IMAGE OF WRIGHT-TYPE HYPERGEOMETRIC AND MITTAG-LEFFLER FUNCTIONS UNDER A GENERALIZED W-E-K FRACTIONAL OPERATOR

By

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Abstract
In the present paper, the approach of the authors is based on the use of a generalized fractional calculus operator namely Wright-Erdelyi-Kober operator (W-E-K) to obtain the images of generalized special functions. Some special cases have also been discussed.

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1. Introduction

The hypergeometric functions play a very important role in solving numerous problems of mathematical physics, engineering and mathematical sciences [see, 6, 9, 10, 14, 15, 16, 17, 24, 26].
The Gauss hypergeometric function is defined [22] as

$$ {}_2F_1\ (a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} z^k, |z| < 1, c \neq 0, -1, -2, \ldots \ . \tag{1}$$

Here in Eqn. (1), the Pochhammer symbol is defined by

$$(\gamma)_n = \gamma(\gamma+1)(\gamma+2)(\gamma+3)\ldots(\gamma+n-1) \forall n = 1,2,3,\ldots \text{ and } (\gamma)_0 = 1. \tag{2}$$

Several generalizations of hypergeometric functions (1) - (2) have been made and also motivated us to make further investigations in this topic. Virchenko *et al.* [27] defined the generalized hypergeometric function ${}_2R_1^{\tau}(z)$ in a different manner, given by

$$ {}_2R_1^{\omega,\mu}(z) = \frac{\Gamma(c)\mu}{\Gamma(c-b)\,\Gamma(b)} \int_0^1 t^{\mu b-1} (1-t)^{c-b-1} (1-zt^{\omega})^{c-b-1}\, dt,$$

where, $\Re(c) > R(b) > 0$. (3)

This is the analogue of Euler's formula for Gauss's hypergeometric functions [4].
Prajapati *et al.* [21], Shukla and Prajapati [25] and Srivastava and Tomovski [26] used the fractional calculus approach in the study of an integral operator and also generalization of the Mittag-Leffler functions [18, 25, 28]. The subject of fractional calculus [1, 2, 6, 9, 10, 14, 15, 16, 17] deals with the investigations of integrals and derivatives of any arbitrary real or complex order, by which, we unify and extend the notions of integer-order derivative and *n*-fold integrals. Kumar [11] has obtained various generalized formulae on application of multiple fractional integral operators analogous to generalized Erdelyi – Kober operators [23]. Again,

Kumar, Pathan and Kumari [13] have evaluated identities for generalized Erdelyi – Kober operators [23] and further obtained generalized results with multiple Mellin – transformations. Kumar and Kumari [12], have investigated some measures analogous to Carlson's Dirichlet Measures on using generalized Erdelyi – Kober operators [23].
The theory of fractional operators defined in Eqns. (11) to (16) has gained importance and popularity during the last four decades or so, mainly due to its vast potential of demonstrated applications in various seemingly diversified fields of science and engineering, such as fluid flow, rheology, diffusion, relaxation, oscillation, anomalous diffusion, reaction-diffusion, turbulence, diffusive transport, electric networks, polymer physics, chemical physics, electrochemistry of corrosion, relaxation processes in complex systems, propagation of seismic waves, dynamical processes in self-similar and porous structures. Recently some interesting results on fractional boundary value problems and fractional partial differential equations were also discussed by Nyamoradi *et al.* [19] and Baleanu *et al.* [1, 2].
The Mittag–Leffler function [18, 28] is a generalization of hypergeometric function which appears as solution of well-known fractional differential and integral equations representing some physical and physiological phenomena like diffusion, transport theory, probability, elasticity and control theory. The purpose of this paper is to increase the accessibility of different dimensions of fractional calculus and generalization of hypergeometric functions to the real world problems of engineering and science (see [6], [9], [15], [16], [17], [24], [26]).

2. Mathematical Preliminaries

Fox's *H*-Function: Fox has defined H-function in terms of a general Mellin-Barnes type integral. He also investigated the most general Fourier kernel associated with the *H*- function and obtained the asymptotic expansions of the kernel for large values of the argument. Fox has also derived theorems about the H - function as asymmetric Fourier kernel and established certain operational properties for this function.

