ON KAEHLERIAN CONHARMONIC* BI-RECURRENT SPACES

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Abstract
In this paper, we define and study Kaehlerian conharmonic* bi-recurrent spaces. Also, we establish the necessary and sufficient conditions for a Kaehlerian conharmonic* bi-recurrent space to be Kaehlerian bi-recurrent and is that the space be Kaehlerian bi-ricci-recurrent, Kaehlerian projective bi-recurrent. Several other theorems are also derived.

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1. Introduction

Recently, Negi [3] defined and studied on almost product and decomposable spaces. In 2009, Negi and Rawat [4] have obtained the relations on almost Kaehlerian spaces with recurrent and symmetric projective curvature tensors. Lichnerowich [2] has called a Riemannian space satisfying the relation $R^h_{ijk,ab} - \lambda_{ab} R^h_{ijk} = 0$, where λ_{ab} is a non-zero tensor a recurrent space of second order. An $n\ (= 2m)$ dimensional Kaehlerian space K^n is a Riemannian space which admits a structure tensor field F_i^h satisfying the relations (see Yano [8]), given as

$$F_j^h F_h^i = -\delta_j^i\ , \tag{1.1}$$

$$F_{ij} = -F_{ji} \text{ and } F_{ij} = F_i^a g_{aj}\ , \tag{1.2}$$

$$F^h_{i,j} = 0. \tag{1.3}$$

Here, the comma (,) followed by an index denotes the operator of covariant differentiation with respect to the metric tensor g_{ij} of the Riemannian space.

The Riemannian curvature tensor, denoted by R^h_{ijk} is given as

$$R^h_{ijk} = \delta_i \begin{Bmatrix} h \\ j \quad k \end{Bmatrix} - \delta_j \begin{Bmatrix} h \\ i \quad k \end{Bmatrix} + \begin{Bmatrix} h \\ i \quad l \end{Bmatrix}\begin{Bmatrix} l \\ j \quad k \end{Bmatrix} - \begin{Bmatrix} h \\ j \quad l \end{Bmatrix}\begin{Bmatrix} l \\ i \quad k \end{Bmatrix}$$

whereas, the Ricci-tensor and the scalar curvature are respectively given by
$R_{ij} = R^h_{hij}$ and $R = R_{ij} g^{ij}$.

It is well known that these tensors satisfy the identities (see Tachibana [7])

$$F_i^a R_a^j = R_i^a F_a^j \tag{1.4}$$

and

$$F_i^a R_{aj} = -R_{ia} F_j^a. \tag{1.5}$$

In view of (1.1), the relation (1.4) gives

$$F_i^a R_a^b F_b^j = -R_i^j. \tag{1.6}$$

Also, multiplying (1.5) by g^{ij}, we obtain

$$F_i^a R_a^i = -R_a^j F_j^a. \tag{1.7}$$

If we define a tensor S_{ij} by

$$S_{ij} = -F_i^a R_{aj}. \tag{1.8}$$

we have

$$S_{ij} = -S_{ji} \tag{1.9}$$

The holomorphically projective curvature tensor P_{ijk}^h and the Bochner curvature tensor $K_{ijk}^h$are respectively given by (see Sinha [6])

$$P_{ijk}^h = R_{ijk}^h + \frac{1}{(n+2)}\left(R_{ik}\delta_j^h - R_{jk}\delta_i^h + S_{ik}F_j^h - S_{jk}F_i^h + 2S_{ij}F_k^h\right), \tag{1.10}$$

$$K_{ijk}^h = R_{ijk}^h + \frac{1}{(n+4)}\left(R_{ik}\delta_j^h - R_{jk}\delta_i^h + g_{ik}R_j^h - g_{jk}R_i^h + S_{ik}F_j^h - S_{jk}F_i^h + F_{ik}S_j^h - F_{jk}S_i^h + 2S_{ij}F_k^h + 2F_{jk}S_k^h\right) - \frac{R}{(n+2)(n+4)}\left(g_{ik}\delta_j^h - g_{jk}\delta_i^h + F_{ik}F_j^h - F_{jk}F_i^h + 2F_{ij}F_k^h\right). \tag{1.11}$$

Also, Behari and Ahuja [1] have defined conharmonic* curvature tensor $^*T_{ijk}^h$ as

$$^*T_{ijk}^h \stackrel{\text{def}}{=} R_{ijk}^h + \frac{1}{n-2}\left(g_{ik}R_j^h - g_{jk}R_i^h\right), \tag{1.12}$$

In our work, we use the followings:

Definition (1.1): A Kaehlerian space K^n satisfying (see Singh, [5])

$$R_{ijk,ab}^h = \lambda_{ab}R_{ijk}^h \tag{1.13}$$

For a non-zero recurrence tensor λ_{ab}, will be called a Kaehlerian recurrent space of second order or briefly a Kaehlerian bi-recurrent space.

