SOME TRANSFORMATION FORMULAE OF RAMANUJAN'S THIRD ORDER AND SIXTH ORDER MOCK THETA FUNCTIONS

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Abstract
In this paper, we define the (p, q) analogues of Ramanujan's mock theta functions of third and sixth order and further generalize the corresponding mock theta functions by adding variables. It is observed that these generalized functions belong to the family of $F_{p,q}$-functions. We have established some transformation formulae of (p, q) −analogues of generalized mock theta functions of third and sixth order by relating them to the (p, q) series. This provides a new insight in the study of third and sixth order mock theta functions.

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1. Introduction

The mock theta functions were Ramanujan's last gift to the world of Mathematics. In his last letter to G.H. Hardy [9, pp. 354-355], Ramanujan mentioned that he had discovered seventeen functions, having peculiar properties which were not exhibited by the usual theta functions. Hence he named these functions 'mock theta functions' and divided them into three classes; four functions of order three, ten functions in two groups , each containing five functions of order five, and three functions of order seven.

The discovery of these functions came to light only in 1935 when Watson mentioned them in his celebrated presidential address delivered at the meeting of London mathematical society. Watson[12] added three more functions to the list of four functions of order 3 given by Ramanujan. Later, some more mock theta functions were discovered by Gordon and McIntosh[7] .

Andrews and Hickerson [3] in 1991 in a long paper pointed out the existence of seven more q-series and some identities given in the 'Lost' notebook and called them mock theta functions of order six, on the basis of certain combinatorial considerations.

The (p, q) number, introduced by Chakrabarty and Jagannathan [14] is a generalization of Heine q- number which occurs in the theory of quantum algebras. Surprisingly, Wachs and White [17] also independently introduced the (p, q) analogue of q number during the process of generalization of Sterling numbers [18], around the same time as [14]. Some applications of the (p, q) hypergeometric series in the context of representations of two-parameter quantum groups have been considered by Nishizawa [15] and Sahai and Srivastava [16].

The main purpose of the present paper is to generalize Ramanujan's mock theta functions to their corresponding (p, q) analogues and to relate them to (p, q) series. In section 2, we define $F_{p,q}$ functions which are extensions of F_q function and also introduce (p, q) analogues of mock theta functions of third and sixth order respectively.

In section 3, we establish that (p,q) analogues of generalized third and sixth order mock theta functions are $F_{p,q}$-functions. In section 4, we derive transformation formulae of (p,q) analogues of generalized mock theta functions of third and sixth order. We also establish some formulae between two generalized mock theta functions of the same group.

Mathematical Preliminaries

We first present the definitions and notations employed in this paper.

The (p,q) −number is defined by [14]

$$[n]_{p,q} = \frac{P^n-q^n}{p-q} \tag{1.1}$$

The q –shifted factorial [6] is given by

$$(a;q)_n = \begin{cases} 1, & n=0 \\ (1-a)(1-aq)\dots(1-aq^{n-1}) & \\ & n=1,2,\dots \end{cases} \tag{1.2}$$

with

$$(a_1,a_2,\dots,a_k;q)_n = (a_1;q)_n(a_2;q)_n\dots(a_k;q)_n \,. \tag{1.3}$$

The basic hypergeometric series is defined by [6]

$$ {}_r\Phi_s \ (a_1,a_2\dots,a_r;b_1,b_2,\dots,b_s;\ q,z)$$

$$= \sum_{n=0}^{\infty} \frac{(a_1,a_2,\dots,a_r;q)_n \left((-1)^n q^{\frac{n(n-1)}{2}}\right)^{1+s-r} z^n}{(b_1,b_2,\dots,b_s;q)_n(q;q)_n} \,. \tag{1.4}$$

