

LOGARITHMIC-SHEFFER POLYNOMIAL SETS

By

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Abstract

In recent papers, new sets of Sheffer and Brenke polynomials based on higher order Bell numbers, and several integer sequences related to them have been studied. In this article new sets of Logarithmic-Sheffer polynomials are introduced.

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1 Introduction

In recent articles [8, 19], new sets of Sheffer [21] and Brenke [7] polynomials, based on higher order Bell numbers [4, 15, 17, 12, 19], have been studied. Furthermore, several integer sequences associated [22] with the considered polynomials sets both of exponential [1, 2] and logarithmic type have been introduced [8].

It is worth to note that exponential and logarithmic polynomials have been recently studied in the multidimensional case [12, 13, 14].

In this article sets of Logarithmic-Sheffer polynomials are considered and some particular cases are analyzed.

2 Sheffer polynomials

The Sheffer polynomials $\{s_n(x)\}$ are introduced [21] by means of the exponential generating function [23] of the type:

$$A(t) \exp(xH(t)) = \sum_{n=0}^{\infty} s_n(x) \frac{t^n}{n!}, \quad (2.1)$$

where

$$A(t) = \sum_{n=0}^{\infty} a_n \frac{t^n}{n!}, \quad (a_0 \neq 0),$$

$$H(t) = \sum_{n=0}^{\infty} h_n \frac{t^n}{n!}, \quad (h_0 = 0). \quad (2.2)$$

According to a different characterization (see [20, p. 18]), the same polynomial sequence can be defined by means of the pair $(g(t), f(t))$, where $g(t)$ is an invertible series and $f(t)$ is

a delta series:

$$\begin{aligned}
 g(t) &= \sum_{n=0}^{\infty} g_n \frac{t^n}{n!}, & (g_0 \neq 0), \\
 f(t) &= \sum_{n=0}^{\infty} f_n \frac{t^n}{n!}, & (f_0 = 0, f_1 \neq 0).
 \end{aligned}
 \tag{2.3}$$

Denoting by $f^{-1}(t)$ the compositional inverse of $f(t)$ (i.e. such that $f(f^{-1}(t)) = f^{-1}(f(t)) = t$), the exponential generating function of the sequence $\{s_n(x)\}$ is given by

$$\frac{1}{g[f^{-1}(t)]} \exp(xf^{-1}(t)) = \sum_{n=0}^{\infty} s_n(x) \frac{t^n}{n!}, \tag{2.4}$$

so that

$$A(t) = \frac{1}{g[f^{-1}(t)]}, \quad H(t) = f^{-1}(t). \tag{2.5}$$

When $g(t) \equiv 1$, the Sheffer sequence corresponding to the pair $(1, f(t))$ is called the associated Sheffer sequence $\{\sigma_n(x)\}$ for $f(t)$, and its exponential generating function is given by

$$\exp(xf^{-1}(t)) = \sum_{n=0}^{\infty} \sigma_n(x) \frac{t^n}{n!}. \tag{2.6}$$

A list of known Sheffer polynomial sequences and their associated ones can be found in [6].

3 New Logarithmic-Sheffer polynomial sets

We introduce, for shortness, the following compact notation.

Put, by definition:

$$\begin{aligned}
 E_0(t) &:= \exp(t) - 1 \\
 E_1(t) &:= E_0(E_0(t)) = \exp(\exp(t) - 1) - 1 \\
 &\dots\dots\dots \\
 E_r(t) &:= E_0(E_{r-1}(t)) = \exp(\dots \exp(\exp(t) - 1) - 1) \dots - 1, & [(r + 1) - \text{times exp}],
 \end{aligned}$$

and in a similar way:

$$\begin{aligned}
 \Lambda_0(t) &:= \log(t + 1) \\
 \Lambda_1(t) &:= \Lambda_0(\Lambda_0(t)) = \log(\log(t + 1) + 1) \\
 &\dots\dots\dots \\
 \Lambda_r(t) &:= \Lambda_0(\Lambda_{r-1}(t)) = \log(\log(\dots(\log(t + 1) + 1) \dots) + 1), & [(r + 1) - \text{times log}].
 \end{aligned}$$