Definition (1.2): A Kaehlerian space K^n will be called a Kaehlerian Ricci-recurrent space of second order or briefly a Kaehlerian bi-Ricci-recurrent space, if its Ricci tensor satisfies the relation (see Singh, [5])

$$R_{ij,ab} = \lambda_{ab}R_{ij} , \tag{1.14}$$

$where\ R_{ij} \neq 0,$ for a non-zero tensor λ_{ab} from (1.14), we deduce

$$R_{,ab} = \lambda_{ab}R. \tag{1.15}$$

Remark (1.1): A Kaehlerian bi-recurrent space is Kaehlerian bi – Ricci – recurrent but the converse is not necessarily true.

Definition (1.3): A Kaehlerian space K^n satisfying (see Singh, [5])

$$P_{ijk,ab}^h = \lambda_{ab}P_{ijk}^h , \tag{1.16}$$

where, λ_{ab} is a non-zero tensor, will be called a Kaehlerian projective recurrent space of second order or briefly a Kaehlerian projective bi-recurrent space.

Definition (1.4): A Kaehlerian space K^n satisfying (see Singh, [5])

$$K^h_{ijk,ab} = \lambda_{ab} K^h_{ijk} \qquad (1.17)$$

for a non-zero tensor λ_{ab}, is called a Kaehlerian Bochner bi-recurrent space.

2. Kaehlerian Conharmonic* Bi - Recurrent Spaces

Definition (2.1): A Kaehlerian space K^n satisfying

$$^*T^h_{ijk,ab} = \lambda_{ab}\ ^*T^h_{ijk}, \qquad (2.1)$$

for some non-zero recurrence tensor field λ_{ab}, will be called a **Kaehlerian conharmonic* recurrent space of second order** or briefly a **Kaehlerian conharmonic* bi-recurrent space.**
We investigate that:

Theorem (2.1): Every Kaehlerian bi-recurrent space is Kaehlerian conharmonic* bi- recurrent.
Proof: A Kaehlerian bi-recurrent space is characterized by (1.13), which yields (1.14). By differentiating (1.12) covariently with respect to x^a and x^b, successively and using equation (1.14), we get after some simplification

$$^*T^h_{ijk,ab} = \lambda_{ab}\ ^*T^h_{ijk}.$$

Which shows that the space is Kaehlerian conharmonic* bi-recurrent.

Theorem (2.2): If in a Kaehlerian space K^n, any two of the following properties are satisfied:

(a) The space is Kaehlerian bi-Ricci-recurrent,

(b) The space is Kaehlerian conharmonic* bi-recurrent,

(c) The space is Kaehlerian Projective bi-recurrent,

Then, it must satisfy the third.

Proof: Equation (1.12), in view of (1.10) becomes

$$^*T^h_{ijk} = P^h_{ijk} + \frac{1}{n-2}\left(g_{ik}R^h_j - g_{jk}R^h_i\right) - \frac{1}{(n+2)}\left(R_{ik}\delta^h_j - R_{jk}\delta^h_i + S_{ik}F^h_j - S_{jk}F^h_i + 2S_{ij}F^h_k\right). \qquad (2.2)$$

Differentiating (2.2) covariently with respect to x^a and x^b successively and using (1.3), we have

$$^*T^h_{ijk,ab} = P^h_{ijk,ab} + \frac{1}{n-2}\left(g_{ik}R^h_{j,ab} - g_{jk}R^h_{i,ab}\right) - \frac{1}{(n+2)}\left(R_{ik,ab}\delta^h_j - R_{jk,ab}\delta^h_i + S_{ik,ab}F^h_j - S_{jk,ab}F^h_i + 2S_{ij,ab}F^h_k\right). \qquad (2.3)$$

Now, multiplying (2.2) by λ_{ab} and subtracting the resulting equation from (2.3), we have

$$^*T^h_{ijk,ab} - \lambda_{ab}\ ^*T^h_{ijk} = P^h_{ijk,ab} - \lambda_{ab}P^h_{ijk} + \frac{1}{(n-2)}\left[g_{ik}\left(R^h_{j,ab} - \lambda_{ab}R^h_j\right) - g_{jk}\left(R^h_{i,ab} - \lambda_{ab}R^h_i\right)\right] - \frac{1}{(n+2)}\left[\delta^h_j\left(R_{ik,ab} - \lambda_{ab}R_{ik}\right) - \delta^h_i\left(R_{jk,ab} - \lambda_{ab}R_{jk}\right) + F^h_j\left(S_{ik,ab} - \lambda_{ab}S_{ik}\right) - F^h_i\left(S_{jk,ab} - \lambda_{ab}S_{jk}\right) + 2F^h_k\left(S_{ij,ab} - \lambda_{ab}S_{ij}\right)\right]. \qquad (2.4)$$

Making use of equations (1.8), (1.14), (1.16), (2.1) and (2.4), the above theorem can be proved.