Twin basic analogues of (1.2) and (1.3) are defined as follows [13]

$$\big((a,b);(p,q)\big)_n = \begin{cases} 1, & n=0 \\ (a-b)(ap-bq)(ap^2-bq^2)\dots(ap^{n-1}-bq^{n-1}) & \\ & n=1,2,\dots \end{cases} \tag{1.5}$$

$$\Big((a_{1p},a_{1q}),(a_{2p},a_{2q}),\dots,(a_{mp},a_{mq});(p,q)\Big)_n$$

$$= \Big((a_{1p},a_{1q});(p,q)\Big)_n\Big((a_{2p},a_{2q});(p,q)\Big)_n\dots\Big((a_{mp},a_{mq});(p,q)\Big)_n \,. \tag{1.6}$$

Particularly, putting $a=1$ and $p=1$ in equation (1.5), it becomes equivalent to the equation (1.2), in view of the relation

$$\big((1,b);(1,q)\big)_n = (b;q)_n \tag{1.7}$$

Then (p,q) analogue of (1.4) or the twin basic hypergeometric series has been defined as [13]

$${}_r\Phi_s \ \Big((a_{1p},a_{1q}),\dots,(a_{rp},a_{rq});\ (b_{1p},b_{1q}),\dots,(b_{sp},b_{sq});(p,q),z\Big)$$

$$= \sum_{n=0}^{\infty} \frac{\big((a_{1p},a_{1q}),\dots,(a_{rp},a_{rq});(p,q)\big)_n}{\big((b_{1p},b_{1q}),\dots,(b_{sp},b_{sq});(p,q)\big)_n\big((p,q);(p,q)\big)_n} \times \big((-1)^n(q/p)^{n(n-1)/2}\big)^{1+s-r} z^n \,. \tag{1.8}$$

where $|q\,/\,p|<1$, assuming that $0<q<p\le 1$. The numbers p and q can also take other values if there is no problem with the convergence of the particular series involved in a result.

Definition of third order mock theta functions and sixth order mock theta functions

The sixth order mock theta functions of Ramanujan [3] are

$$\Phi(q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2} (q;q^2)_n}{(-q)_{2n}},$$

$$\Psi(q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{(n+1)^2} (q;q^2)_n}{(-q)_{2n+1}},$$

$$\rho(q) = \sum_{n=0}^{\infty} \frac{q^{\frac{n(n+1)}{2}} (-q)_n}{(q;q^2)_{n+1}},$$

$$\sigma(q) = \sum_{n=0}^{\infty} \frac{q^{\frac{(n+1)(n+2)}{2}} (-q)_n}{(q;q^2)_{n+1}},$$

$$\lambda(q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^n (q;q^2)_n}{(-q)_n},$$

$$\mu(q) = \sum_{n=0}^{\infty} \frac{(-1)^n (q;q^2)_n}{(-q)_n},$$

and

$$\gamma(q) = \sum_{n=0}^{\infty} \frac{q^{n^2} (q)_n}{(q^3;q^3)_n}.$$

The third order mock theta functions of Ramanujan [9,12] are

$$f(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q)_n^2},$$

$$\phi(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q^2;q^2)_n},$$

$$\psi(q) = \sum_{n=1}^{\infty} \frac{q^{n^2}}{(q;q^2)_n},$$

$$\chi(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1-q+q^2)\dots(1-q^n+q^{2n})},$$

$$\omega(q) = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q;q^2)_{n+1}^2},$$

$$\nu(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(-q;q^2)_{n+1}},$$

and

$$\rho(q) = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(1+q+q^2)\dots(1+q^{2n+1}+q^{4n+2})}.$$

In next section we shall define their (p,q) analogues.

2. Definition of $F_{p,q}$ functions

The functions which satisfy the functional equation

$$D_{p,q,z}\, F(z,\alpha,\beta) = F(z,\alpha+1,\beta+1), \tag{2.1}$$

where

$$zD_{p,q,z}\, F(z,\alpha,\beta) = F(z,\alpha,\beta) - F(zpq,\alpha,\beta)$$

are called $F_{p,q}$ functions.