Remark 1. Note that, for every integers r, k, h ,

$$\begin{aligned}
 E_r(\Lambda_r(t)) &= t, & \Lambda_r(E_r(t)) &= t, \\
 (\text{if } k > h) & \quad E_k(\Lambda_h(t)) = E_{k-h-1}(t), & E_h(\Lambda_k(t)) &= \Lambda_{k-h-1}(t), \\
 (\text{if } k > h) & \quad \Lambda_k(E_h(t)) = \Lambda_{k-h-1}(t), & \Lambda_h(E_k(t)) &= E_{k-h-1}(t), \\
 e^{E_r(t)} &= E_{r+1}(t) + 1, & e^{\Lambda_r(t)} &= \Lambda_{r-1}(t) + 1.
 \end{aligned}$$

Remark 2. Note that the coefficients of the Taylor expansion of $H_1(t)$ are given by the Bell numbers $b_n = b_n^{[1]}$

$$E_1(t) = \sum_{n=1}^{\infty} b_n^{[1]} \frac{t^n}{n!},$$

and, in general the coefficients of the Taylor expansion of $H_r(t)$ are given by the higher order Bell numbers $b_n^{[r]}$

$$E_r(t) = \sum_{n=1}^{\infty} b_n^{[r]} \frac{t^n}{n!}.$$

The higher order Bell numbers, also known as higher order exponential numbers [1, 2], have been considered in [15, 16, 17], and used in [19], in the framework of Brenke and Sheffer polynomials [7, 21].

Remark 3. Note that the coefficients of the Taylor expansion of $\Lambda_0(t)$ are given by the logarithmic numbers $l_n^{[1]} = (-1)^{n-1}(n-1)!$

$$\Lambda_0(t) = \sum_{n=1}^{\infty} l_n^{[1]} \frac{t^n}{n!} = \sum_{n=1}^{\infty} (-1)^{n-1}(n-1)! \frac{t^n}{n!},$$

and, in general the coefficients of the Taylor expansion of $\Lambda_{r-1}(t)$ are given by the higher order logarithmic numbers $l_n^{[r]}$

$$\Lambda_{r-1}(t) = \sum_{n=1}^{\infty} l_n^{[r]} \frac{t^n}{n!}.$$

The higher order logarithmic numbers, which are the counterpart of the higher order Bell (exponential) numbers, have been considered in [8], and used there in the framework of new sets of Sheffer polynomials.

3.1 The polynomials $\mathcal{L}_k^{(1)}(x)$

Therefore, we consider the Sheffer polynomials, defined through their generating function, by putting

$$\begin{aligned} A(t) &= (t+1)^\beta, & H(t) &= \Lambda_0(t), \\ G(t, x) &= \exp[(\beta+x)\Lambda_0(t)] = (t+1)^{\beta+x} = \sum_{k=0}^{\infty} \mathcal{L}_k^{(1)}(x) \frac{t^k}{k!}. \end{aligned} \quad (3.1)$$

Remark 4. Of course, the above polynomials depend on β and they should be denoted by $\mathcal{L}_k^{(1)}(\beta, x)$. But here and below, we will use for them the simpler notation $\mathcal{L}_k^{(1)}(x)$. A similar notation (avoiding β) will be used, for shortness, even for their iterations $\mathcal{L}_k^{(2)}(x), \dots, \mathcal{L}_k^{(r)}(x)$, introduced in Sections 4 and 5.

