(\frac{2}{n\sqrt{k}})) + 3(1+\alpha^3)\sqrt{k} I_{\frac{3}{n}}(\frac{2}{n\sqrt{k}})}, \tag{4.15}$$

$$c_3 = \frac{6\sqrt{k}}{(1-\alpha^3)(I_{\frac{3}{n}+1}(\frac{2}{n\sqrt{k}}) + I_{\frac{3}{n}-1}(\frac{2}{n\sqrt{k}})) + 3(1+\alpha^3)\sqrt{k} I_{\frac{3}{n}}(\frac{2}{n\sqrt{k}})}. \tag{4.16}$$

Solution for $n = 0$

In this case the solution of modified Brinkman equation (11) is

$$f_2(r) = c_3 r^{\frac{-\sqrt{k}+\sqrt{4+9k}}{2\sqrt{k}}}. \tag{4.17}$$

Constants c_1, c_2 and c_3 are obtained by applying the boundary conditions (13, 14, 15), thus we get the following system of equations

$$c_1 + c_2 = c_3, \tag{4.18}$$

$$c_1 - 2c_2 = c_3(\frac{-\sqrt{k} + \sqrt{4+9k}}{2\sqrt{k}})), \tag{4.19}$$

$$c_1 + \alpha^3 c_2 = 1. \tag{4.20}$$

The constants involved in above system of equations are found to be

$$c_1 = \frac{3\sqrt{k} + \sqrt{4+9k}}{3(1+\alpha^3)\sqrt{k} + (1-\alpha^3)\sqrt{4+9k}}, \tag{4.21}$$

$$c_2 = \frac{3\sqrt{k} - \sqrt{4+9k}}{3(1+\alpha^3)\sqrt{k} + (1-\alpha^3)\sqrt{4+9k}}, \tag{4.22}$$

$$c_3 = \frac{6\sqrt{k}}{3(1+\alpha^3)\sqrt{k} + (1-\alpha^3)\sqrt{4+9k}}. \tag{4.23}$$

5 Torque on Porous Sphere

The hydrodynamic couple acting on the porous sphere of variable permeability in a spherical cavity is given by(see Landau and Lifshitz [12])

$$M' = 2\pi a^3 \int_0^\pi \tau'_{r'\phi(1)} \sin^2\theta d\theta, \tag{5.1}$$

where $\tau'_{r'\phi(1)}$ is the tangential stress on the porous sphere is given by

$$\tau_{r\phi(1)} = \frac{\tau'_{r'\phi(1)}}{\mu\Omega} = (\frac{\partial u_{1\phi}}{\partial r} - \frac{u_{1\phi}}{r}) = (f_1'(1) - f_1(1))\sin\theta = -3c_2\sin\theta. \tag{5.2}$$

Thus, on evaluating the integral (32), non-dimensional torque comes out

$$M = \frac{M'}{8\pi\mu\Omega a^3} = c_2, \tag{5.3}$$

where $c_2$is given in (23) for $n > 2$ and (30) for $n = 0$.

The torque on the porous sphere in an unbounded medium is

$$M^\infty = \frac{I_{\frac{3}{n}-1}(\frac{2}{n\sqrt{k}}) - 3\sqrt{k}I_{\frac{3}{n}}(\frac{2}{n\sqrt{k}}) + I_{\frac{3}{n}+1}(\frac{2}{n\sqrt{k}})}{I_{\frac{3}{n}-1}(\frac{2}{n\sqrt{k}}) + I_{\frac{3}{n}+1}(\frac{2}{n\sqrt{k}})) + 3\sqrt{k}I_{\frac{3}{n}}(\frac{2}{n\sqrt{k}})}(n > 0), \tag{5.4}$$

$$M^\infty = \frac{\sqrt{9k+4} - 3\sqrt{k}}{\sqrt{9k+4} + 3\sqrt{k}}(n = 0). \tag{5.5}$$

6 Wall Effect

The wall correction factor K defined as (see Happle and Brenner [6]). (The actual couple experienced by the particle in the enclosure)/(the couple on a particle in an infinite expanse of fluid) i.e.