Definition of (p,q) analogue of third order mock theta function and sixth order mock theta function:

We define (p,q) analogues of third order mock theta functions as

$$f(p,q) = \sum_{n=0}^{\infty} \frac{p^{n^2} q^{n^2}}{((1,-q);(1,q))_n^2}, \tag{2.2}$$

$$\phi(p,q) = \sum_{n=0}^{\infty} \frac{p^{n^2} q^{n^2}}{((1,-q^2);(1,q^2))_n}, \tag{2.3}$$

$$\psi(p,q) = \sum_{n=1}^{\infty} \frac{p^{n^2} q^{n^2}}{((1,q);(1,q^2))_n}, \tag{2.4}$$

$$\omega(p,q) = \sum_{n=0}^{\infty} \frac{p^{2n(n+1)} q^{2n(n+1)}}{((1,q);(1,q^2))_{n+1}^2}, \tag{2.5}$$

$$\nu(p,q) = \sum_{n=0}^{\infty} \frac{p^{n(n+1)} q^{n(n+1)}}{((1,-q);(1,q^2))_{n+1}}. \tag{2.6}$$

The (p,q) analogue of functions $\chi(q)$ and $\rho(q)$ have not been considered as they take the form of Ramanujan functions $\chi(q)$ and $\rho(q)$ respectively on taking $pq = r,\ |r| < 1$. Thus the $\chi(p,q)$ and $\rho(p,q)$ are essentially the same as $\chi(q)$ and $\rho(q)$.

We further generalize third order (p,q) mock theta functions by adding independent variables and parameters as follows:

$$f\big(t,\alpha,\beta,\delta,\varepsilon,z,x;(p,q)\big) = \frac{1}{(t)_\infty} \sum_{n=0}^{\infty} \frac{(t)_n p^{n^2-4n+n\delta} q^{n^2-4n+n\beta} \alpha^n \varepsilon^n z^{2n} x^{2n}}{((1,-z);(1,q))_n \left(\left(1,-\frac{\alpha z}{q}\right);(1,q)\right)_n}, \tag{2.7}$$

$$\phi\big(t,\alpha,\beta,\delta,z,x;(p,q)\big) = \frac{1}{(t)_\infty} \sum_{n=0}^{\infty} \frac{(t)_n p^{n^2-3n+n\delta} q^{n^2-3n+n\beta} z^{2n} x^{2n}}{\left(\left(1,-\frac{\alpha z^2}{q}\right);(1,q^2)\right)_n}, \tag{2.8}$$

$$\psi\big(t,\alpha,\beta,\delta,z,x;(p,q)\big) = \frac{1}{(t)_\infty} \sum_{n=0}^{\infty} \frac{(t)_n p^{n^2-n+n\delta} q^{n^2-n+n\beta} z^{2n+1} x^{2n+1}}{\left(\left(1,\frac{\alpha z^2}{q^2}\right);(1,q^2)\right)_{n+1}}, \tag{2.9}$$

$$\nu\big(t,\alpha,\beta,\delta,z,x;(p,q)\big) = \frac{1}{(t)_\infty} \sum_{n=0}^{\infty} \frac{(t)_n\, q^{n^2-2n+n\beta} z^{2n} p^{n^2-2n+n\delta} x^{2n}}{\left(\left(1,-\frac{\alpha^2 z^2}{q^3}\right);(1,q^2)\right)_{n+1}}, \tag{2.10}$$

$$\omega(t,\alpha,\beta,\delta,\varepsilon,z,x;(p,q) = \frac{1}{(t)_\infty} \sum_{n=0}^{\infty} \frac{(t)_n p^{2n^2-5n-4+n\delta} x^{4(n+1)} q^{2n^2-5n+n\beta-4} \alpha^{2n} z^{4(n+1)} \varepsilon^{2n}}{\left(\left(1,\frac{z^2}{q}\right);(1,q^2)\right)_{n+1} \left(\left(1,\frac{\alpha^2 z^2}{q^3}\right);(1,q^2)\right)_{n+1}}. \tag{2.11}$$

Setting $t = 0, \alpha = q, \beta = 1, \delta = 1, z = q, x = p, \varepsilon = p$ these generalized twin basic functions reduce to corresponding third order (p,q) mock theta functions.