3.2 Recurrence relation for the $\mathcal{L}_k^{(1)}(x)$

Theorem 3.1. - For any $k \geq 0$, the polynomials $\mathcal{L}_k^{(1)}(x)$ satisfy the following recurrence relation:

$$\mathcal{L}_{k+1}^{(1)}(x) = (\beta + x - k) \mathcal{L}_k^{(1)}(x). \quad (3.2)$$

Proof. - Differentiating $G(t, x)$ with respect to t , we have

$$\frac{\partial G(t, x)}{\partial t} = G(t, x) \frac{\beta + x}{t + 1}, \quad (3.3)$$

and therefore

$$(t + 1) \sum_{k=1}^{\infty} \mathcal{L}_k^{(1)}(x) \frac{t^{k-1}}{(k-1)!} = (\beta + x) \sum_{k=0}^{\infty} \mathcal{L}_k^{(1)}(x) \frac{t^k}{k!},$$

i.e.

$$\sum_{k=0}^{\infty} \mathcal{L}_k^{(1)}(x) \frac{k t^k}{k!} + \sum_{k=0}^{\infty} \mathcal{L}_{k+1}^{(1)}(x) \frac{t^k}{k!} = (\beta + x) \sum_{k=0}^{\infty} \mathcal{L}_k^{(1)}(x) \frac{t^k}{k!},$$

so that the recurrence relation (3.2) follows.

Remark 5. Note that the recurrence relation (3.2), taking into account the initial condition $\mathcal{L}_0^{(1)}(x) = 1$, implies that the Logarithmic-Sheffer polynomials $\mathcal{L}_k^{(1)}$, are nothing but the polynomials used in the Lagrange interpolation formula (with knots at the points: $0, 1, 2, \dots, k$).

In fact, by putting $\beta + x = X$, we find

$$\mathcal{L}_k^{(1)}(X - \beta) = X(X - 1)(X - 2) \cdots (X - k).$$

This is confirmed in Section 3.6, where the first values of $\mathcal{L}_k^{(1)}(x)$ are shown.

3.3 Generating function's PDE

Theorem 3.2. The generating function (3.1)₂ satisfies the homogeneous linear PDE's

$$(t + 1) \Lambda_0(t) \frac{\partial G(t, x)}{\partial t} = (\beta + x) \frac{\partial G(t, x)}{\partial x}, \quad (3.4)$$

Proof. Differentiating $G(t, x)$ with respect to x , we have

$$\frac{\partial G(t, x)}{\partial x} = G(t, x) \Lambda_0(t). \quad (3.5)$$

By taking the ratio between the members of equations (3.3) and (3.5), we find equation (3.4). \square

3.4 Shift operators

We recall that a polynomial set $\{p_n(x)\}$ is called quasi-monomial if and only if there exist two operators \hat{P} and \hat{M} such that

$$\hat{P}(p_n(x)) = np_{n-1}(x), \quad \hat{M}(p_n(x)) = p_{n+1}(x), \quad (n = 1, 2, \dots). \quad (3.6)$$

\hat{P} is called the *derivative* operator and \hat{M} the *multiplication* operator, as they act in the same way of classical operators on monomials.

This definition traces back to a paper by J.F. Steffensen [25], recently improved by G. Dattoli [9] and widely used in several applications.

Y. Ben Cheikh [3] proved that every polynomial set is quasi-monomial under the action of suitable derivative and multiplication operators. In particular, in the same article (Corollary 3.2), the following result is proved

Theorem 3.3. *Let $(p_n(x))$ denote a Boas-Buck polynomial set [5, 6], i.e. a set defined by the generating function*

$$A(t)\psi(xH(t)) = \sum_{n=0}^{\infty} p_n(x) \frac{t^n}{n!}, \quad (3.7)$$

where

$$\begin{aligned} A(t) &= \sum_{n=0}^{\infty} a_n t^n, & (a_0 \neq 0), \\ \psi(t) &= \sum_{n=0}^{\infty} \gamma_n t^n, & (\gamma_n \neq 0 \quad \forall n), \end{aligned} \quad (3.8)$$

with $\psi(t)$ not a polynomial, and lastly

$$H(t) = \sum_{n=0}^{\infty} h_n t^{n+1}, \quad (h_0 \neq 0). \quad (3.9)$$

Let $\sigma \in \Lambda^{(-)}$ the lowering operator defined by

$$\sigma(1) = 0, \quad \sigma(x^n) = \frac{\gamma_{n-1}}{\gamma_n} x^{n-1}, \quad (n = 1, 2, \dots). \quad (3.10)$$