$$K = \frac{M}{M^\infty} \tag{6.1}$$

Wall effect for $n > 0$ and $n = 0$ respectively given by

$$K = \frac{I_{\frac{3}{n}-1}(\frac{2}{n\sqrt{k}}) + 3\sqrt{k}I_{\frac{3}{n}}(\frac{2}{n\sqrt{k}}) + I_{\frac{3}{n}+1}(\frac{2}{n\sqrt{k}})}{(1-\alpha^3)(I_{\frac{3}{n}-1}(\frac{2}{n\sqrt{k}}) + I_{\frac{3}{n}+1}(\frac{2}{n\sqrt{k}})) + 3(1+\alpha^3)\sqrt{k}I_{\frac{3}{n}}(\frac{2}{n\sqrt{k}})}(n > 0), \tag{6.2}$$

$$K = \frac{2}{2 - \alpha^3(2 + 9k - 3\sqrt{k}\sqrt{9k+4})}(n = 0). \tag{6.3}$$

7 Limiting Cases

(i) When $k \to 0$ then we get the result for solid sphere

$$M = \frac{1}{1-\alpha^3}. \tag{7.1}$$

This result agrees with the classical case of rotation of the cavity rotating about a solid sphere, the same as obtained in [8, 9 and 11].

(ii) If $n = 2$, the torque on the porous sphere in a spherical cavity is reduces to

$$M = \frac{3\sqrt{k}\cosh(1/\sqrt{k}) - (1+3k)\sinh(1/\sqrt{k})}{3\alpha^3\sqrt{k}\cosh(1/\sqrt{k}) - (\alpha^3(1+3k) - 1)\sinh(1/\sqrt{k})}. \tag{7.2}$$

This matches with the result of [8, 9].

8 Results and Discussion

We present below some graphs to show the effects of various parameters on the velocity field $u_{1\phi}/(\Omega r \sin\theta)$ in the Stokes region and $u_{2\phi}/(\Omega r \sin\theta)$ in the Brinkman's region, on torque M and wall correction factor K.

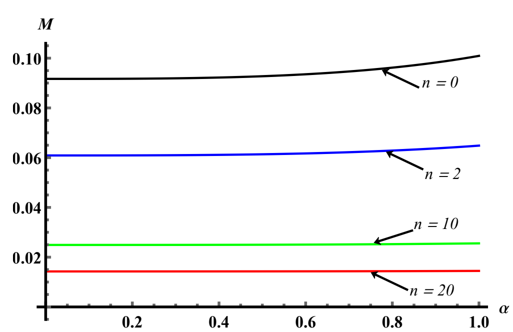

Figure 2: Variation of torque M with separation parameter $\alpha = a/b$ for various values of n when $k = 1$.

Fig.2 shows the variation of torque M with separation parameter α for various values of n. In this figure graphs show that torque increases as n decreases but in each graph rate of variation of M with separation parameter diminishes as n increases.

Fig.3 shows the variation of torque M with index parameter n for various values of separation parameter α when $k = 0.01$. In this figure each of the graphs shows that torque M is decreases with n. Also, M increases when α decreases but in each graph rate of

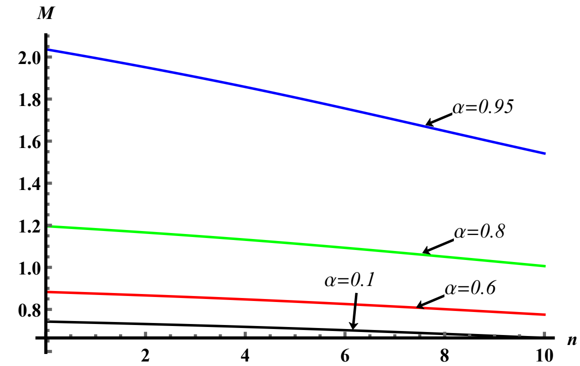

Figure 3: Variation of torque M with index parameter n for various values of separation parameter α when $k = 0.01$.