Similarly (p,q) −analogues of seven sixth order mock theta function can be defined as follows:

$$\Phi(p,q)=\sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2}p^{n^2}((1,q);(1,q^2))_n}{((1,-q);(1,q))_{2n}}, \tag{2.12}$$

$$\Psi(p,q)=\sum_{n=0}^{\infty}\frac{(-1)^n p^{(n+1)^2}q^{(n+1)^2}((1,q);(1,q^2))_n}{((1,-qp^{2n});(1,q))_{2n+1}}, \tag{2.13}$$

$$\rho(p,q)=\sum_{n=0}^{\infty}\frac{p^{\frac{n(n+1)}{2}}q^{\frac{n(n+1)}{2}}((1,-q);(1,q))_n}{((1,q);(1,q^2))_{n+1}}, \tag{2.14}$$

$$\sigma(p,q)=\sum_{n=0}^{\infty}\frac{p^{\frac{(n+1)(n+2)}{2}}q^{\frac{(n+1)(n+2)}{2}}((1,-q);(1,q))_n}{((1,q);(1,q^2))_{n+1}}, \tag{2.15}$$

$$\lambda(p,q)=\sum_{n=0}^{\infty}\frac{(-1)^n p^n q^n((1,q);(1,q^2))_n}{((1,-q);(1,q))_n}, \tag{2.16}$$

$$\mu(p,q)=\sum_{n=0}^{\infty}\frac{(-1)^n((1,q);(1,q^2))_n}{((1,-q);(1,q))_n}, \tag{2.17}$$

$$\gamma(p,q)=\sum_{n=0}^{\infty}\frac{p^{n^2}q^{n^2}((1,q);(1,q))_n}{((1,q^3);(1,q^3))_n}. \tag{2.18}$$

Next we define further generalization of seven sixth order (p,q) −mock theta functions as follows :

$$\Phi(t,\alpha,\beta,z,x;(p,q))=\frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n(-1)^n q^{n(n-3)+n\beta}p^{n(n-3)+n\alpha}z^{2n}x^{2n}\left(\left(1,\frac{z^2}{q}\right);(1,q^2)\right)_n}{\left(\left(1,\frac{-z^2}{q}\right);(1,q)\right)_{2n}}, \tag{2.19}$$

$$\Psi(t,\alpha,\beta,z,x;(p,q))=\frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n(t)_n\, p^{n(n-1)+n\alpha}q^{n(n-1)+n\beta}z^{2n+1}x^{2n+1}\left(\left(1,\frac{z^2}{q}\right);(1,q^2)\right)_n}{\left(\left(1,\frac{-z^2p^{2n}}{q}\right);(1,q)\right)_{2n+1}}, \tag{2.20}$$

$$\rho(t,\alpha,\beta,z,x;(p,q))=\frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n p^{\frac{n(n-3)}{2}+n\alpha}q^{\frac{n(n-3)}{2}+n\beta}z^n x^n((1,-z);(1,q))_n}{\left(\left(1,\frac{z^2}{q}\right);(1,q^2)\right)_{n+1}}, \tag{2.21}$$

$$\sigma((t,\alpha,\beta,z,x;(p,q))=\frac{1}{2(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n\, p^{\frac{n(n-1)}{2}+n\alpha}z^{n+1}\left(\left(1,\frac{-z}{q}\right);(1,q)\right)_{n+1}q^{\frac{n(n-1)}{2}+n\beta}x^{n+1}}{\left(\left(1,\frac{z^2}{q}\right);(1,q^2)\right)_{n+1}}, \tag{2.22}$$

$$\lambda(t,\alpha,\beta,z,x;(p,q))=\frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n(-1)^n p^{n\alpha}q^{n\beta}\left(\left(1,\ \frac{q^3}{z^2}\right);(1,q^2)\right)_n}{\left(\left(1,\frac{-q^2}{z}\right);(1,q)\right)_n}, \tag{2.23}$$