Put

$$\sigma^{-1}(x^n) = \frac{\gamma_{n+1}}{\gamma_n} x^{n+1} \quad (n = 0, 1, 2, \dots). \quad (3.11)$$

Denoting, as before, by $f(t)$ the compositional inverse of $H(t)$, the Boas-Buck polynomial set $\{p_n(x)\}$ is quasi-monomial under the action of the operators

$$\hat{P} = f(\sigma), \quad \hat{M} = \frac{A'[f(\sigma)]}{A[f(\sigma)]} + xD_x H'[f(\sigma)]\sigma^{-1}, \quad (3.12)$$

where prime denotes the ordinary derivatives with respect to t .

Note that in our case we are dealing with a Sheffer polynomial set, so that since we have $\psi(t) = e^t$, the operator σ defined by equation (3.10) simply reduces to the derivative operator D_x . Furthermore, we have:

$$A(t) = (t + 1)^\beta, \quad H(t) = \Lambda_0(t),$$

$$f(t) = H^{-1}(t) = E_0(t),$$

$$\frac{A'(t)}{A(t)} = \frac{\beta}{t+1}, \quad H'(t) = \frac{1}{t+1},$$

and consequently

$$f(\sigma) = E_0(D_x), \quad \frac{A'[E_0(D_x)]}{A[E_0(D_x)]} = \frac{\beta}{E_0(D_x)+1},$$

$$H'[f(\sigma)] = H'[E_0(D_x)] = \frac{1}{E_0(D_x)+1}.$$

Theorem 3.4. *The Logarithmic-Sheffer polynomials $\{\mathcal{L}_k^{(1)}(x)\}$ are quasi-monomial under the action of the operators*

$$\hat{P} = E_0(D_x) = \sum_{k=0}^{\infty} \frac{D_x^{k+1}}{(k+1)!}, \tag{3.13}$$

$$\hat{M} = \frac{\beta+x}{E_0(D_x)+1} = (\beta+x) \sum_{k=0}^{\infty} (-1)^k \frac{D_x^k}{k!},$$

where we have noticed that $E_0(D_x) + 1 = e^{D_x}$.

3.5 Differential equation for the $\mathcal{L}_k^{(1)}(x)$

According to the results of monomiality principle [9, 11], the quasi-monomial polynomials $\{p_n(x)\}$ satisfy the differential equation

$$\hat{M}\hat{P}p_n(x) = np_n(x). \tag{3.14}$$

In the present case, recalling equations (3.13), we have

Theorem 3.5. *The Logarithmic-Sheffer polynomials $\{\mathcal{L}_k^{(1)}(x)\}$ satisfy the differential equation*

$$(\beta+x) \sum_{k=0}^{\infty} k = 0^{n-1} \sum_{h=0}^k \frac{(-1)^{k-h}}{(h+1)!} \frac{D_x^{k+1}}{(k-h)!} \mathcal{L}_n^{(1)}(x) = n \mathcal{L}_n^{(1)}(x), \tag{3.15}$$

Proof - Equation (3.14), by using equations (3.13), becomes

$$\begin{aligned} \hat{M}\hat{P}\mathcal{L}_n^{(1)}(x) &= (\beta+x) \sum_{k=0}^{\infty} (-1)^k \frac{D_x^k}{k!} \sum_{k=0}^{\infty} \frac{D_x^{k+1}}{(k+1)!} \mathcal{L}_n^{(1)}(x) = \\ &= (\beta+x) \sum_{k=0}^{\infty} \sum_{h=0}^k \frac{(-1)^{k-h}}{(h+1)!} \frac{D_x^{k+1}}{(k-h)!} \mathcal{L}_n^{(1)}(x) = n \mathcal{L}_n^{(1)}(x), \end{aligned}$$

and furthermore, for any fixed n , the series expansions reduce to finite sums, with upper limit at most $n-1$, when it is applied to a polynomial of degree n , because the last not vanishing term contains the derivative of order n .