Theorem (2.3): If in a Kaehlerian space K^n, any two of the following properties are satisfied:

(a) The space is Kaehlerian bi-Ricci-recurrent,

(b) The space is Kaehlerian conharmonic* bi-recurrent,

(c) The space is Kaehlerian Bochner bi-recurrent. Then it must satisfy the third.

Proof: With the help of equations (1.11) and (1.12), we obtain

$$K_{ijk}^{h} = {}^{*}T_{ijk}^{h} - \frac{6}{(n-2)(n+4)}\left(g_{ik}R_j^h - g_{jk}R_i^h\right) + \frac{1}{(n+4)}\left(R_{ik}\delta_j^h - R_{jk}\delta_i^h + S_{ik}F_j^h - S_{jk}F_i^h + F_{ik}S_j^h - F_{jk}S_i^h + 2S_{ij}F_k^h + 2F_{ij}S_k^h\right) - \frac{R}{(n+2)(n+4)}\left(g_{ik}\delta_j^h - g_{jk}\delta_i^h + F_{ik}F_j^h - F_{jk}F_i^h + 2F_{ij}F_k^h\right). \quad (2.5)$$

Differentiating (2.5) covariently with respect to x^a and x^b successively and using (1.3), we obtain

$$K_{ijk,ab}^{h} = {}^{*}T_{ijk,ab}^{h} - \frac{6}{(n-2)(n+4)}\left(g_{ik}R_{j,ab}^h - g_{jk}R_{i,ab}^h\right) + \frac{1}{(n+4)}\left(R_{ik,ab}\delta_j^h - R_{jk,ab}\delta_i^h + S_{ik,ab}F_j^h - S_{jk,ab}F_i^h + F_{ik}S_{j,ab}^h - F_{jk}S_{i,ab}^h + 2S_{ij,ab}F_k^h + 2F_{ij,ab}S_k^h\right) - \frac{R_{,ab}}{(n+2)(n+4)}\left(g_{ik}\delta_j^h - g_{jk}\delta_i^h + F_{ik}F_j^h - F_{jk}F_i^h + 2F_{ij}F_k^h\right). \quad (2.6)$$

Multiplying (2.5) by λ_{ab} and subtracting the resulting equation from (2.6), we get

$$K_{ijk,ab}^{h} - \lambda_{ab}K_{ijk}^{h} = {}^{*}T_{ijk,ab}^{h} - \lambda_{ab}\,{}^{*}T_{ijk}^{h} - \frac{6}{(n-2)(n+4)}\left[g_{ik}\left(R_{j,ab}^h - \lambda_{ab}R_j^h\right) - g_{jk}\left(R_{i,ab}^h - \lambda_{ab}R_i^h\right)\right] + \frac{1}{(n+4)}\Big(\delta_j^h\left(R_{ik,ab} - \lambda_{ab}R_{ik}\right) - \delta_i^h\left(R_{jk,ab} - \lambda_{ab}R_{jk}\right) + F_j^h\left(S_{ik,ab} - \lambda_{ab}S_{ik}\right) - F_i^h\left(S_{jk,ab} - \lambda_{ab}S_{jk}\right) + F_{ik}\left(S_{j,ab}^h - \lambda_{ab}S_j^h\right) - F_{jk}\left(S_{i,ab}^h - \lambda_{ab}S_i^h\right) + 2F_k^h\left(S_{ij,ab} - \lambda_{ab}S_{ij}\right) + 2S_k^h\left(F_{ij,ab} - \lambda_{ab}F_{ij}\right)\Big) - \frac{\left(R_{,ab} - \lambda_{ab}R\right)}{(n+2)(n+4)}\left(g_{ik}\delta_j^h - g_{jk}\delta_i^h + F_{ik}F_j^h - F_{jk}F_i^h + 2F_{ij}F_k^h\right). \quad (2.7)$$

Making use of equations (1.8), (1.14), (1.15), (1.17) and (2.7), we can obtain the proof of the above theorem.