The H - function is defined by Fox [5] as follows

$$H(z) = H_{p,q}^{m,n}\left[\left(z\left|\begin{matrix}(a_1,\alpha_1),\ldots,(a_p,\alpha_p)\\(b_1,\beta_1),\ldots,(b_q,\beta_q)\end{matrix}\right.\right)\right] = \frac{1}{2\pi i}\int_L \varphi(s) z^s ds \tag{4}$$

where, $\varphi(s) = \dfrac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+\beta_j s)\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j s)}$, the point $z = 0$ is tacitly excluded,

$$\Omega = \sum_{j=1}^{n}\alpha_j - \sum_{j=n+1}^{p}\alpha_j + \sum_{j=1}^{m}\beta_j - \sum_{j=m+1}^{q}\beta_j > 0,\ |arg\ (z)| < \frac{1}{2}\Omega\pi. \tag{5}$$

Kalla et al. [7, 8] have used H – function (4) – (5) to derive various results in fractional calculus theory.

Wright-type Hypergeometric Function: The generalized form of the hypergeometric function has been investigated by Dotsenko [3] and Malovichko [14] and the series form of Eqn. (3) due to Dotsenko [3] is given by

$${}_2R_1^{\omega,\mu}(z) = {}_2R_1\,(a,b,c,\omega,\mu,z) = \frac{\Gamma(c)}{\Gamma(a)\,\Gamma(b)}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma\left(b+\frac{\omega}{\mu}n\right)}{\Gamma\left(c+\frac{\omega}{\mu}n\right)}\frac{z^n}{n!} \tag{6}$$

In 2001 Virchenko et al [27] have defined the Wright type Hypergeometric function by taking $\frac{\omega}{\mu} = \tau > 0$ in above equation (6) as

$$ {}_2R_1^{\tau}(z) = {}_2R_1\,(a, b, c, \tau, z)) = \frac{\Gamma(c)}{\Gamma(b)} \sum_{k=0}^{\infty} \frac{(a)_k \Gamma(b+\tau k)}{\Gamma(c+\tau k)} \frac{z^k}{k!} \; ; \; \tau > 0, |z| < 1. \qquad (7) $$

If $\boldsymbol{\tau}$ =1, then (7) reduces to Gauss's hypergeometric function (1) – (2).

Mittag-Leffler Functions:

The single parameter Mittag-Leffler function is defined by Mittag –Leffler ([15], [18]), as follows:

$$ E_{\alpha}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(1+\alpha n)}, \text{ for } \alpha \in \mathbb{C}, \Re(\alpha) > 0. \qquad (8) $$

Its generalization with two complex parameters was introduced by Wiman [28] as follows:

$$ E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\beta+\alpha n)}, \text{ for } \alpha, \beta \in \mathbb{C}, \Re(\alpha, \beta) > 0. \qquad (9) $$

In 1971, Prabhakar [20] introduced the generalized triple parameter ML function $E_{\alpha,\beta}^{\gamma}(z)$ as follows:

$$ E_{\alpha,\beta}^{\gamma}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n z^n}{\Gamma(\beta+\alpha n)}, \text{ for } \alpha, \beta, \gamma \in \mathbb{C}, \Re(\alpha, \beta, \gamma) > 0. \qquad (10) $$

3. Fractional Operators

Riemann-Liouville Fractional Operator: The Riemann-Liouville fractional integrals of arbitrary order μ for a function $f(t)$, is a natural consequence of the well-known formula (Cauchy-Dirichlet formula) that reduces the calculation of the μ- fold primitive of a function $f\,(x)$ to a single integral of convolution type (see [11, 12, 13, 24])

$$ I_a^{\mu} f(x) = \frac{1}{(\mu-1)!} \int_a^x (x-y)^{\mu-1} f(y) dy \,, (x > a). \qquad (11) $$

The above integral is meaningful for any number μ provided its real part is greater than zero.