3.6 First values of the $\mathcal{L}_k^{(1)}(x)$

Here we show the first values for the Logarithmic-Sheffer polynomials $\mathcal{L}_k^{(1)}(x)$, defined by the generating function (3.1)₂

$$\begin{aligned} \mathcal{L}_0^{(1)}(x) &= 1, \\ \mathcal{L}_1^{(1)}(x) &= (\beta + x), \\ \mathcal{L}_2^{(1)}(x) &= (\beta + x)^2 - (\beta + x), \\ \mathcal{L}_3^{(1)}(x) &= (\beta + x)^3 - 3(\beta + x)^2 + 2(\beta + x), \\ \mathcal{L}_4^{(1)}(x) &= (\beta + x)^4 - 6(\beta + x)^3 + 11(\beta + x)^2 - 6(\beta + x), \\ \mathcal{L}_5^{(1)}(x) &= (\beta + x)^5 - 10(\beta + x)^4 + 35(\beta + x)^3 - 50(\beta + x)^2 + 24(\beta + x), \\ \dots & \quad \dots \quad \dots \quad \dots \end{aligned}$$

Further values can be easily achieved by using Wolfram Alpha[©].

Remark 6. Note that the polynomials $\mathcal{L}_k^{(1)}(x)$ are strictly connected with the dual Actuarial polynomials $\tilde{\alpha}_k^{(\beta)}(x)$, introduced in [18], and related to the Actuarial ones [24, 26].

Namely, the first ones are obtained by the second ones changing β into $-\beta$ and writing the second ones with alternating sign. This is a trivial consequence of the relevant generating functions.

However, the $\mathcal{L}_k^{(1)}(x)$ are more interesting, because they are suitable for iteration, as it is shown in the subsequent Sections.

4 Iterated Logarithmic-Sheffer polynomial sets

Here we iterate the procedure introduced in Sect. 3, by considering the Sheffer polynomial sets defined by putting

$$\begin{aligned} A(t) &= (t + 1)^\beta, & H(t) &= \Lambda_1(t), \\ G(t, x) &= \exp [(\beta + x) \Lambda_1(t)] = \sum_{k=0}^{\infty} \mathcal{L}_k^{(2)}(x) \frac{t^k}{k!}. \end{aligned} \tag{4.1}$$

We find: $f(t) = H^{-1}(t) = E_1(t)$

$$\frac{A'(t)}{A(t)} = \frac{\beta}{t + 1}, \quad H'(t) = \frac{1}{(\Lambda_0(t) + 1)(t + 1)},$$

and consequently

$$\begin{aligned} f(\sigma) &= E_1(D_x), & \frac{A'[E_1(D_x)]}{A[E_1(D_x)]} &= \frac{\beta}{E_1(D_x) + 1} \\ H'[f(\sigma)] &= \frac{1}{(\Lambda_0(E_1(D_x)) + 1)(E_1(D_x) + 1)} = \frac{1}{e^{D_x}(E_1(D_x) + 1)}. \end{aligned}$$

Theorem 4.1. *The Logarithmic-Sheffer polynomials $\{\mathcal{L}_k^{(2)}(x)\}$ are quasi-monomial under the action of the operators*

$$\hat{P} = E_1(D_x), \tag{4.2}$$

$$\hat{M} = \frac{\beta[E_0(D_x)+1]+x}{[E_0(D_x)+1][E_1(D_x)+1]},$$

where we have put, for convenience, $e^{D_x} = [E_0(D_x) + 1]$.

4.1 Differential equation for the $\mathcal{L}_k^{(2)}(x)$

According to the results of monomiality principle, the quasi-monomial polynomials $\{p_n(x)\}$ satisfy the differential equation

$$\hat{M}\hat{P}p_n(x) = np_n(x). \tag{4.3}$$

In the present case, recalling equations (3.13), we have

Theorem 4.2. *The Logarithmic-Sheffer polynomials $\{\mathcal{L}_k^{(2)}(x)\}$ satisfy the differential equation*

$$\frac{\beta [E_0(D_x)+1]+x}{[E_0(D_x)+1] [E_1(D_x)+1]} \mathcal{L}_n^{(2)}(x) = n \mathcal{L}_n^{(2)}(x). \tag{4.4}$$