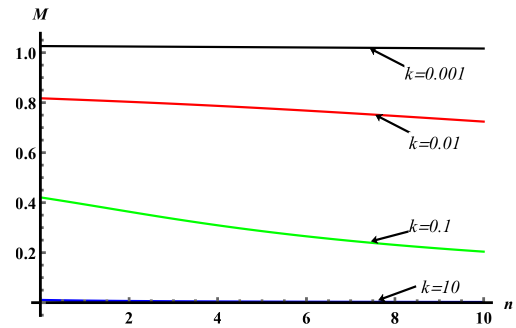

Figure 4: Variation of torque M with index parameter n for various values of k when $\alpha = 0.5$.

variation of M with respect to n diminishes when α decreases.

Fig.4 shows the variation of torque M with index parameter n for various values of k when separation parameter $\alpha = 0.5$. In this figure each of the graphs shows that torque decreases when n increases. Also, torque decreases when k increases but in each graph rate of variation of M with n diminishes as k decreases.

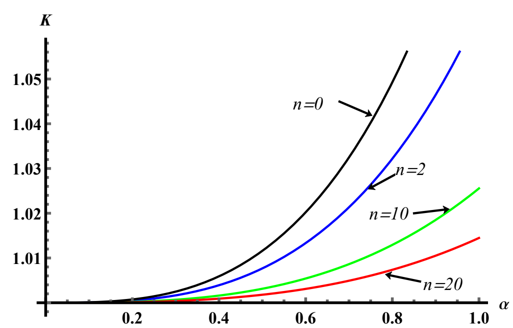

Figure 5: Variation of wall correction factor K with separation parameter α for various values of index parameter n when $k = 1$.

In fig. 5 the graphs of the wall correction factor K with α for various values of n are depicted. We see that the wall correction factor increases when α increases; also that when n decreases wall correction factor increases for fixed α.

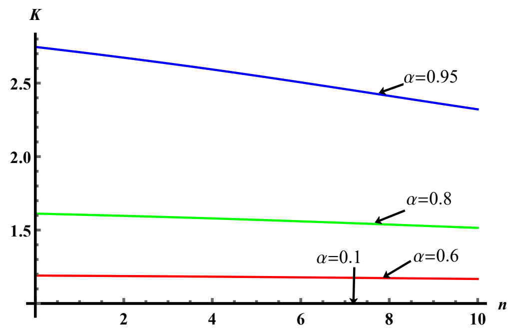

Figure 6: Variation of wall correction factor K with index parameter n for various values of separation parameter $\alpha = a/b$ when $k = 0.01$.

Fig.6 shows the variation of wall correction factor K with index parameter n for various values of separation parameter α when $k = 0.01$. In this figure each graph shows that wall correction factor K increases as n decreases. Also, wall correction factor increases when separation between porous sphere and cavity wall is decreases but in each graph its rate of variation with n diminishes when α decreases.

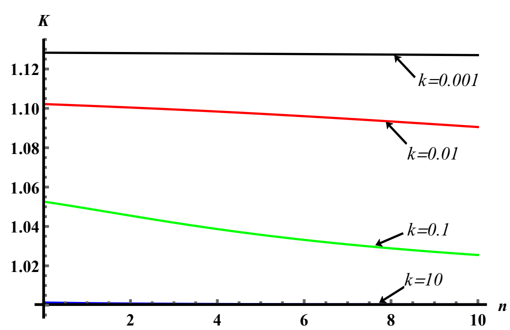

Figure 7: Variation of wall correction factor K with index parameter n for various values of k when $\alpha = 0.5$.

Fig.7 shows the variation of wall correction factor K with index parameter n for various values of k when separation parameter $\alpha = 0.5$. In this figure individual graph shows that wall correction factor decreases as n increases. Also, wall correction factor increases when k decreases but in each graph rate of variation of K with n diminishes when k decreases.