$$\mu(t,\alpha,\beta,z,x;(p,q))=\frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n(-1)^n\, p^{n(\alpha-1)}q^{n(\beta-1)}\left(\left(1,\frac{q^3}{z^2}\right);(1,q^2)\right)_n}{\left(\left(1,\frac{-q^2}{z}\right);(1,q)\right)_n}, \tag{2.24}$$

$$\gamma(t,\alpha,\beta,z,x;(p,q)) = \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n\, p^{n(n-3)+n\alpha}q^{n(n-3)+n\beta}z^{2n}x^{2n}}{((1,\nu^2 z);(1,q))_n\big((1,\nu^4 z);(1,q)\big)_n}, \tag{2.25}$$

where $\nu = e^{\frac{\pi i}{3}}$.

Putting $t = 0, \alpha = 1, \beta = 1, z = q$ and $x = p$ these functions take the form of (p,q) −analogues of sixth order mock theta functions.

3. Generalized (p,q) mock theta functions are $F_{p,q}$ –functions:

In this section, we shall show that our generalized (p,q) −mock theta functions are indeed $F_{p,q}$ –functions.

Here we give the detailed proof for the function $\Phi((t,\alpha,\beta,z,x;(p,q))$ only; the proofs for other functions are similar and hence omitted.

Theorem 1 : The generalized third and sixth order mock theta functions defined by (2.7) to (2.11) and (2.19) to (2.25) are $F_{p,q}$ functions.

Proof: We consider the generalized mock theta function $\Phi((t,\alpha,\beta, z, x;(p, q))$.

Applying the difference operator $D_{p,q,t}$ *to* $\Phi((t,\alpha,\beta,z, x;(p, q))$ we have

$$tD_{p,q,t}\,\Phi((t,\alpha,\beta,z, x;(p, q)) = \Phi((t,\alpha,\beta,z, x;(p, q)) - \Phi((tpq,\alpha,\beta,z, x;(p, q))$$

$$= \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n(t)_n\, p^{n(n-3)+n\alpha}q^{n(n-3)+n\beta}x^{2n}z^{2n}\left(\left(1,\frac{z^2}{q}\right);(1,q^2)\right)_n}{\left(\left(1,\frac{-z^2}{q}\right);(1,q)\right)_{2n}}$$

$$-\frac{1}{(tpq)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n(tpq)_n\, p^{n(n-3)+n\alpha}q^{n(n-3)+n\beta}z^{2n}x^{2n}\left(\left(1,\frac{z^2}{q}\right);(1,q^2)\right)_n}{\left(\left(1,\frac{-z^2}{q}\right);(1,q)\right)_{2n}}$$

$$= \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n(t)_n\, p^{n(n-3)+n\alpha}q^{n(n-3)+n\beta}x^{2n}z^{2n}\left(\left(1,\frac{z^2}{q}\right);(1,q^2)\right)_n}{\left(\left(1,\frac{-z^2}{q}\right);(1,q)\right)_{2n}}$$

$$-\frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n(t)_n\, p^{n(n-3)+n\alpha}z^{2n}x^{2n}q^{n(n-3)+n\beta}\left(\left(1,\frac{z^2}{q}\right);(1,q^2)\right)_n(1-tp^nq^n)}{\left(\left(1,\frac{-z^2}{q}\right);(1,q)\right)_{2n}}$$

$$= \frac{t}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n(t)_n\, p^{n(n-3)+n(\alpha+1)}q^{n(n-3)+n(\beta+1)}x^{2n}z^{2n}\left(\left(1,\frac{z^2}{q}\right);(1,q^2)\right)_n}{\left(\left(1,\frac{-z^2}{q}\right);(1,q)\right)_{2n}}$$

$= t\Phi((t, \alpha+1, \beta+1, z, x;(p, q))$.