4.2 First values of the $\mathcal{L}_k^{(2)}(x)$

Here we show the first values for the Logarithmic-Sheffer polynomials $\mathcal{L}_k^{(2)}(x)$, defined by the generating function (3.1)₂

$$\begin{aligned} \mathcal{L}_0^{(2)}(x) &= 1, \\ \mathcal{L}_1^{(2)}(x) &= (\beta + x), \\ \mathcal{L}_2^{(2)}(x) &= (\beta + x)^2 - 3(\beta + x), \\ \mathcal{L}_3^{(2)}(x) &= (\beta + x)^3 - 9(\beta + x)^2 + 15(\beta + x), \\ \mathcal{L}_4^{(2)}(x) &= (\beta + x)^4 - 18(\beta + x)^3 + 87(\beta + x)^2 - 105(\beta + x), \\ \mathcal{L}_5^{(2)}(x) &= (\beta + x)^5 - 30(\beta + x)^4 + 285(\beta + x)^3 - 975(\beta + x)^2 + 947(\beta + x), \\ \dots & \qquad \dots \qquad \dots \qquad \dots \end{aligned}$$

Further values can be easily achieved by using Wolfram Alpha[©].

5 The general case

In general, by putting

$$\begin{aligned} A(t) &= (t + 1)^\beta, & H(t) &= \Lambda_r(t), \\ G(t, x) &= \exp [(\beta + x) \Lambda_r(t)] = \sum_{k=0}^{\infty} \mathcal{L}_k^{(r+1)}(x) \frac{t^k}{k!}, \end{aligned} \tag{5.1}$$

we find:

$$f(t) = H^{-1}(t) = E_r(t),$$

$$\frac{A'(t)}{A(t)} = \frac{\beta}{t + 1}, \quad H'(t) = \prod_{\ell=0}^{r-1} \frac{1}{[\Lambda_\ell(t)+1] (t+1)},$$

and consequently

$$f(\sigma) = E_r(D_x), \quad \frac{A'[E_r(D_x)]}{A[E_r(D_x)]} = \frac{\beta}{E_r(D_x)+1},$$

$$H'[E_r(D_x)] = \prod_{\ell=0}^{r-1} \frac{1}{[\Lambda_\ell(E_r(D_x))+1] [E_r(D_x)+1]}.$$

Recalling Remark 3.1 and changing the indexes, we find

$$H'[E_r(D_x)] = \prod_{\ell=0}^{r-1} \frac{1}{[E_{r-\ell-1}(D_x)+1] [E_r(D_x)+1]} = \prod_{h=0}^r \frac{1}{[E_h(D_x)+1]}.$$

Theorem 5.1. *The Logarithmic-Sheffer polynomials $\{\mathcal{L}_k^{(r+1)}(x)\}$ are quasi-monomial under the action of the operators*

$$\hat{P} = E_r(D_x), \quad \hat{M} = \frac{\beta \prod_{h=0}^{r-1} [E_h(D_x) + 1] + x}{\prod_{h=0}^r [E_h(D_x) + 1]}. \quad (5.2)$$

5.1 Differential equation for the $\mathcal{L}_k^{(r+1)}(x)$

According to the results of monomiality principle [9, 10], the quasi-monomial polynomials $\{p_n(x)\}$ satisfy the differential equation

$$\hat{M}\hat{P}p_n(x) = np_n(x). \quad (5.3)$$

In the present case, recalling equations (3.13), we have

Theorem 5.2. *The Logarithmic-Sheffer polynomials $\{\mathcal{L}_k^{(r+1)}(x)\}$ satisfy the differential equation*

$$\left\{ \frac{\beta \prod_{h=0}^{r-1} [E_h(D_x) + 1] + x}{\prod_{h=0}^r [E_h(D_x) + 1]} \right\} E_r(D_x) \mathcal{L}_n^{(r+1)}(x) = n \mathcal{L}_n^{(r+1)}(x). \quad (5.4)$$

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