Weyl Fractional Integral Operator:

The Weyl fractional integral of f(x) of order α, is defined as (see [11, 12, 13, 24])

$W_{\infty}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^{\infty} (t-x)^{\alpha-1} f(t)\, dt\,,\ -\infty < x < \infty,$ where $\alpha \in \mathbb{C}, \Re(\alpha) > 0$, is also denoted by $I_x^{\alpha} f(x)$. (12)

Kober Fractional Integral Operator:

The Kober operator is the generalization of Riemann-Liouville and Weyl operators which was given by Saxena in (1967). These operators have been used by many authors in deriving the solution of single, dual and triple integral equations involving different special functions as their kernels. The operator is defined by (see [11, 12, 13])

$$ E_{0,x}^{\alpha,\eta} f(x) = \frac{(x)^{-\eta-\alpha}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1}\, t^{\eta} f(t)\, dt \;, \quad \Re(\alpha) > 0. \qquad (13) $$

Erdelyi-Kober Fractional Integral Operator:

Generalization of Kober operator was introduced by Kalla and Saxena (1969) given as follows

$$ I(\alpha, \eta{:}\, m) f(x) = \frac{m(x)^{-\eta-m\alpha+m+1}}{\Gamma(\alpha)} \times \left\{ \int_0^x (x^m - t^m)^{\alpha-1}\, t^{\eta} f(t) dt \right\}, \Re(\alpha) > 0. \qquad (14) $$

Saigo Fractional Operator: Useful and interesting generalization of both the Riemann-Liouville and Erdlyi- Kober fractional integration operators is introduced by Saigo [23], in terms of Gauss's hypergeometric function as given below

Let α, β and $\eta \in \mathbb{C}$ (set of complex numbers) and let $x \in \mathbb{R}_+$, $\Re(\alpha) > 0$ and the fractional derivative of the first kind of a function

$$I_{0,x}^{\alpha,\beta,\eta} f(x) = \frac{x^{-\alpha-\beta}}{\Gamma\alpha} \int_0^x (x-t)^{\alpha-1} {}_2F_1 \left(\alpha+\beta, -\eta\,; 1-\frac{t}{x}\right) f(t)\,dt,\ \Re(\alpha) > 0$$
$$= \frac{d^n}{dx^n} I_{0,x}^{\alpha+n,\beta+n,\eta-n} f(x)\,, 0 < \Re(\alpha)+\eta \le 1\ (n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}). \tag{15}$$

Wright-Erdelyi-Kober Operators (W-E-K): (W-E-K) operators of fractional integration introduced by Kalla, Galue and Srivastava [7, 8]. For integer $m \ge 1$ and real parameters $\beta_k > 0,\ \lambda_k > 0, \delta_k \ge 0\ and\ \gamma_k$; k= 1. …. m. the multiplicity W-E-K fractional integral is defined by

$$\overline{I} f(z) = I_{\beta_k,\lambda_k,m}^{\gamma_k,\delta_k} f(z) = \int_0^\infty H_{m,m}^{m,0} \left[\left(\sigma \left| \begin{matrix} \left(\gamma_k+\delta_k+1-\frac{1}{\beta_k}, \frac{1}{\beta_k}\right)_{1,m} \\ \left(\gamma_k+1-\frac{1}{\lambda_k}, \frac{1}{\lambda_k}\right)_{1,m} \end{matrix} \right.\right)\right] f(z\sigma)\,d\sigma,$$

$if\ \sum_{k=1}^m \delta_k > 0.$

In the case, when $\forall\delta_k = 0$ and $\forall\lambda_k = \beta_k, k = 1,2, \ldots .m$, then it is an identity operator : $\overline{I} f(z) = f(z)$.

The W-E-K fractional integral operator of the power function is given in [7, 8] as follows:

$I_{\beta_k,\lambda_k,m}^{\gamma_k,\delta_k}(z^p) = c_p z^p$, where, $c_p = \prod_{k=1}^m \frac{\Gamma(\gamma_k+1+p/\lambda_k)}{\Gamma(\gamma_k+\delta_k+1+\frac{p}{\beta_k})}$;

including for $p = 0$, if for all $\gamma_k \ge -1$. (16)