So $D_{p,q,t}\Phi((t, \alpha, \beta, z, x ; (p, q)) = \Phi((t, \alpha+1, \beta+1, z, x; (p, q))$.

Hence $\Phi((t, \alpha, \beta, z, x; (p, q))$ is a $F_{p,q}$ -function.

Similarly all other functions in theorem 1 can be shown to be $F_{p,q}$ functions.

4. Transformation Formulae of Generalized (p, q) Mock Theta Functions:

In this section we shall establish some transformation formulae to show that a generalized (p, q) mock theta functions can be expressed in terms of another (p, q) mock theta functions.

Theorem : 2

$(i)\,\Phi\big(t, \alpha, \beta, z, x; (p, q)\big)$

$$= \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n (t)_n p^{n(n-3)+n\alpha} x^{2n} z^{2n}\, q^{n(n-3)+n\beta}\left(\left(1, \frac{z^2}{q}\right); (1, q^2)\right)_n}{\left(\left(1, \frac{-z^2 p^{2n}}{q}\right); (1, q)\right)_{2n+1}} + \frac{z}{qx}\;\Psi\big(t, \alpha, \beta, z, x; (p, q)\big),$$

$$(ii)\,\sigma\big(t, \alpha, \beta, z, x; (p, q)\big) = \frac{zx}{2}\left(1 + \frac{z}{q}\right) D_{p,q,t}\,\rho\big(t, \alpha, \beta, z, x; (p, q)\big),$$

$$(iii)\,D_{p,q,t}\,\Phi\big(t, \alpha^2, \beta, \delta, z, x; (p, q)\big) = \left(1 + \frac{\alpha^2 z^2}{q^3}\right)\nu\big(t, \alpha, \beta, \delta, z, x; (p, q)\big),$$

$$(iv)\,\psi\left(t, \frac{-\alpha^2}{q}, \beta, \delta, z, x; (p, q)\right) = xz\, D_{p,q,t}\,\nu\big(t, \alpha, \beta, \delta, z, x; (p, q)\big).$$

Proof of (i)

$\Phi\big(t, \alpha, \beta, z, x; (p, q)\big)$

$$= \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n (t)_n\, p^{n(n-3)+n\alpha}\, q^{n(n-3)+n\beta} x^{2n} z^{2n}\left(\left(1, \frac{z^2}{q}\right); (1, q^2)\right)_n (1 + z^2 p^{2n} q^{2n-1})}{\left(\left(1, \frac{-z^2 p^{2n}}{q}\right); (1, q)\right)_{2n+1}}$$

$$= \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n (t)_n\, p^{n(n-3)+n\alpha}\, q^{n(n-3)+n\beta} x^{2n} z^{2n}\left(\left(1, \frac{z^2}{q}\right); (1, q^2)\right)_n}{\left(\left(1, \frac{-z^2 p^{2n}}{q}\right); (1, q)\right)_{2n+1}}$$

$$+ \frac{z}{qx}\frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n (t)_n\, p^{n(n-1)+n\alpha}\, q^{n(n-1)+n\beta} z^{2n+1} x^{2n+1}\left(\left(1, \frac{z^2}{q}\right); (1, q^2)\right)_n}{\left(\left(1, \frac{-z^2 p^{2n}}{q}\right); (1, q)\right)_{2n+1}}$$

$$= \frac{1}{(t)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n (t)_n \, p^{n(n-3)+n\alpha} \, q^{n(n-3)+n\beta} x^{2n} z^{2n} \left(\left(1, \frac{z^2}{q}\right); (1, q^2)\right)_n}{\left(\left(1, \frac{-z^2 p^{2n}}{q}\right); (1, q)\right)_{2n+1}} + \frac{z}{qx} \; \Psi\big(t, \alpha, \beta, x, z; (p, q)\big),$$

which proves Theorem 2(i).