Theorem (2.4): A Kaehlerian conharmonic* bi-recurrent space will Kaehlerian bi-recurrent provided that it is Kaehlerian bi-Ricci-recurrent i.e.

$${}^{*}T_{ijk,ab}^{h} - \lambda_{ab}\,{}^{*}T_{ijk}^{h} = R_{ijk,ab}^{h} - \lambda_{ab}R_{ijk}^{h}.$$

Proof: Differentiating (1.12) covariantly with respect to x^a and x^bsuccessively, we get

$${}^{*}T_{ijk,ab}^{h} = R_{ijk,ab}^{h} + \frac{1}{(n-2)}\left[g_{ik}R_{j,ab}^h - g_{jk}R_{i,ab}^h\right]. \quad (2.8)$$

Multiplying (1.12) by λ_{ab} and subtracting the resulting equation from (2.8), we get

$${}^{*}T_{ijk,ab}^{h} - \lambda_{ab}\,{}^{*}T_{ijk}^{h} = R_{ijk,ab}^{h} - \lambda_{ab}R_{ijk}^{h} + \frac{1}{(n-2)}\left[g_{ik}\left(R_{j,ab}^h - \lambda_{ab}R_j^h\right) - g_{jk}\left(R_{i,ab}^h - \lambda_{ab}R_i^h\right)\right] \quad (2.9)$$

Let the space be Kaehlerian bi-Ricci-recurrent then, (2.9), in view of (1.8) and (1.14), yields

$${}^{*}T_{ijk,ab}^{h} - \lambda_{ab}\,{}^{*}T_{ijk}^{h} = R_{ijk,ab}^{h} - \lambda_{ab}R_{ijk}^{h}.$$

This relation shows that the Kaehlerian conharmonic* bi-recurrent space is Kaehlerian bi-recurrent.

Theorem (2.5): If in a Kaehlerian space K^n any two of the following properties are satisfied:

(a) The space is Kaehlerian bi-recurrent,

(b) The space is Kaehlerian bi-Ricci-recurrent,

(c) The space is Kaehlerian conharmonic* bi-recurrent,

Then it must satisfy the third property.

Proof: Kaehlerian bi-recurrent, Kaehlerian bi-Ricci-recurrent and Kaehlerian conharmonic* bi-recurrent space are respectively characterized by equations (1.13), (1.14) and (2.1).The statement of the theorem follows in view of equations (1.13), (1.14),(2.1) and (2.9).

Theorem (2.6): The necessary and sufficient condition that a K^n be Kaehlerian bi-Ricci-recurrent, is that

$$ {}^{*}T^{h}_{ijk,ab} - \lambda_{ab}\ {}^{*}T^{h}_{ijk} = R^{h}_{ijk,ab} - \lambda_{ab} R^{h}_{ijk}. \tag{2.10} $$

Proof: Let K^n be Kaehlerian bi-Ricci-recurrent, then the relation (1.14) is satisfied and so the equation (2.9) reduces to (2.10).

Conversely, if in a K^n, (2.10) is satisfied , then the equation (2.9) yields

$$ g_{ik}\left(R^{h}_{j,ab} - \lambda_{ab} R^{h}_{j}\right) - g_{jk}\left(R^{h}_{i,ab} - \lambda_{ab} R^{h}_{i}\right) = 0, \tag{2.11} $$

after doing some simplification in the above equation, it gives us

$$ R^{h}_{ijk,ab} - \lambda_{ab} R^{h}_{ijk} = 0 \text{ that is } R^{h}_{ijk,ab} = \lambda_{ab} R^{h}_{ijk}. $$

This shows that the space K^n is Kaehlerian bi-Ricci-recurrent,

This completes the Proof.

References

[1] R. Behari and L. R. Ahuja, On conharmonic* curvature tensor Kaehlerian manifold. (To appear).

[2] A. Lichnerowich, Courbure, numbers de Betti et espaces symmetriques, *Proc. Int. Cong. Math.* **2** (1960), 216-222.

[3] U. S. Negi, Theorems on Almost product and decomposable spaces, *Aryabhatta J. Math. Inform.* **9** (1) (2017), 105-110.

[4] U. S. Negi and A. Rawat, Some theorems on almost Kaehlerian spaces with recurrent and symmetric projective curvature Tensors. *Acta Ciencia Indica*, **XXXVM**, (3) (2009), 947-951.

[5] S. S. Singh, On Kaehlerian recurrent and Ricci-recurrent spaces of second order, *Est. Dag. Atti. della. Accad. della. Sci. di.Torino*, **106** (1971-72), 509-518.

[6] B. B. Sinha, On H-curvature tensors in Kaehler manifolds. *Kyungpook Math. J.* **13** (2) (1973), 185-189.

[7] S. Tachibana, On the Bochner curvature tensor, *Nat. Sci. Rep. Ochanomizu University* **18** (1) (1967), 15-19.

[8] K. Yano, *Differential Geometry on Complex and Almost Complex Spaces,* Pergamon Press, London, 1965.