Fig. 8 shows the dependence of $u_{1\phi}$ in the Stokes region $(1 \leqslant r \leqslant 2)$ and $u_{2\phi}$ in the Brinkman region $(0 \leqslant r \leqslant 1)$ for $\theta = \pi/2$ for different values of the permeability parameter $n(0, 2, 15, 20)$ when permeability parameter $k = 0.01$ with fixed value of separation parameter $\alpha = 0.5$.

Fig. 9 shows the dependence of $u_{1\phi}$ in the Stokes region $(1 \leqslant r \leqslant 2)$ and $u_{1\phi}$ in the Brinkman's region $(0 \leqslant r \leqslant 1)$ for different values of the permeability parameter $k(0.01, 0.08, 0.5, 5)$ when permeability parameter $n = 5$ with fixed value of separation parameter $\alpha = 0.5$.

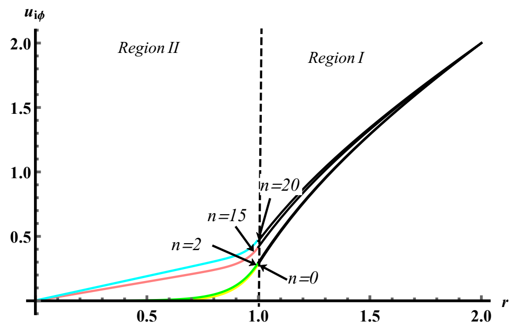

Figure 8: Variation of $u_{i\phi}(i = 1, 2)$ with r for various values of n at $k = 0.01, \alpha = 0.5$ and $\theta = \pi/2$

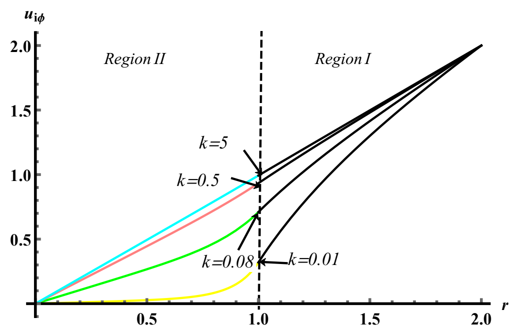

Figure 9: Variation of $u_{i\phi}(i = 1, 2)$ with r for various values of k at $n = 5, \alpha = 0.5$ and $\theta = \pi/2$

In figs. 8 and 9, $u_{1\phi}(1 \leqslant r \leqslant 2)$ and $u_{2\phi}(0 \leqslant r \leqslant 1)$ increase when r increases this is due to the rotation starts from the outer sphere at $r = 2$ and tends to zero as r tends to zero. It may be observed that in the Stokes region $(1 \leqslant r \leqslant 2)$ while there is no permeability term yet permeability effect is there which dies down as r tends to 2. Also it is seen that the permeability effect decreases as n and k increase.

9 Conclusion

An analytic solution of the governing equations for the problem of the motion of a porous sphere of variable permeability in a spherical cavity filled with an incompressible viscous fluid has been obtained. Brinkman model is used in porous region and Stokes in the clear fluid region to solve the problem. An expression for the hydrodynamic torque on the porous sphere of variable permeability in the spherical cavity is obtained. The wall effect is computed and presented graphically. Also the limiting cases obtained by letting $k \to 0$ and $k \to \infty$ correspond respectively to those obtained earlier in [8, 9, 11]. It has been found that, the wall correction factor of the porous sphere is an increasing function of separation parameter α. Graphs have been drawn to depict the effect of various parameters on torque M , wall correction factor K (Figs. 2 to 7) and $u_{i\phi}(i = 1, 2)$ for $\theta = \pi/2$. It may be noted that the effect of permeability is governed by the term r^n/k in equation (11). Since $0 \leqslant r \leqslant 1$, the effect wanes as $n \geqslant 0$ increases akin to the effect of increasing k. The separation parameter $\alpha = a/b$ determines the closeness of the inner sphere and the outer, as $\alpha \to 1$ the outer sphere comes closer to the inner resulting in increasing torque M as well as wall correction factor K. In Figs.2 to 7 the depicted variations can be traced to above factors. It is concluded that in general increasing the index n, permeability effects decrease.

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