Proof of (ii)

$$\sigma\big(t, \alpha, \beta, z, x; (p, q)\big) = \frac{1}{2(t)_\infty} \sum_{n=0}^{\infty} \frac{(t)_n \, p^{\frac{n(n-1)+n\alpha}{2}} \, z^{n+1} \, q^{\frac{n(n-1)+n\beta}{2}} \, x^{n+1} \left(\left(1, \frac{-z}{q}\right); (1, q)\right)_{n+1}}{\left(\left(1, \frac{z^2}{q}\right); (1, q^2)\right)_{n+1}}$$

$$= \frac{1}{2(t)_\infty} \sum_{n=0}^{\infty} \frac{(t)_n \, p^{\frac{n^2-n+2n\alpha}{2}} \, z^{n+1} \, q^{\frac{n^2-n+2n\beta}{2}} x^{n+1} \left(\left(1, \frac{-z}{q}\right); (1, q)\right)_{n+1}}{\left(\left(1, \frac{z^2}{q}\right); (1, q^2)\right)_{n+1}}$$

$$= \frac{zx}{2(t)_\infty} \left(1 + \frac{z}{q}\right) \sum_{n=0}^{\infty} \frac{p^{\frac{n(n-1)+n\alpha}{2}} \, q^{\frac{n(n-1)+n\beta}{2}} \, z^n \left((1, -z); (1, q)\right)_n x^n}{\left(\left(1, \frac{z^2}{q}\right); (1, q^2)\right)_{n+1}}$$

$$= \frac{zx}{2} \left(1 + \frac{z}{q}\right) D_{p,q,t} \, \rho\big(t, \alpha, \beta, z, x; (p, q)\big),$$

which proves theorem 2 (ii).

Proof of (iii)

Replacing α by α^2 in $\Phi(\big(t, \alpha, \beta, \delta, z, x; (p, q)\big)$, we have

$$D_{p,q,t} \, \Phi(\big(t, \alpha^2, \beta, \delta, z, x; (p, q)\big) = \frac{1}{(t)_\infty} \sum_{n=0}^{\infty} \frac{(t)_n \, p^{n^2-2n+n\alpha} \quad q^{n^2-2n+n\beta)} z^{2n} x^{2n}}{\left(\left(1, \frac{-\alpha^2 z^2}{q}\right); (1, q^2)\right)_n}$$

$$= \left(1 + \frac{\alpha^2 z^2}{q^3}\right) \nu \, (\big(t, \alpha, \beta, \delta, z, x; (p, q)\big),$$

which proves theorem 2 (iii).

Proof of (iv)

$$\psi\big(t, \alpha, \beta, \delta, z, x; (p, q)\big) = \frac{1}{(t)_\infty} \sum_{n=0}^{\infty} \frac{(t)_n \, p^{n^2-n+n\delta} \, q^{n^2-n+n\beta} z^{2n+1} x^{2n+1}}{\left(\left(1, \frac{\alpha z^2}{q^2}\right); (1, q^2)\right)_{n+1}},$$

Replacing α by $\frac{-\alpha}{q}$ and then writing α^2 for α in $\psi\big(t, \alpha, \beta, \delta, z, x; (p, q)\big)$,

$$\psi\left(t,\frac{-\alpha^2}{q},\beta,\delta,z,x;(p,q)\right)=\frac{zx}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n\,p^{n^2-n+n\delta}\,q^{n^2-n+n\beta}z^{2n}x^{2n}}{\left(\left(1,\frac{-\alpha^2z^2}{q^3}\right);(1,q^2)\right)_{n+1}}$$

$= xz\,D_{p,q,t}\,\nu\big((t,\alpha,\beta,\delta,z,x;(p,q)\big),$
which proves theorem 2(iv).

Conclusion

In this paper we have defined and further generalized (p,q) analogues of Ramanujan's third and sixth order mock-theta functions and have shown that these functions are $F_{p,q}$ functions. Subsequently we have established some transformation formulae between generalized functions of same order. These results are quite general in nature and reduce to earlier established relations on specializing the parameters.

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