4. Main Results

Theorem 1: For integer $m \ge 1$ and real parameters $\beta > 0,\ \lambda > 0, \delta \ge 0\ and\ \gamma$, the multiple W-E-K fractional integral operator for Wright-type hypergeometric function is as follows:

$$I_{\beta,\lambda,1}^{\gamma,\delta}\left\{ {}_2R_1(a,b,c,\tau,z) \right\} = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\ {}_3\Psi_3 \left(\begin{matrix} (a,1),(b,\tau),\left(\gamma+1,\frac{1}{\lambda}\right) \\ (c,\tau),\left(\gamma+\delta+1,\frac{1}{\beta}\right),(1,1) \end{matrix} \middle| z \right). \tag{17}$$

Proof: For $\beta > 0, \lambda > 0, \delta \ge 0$, use Eqn. (16) in left hand side of Eqn. (17), the image of a Wright-type hypergeometric function under a multiple W-E-K operator is

$$I_{\beta,\lambda,1}^{\gamma,\delta}\left\{ {}_2R_1(a,b,c,\tau,z) \right\} = \frac{\Gamma(c)}{\Gamma(b)} \sum_{k=0}^\infty \frac{(a)_k \Gamma(b+\tau k)}{\Gamma(c+\tau k)\,k!} I_{\beta,\lambda,1}^{\gamma,\delta}\{z^k\}$$

which implies that

$$I_{\beta,\lambda,1}^{\gamma,\delta}\left\{ {}_2R_1(a,b,c,\tau,z) \right\} = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\ {}_3\Psi_3 \left[\left(\begin{matrix} (a,1)(b,\tau)\left(\gamma+1,\frac{1}{\lambda}\right) \\ (c,\tau)\left(\gamma+\delta+1,\frac{1}{\beta}\right)(1,1) \end{matrix} \middle| z \right)\right]. \tag{18}$$

The Eqn. (18) proves the theorem 1.

Theorem 2: For integer $m \ge 1$ and real parameters $\beta > 0,\ \lambda > 0, \delta \ge 0\ and\ \gamma$, the multiple W-E-K fractional integral operator for Mittag-Leffler function is as follows:

$$I_{\beta,\lambda,1}^{\gamma,\delta}\{E^\gamma{}_{\alpha,\beta}(z)\} = {}_2\Psi_4 \left[\left(\begin{matrix} (\gamma,1)\left(\gamma+1,\frac{1}{\lambda}\right) \\ (\gamma,0)(\beta,\alpha)\left(\gamma+\delta+1,\frac{1}{\beta}\right)(1,1) \end{matrix} \middle| z \right)\right]. \tag{19}$$

Proof: For $\beta > 0, \lambda > 0, \delta \ge 0$, use Eqn. (16) in left hand side of Eqn. (19), the image of a Mittag-Leffler function under a multiple W-E-K operator is

$$I_{\beta,\lambda,1}^{\gamma,\delta}\{E^\gamma{}_{\alpha,\beta}(z)\} = I_{\beta,\lambda,1}^{\gamma,\delta}\left\{\sum_{n=0}^\infty \frac{(\gamma)_n}{\Gamma(n\alpha+\beta)} \frac{z^n}{n!}\right\}$$

which implies that

$$I_{\beta,\lambda,1}^{\gamma,\delta}\{E^\gamma{}_{\alpha,\beta}(z)\} = {}_2\Psi_4 \left[\left(\begin{matrix} (\gamma,1)\left(\gamma+1,\frac{1}{\lambda}\right) \\ (\gamma,0)(\beta,\alpha)\left(\gamma+\delta+1,\frac{1}{\beta}\right)(1,1) \end{matrix} \middle| z \right)\right]. \tag{20}$$

The Eqn. (20) proves the theorem 2.

5. Special Cases

1. If in equation (17), we take $\boldsymbol{\tau}=1$, $\delta = 0$, and $\lambda = \beta$, then we get well known result of Gauss's hypergeometric function given in Eqn. (1).

2. If in equation (18), we take $\delta = 0$, and $\lambda = \beta$, then we get well known results of Mittag-Leffler functions given in Eqns. (8) – (10).

Conclusion

The results proved in this paper give some contributions to the theory of the fractional calculus, especially Wright-type hypergeometric function and Mittag-Leffler function. The results proved in this paper appear to be new and likely to have useful applications to a wide range of problems of mathematics, statistics and physical sciences.

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