ON A BI DIMENSIONAL BASIS INVOLVING SPECIAL FUNCTIONS FOR PARTIAL IN SPACE AND TIME FRACTIONAL WAVE MECHANICAL PROBLEMS AND APPROXIMATION

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Abstract
In this work, we construct a fractional time dependent wave mechanical problem consisting partial in space and time fractional derivatives and solved it on introducing a bi dimensional basis function involving Hermite and Mittag – Leffler functions. Then, we use it to approximate the solution of above two variable wave mechanical problem and then discuss its various cases in which it is computable.

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1. Introduction

The Hermite functions play an important role in the wave mechanical treatment of the harmonic oscillator. (see Mott and Sneddon [11, p. 50], Eravanis [1]). It also has great importance in the application to the quantum theory of radiation [19]. In this connection, the time independent Schrödinger equation corresponding to harmonic oscillator of point mass m with vibrational frequency ν is given by (Sneddon [17])

$$\frac{d^2\Psi_n}{dx^2} + \left(\frac{2T}{h\nu} - x^2\right)\Psi_n = 0, \tag{1}$$

$where,$ T is the total energy of the oscillator and h is the Plank's constant. Here, the wave functions Ψ_n have the property that

$$\Psi_n \to 0 \text{ as } |x| \to \infty; \tag{2}$$

and

$$\int_{-\infty}^{\infty} |\Psi_n|^2 dx = 2\pi\sqrt{\frac{m\nu}{h}} \tag{3}$$

In Eqn. (1), there is the Hermite function $\Psi_n = e^{-\frac{x^2}{2}} H_n(x)$ and $\frac{2T}{h\nu} = 2n+1$, where, $H_n(x)$ being the Hermite polynomials [13] for all $n = 0, 1, 2, ...$, satisfying the differential equation

$$\frac{d^2 H_n(x)}{dx^2} - 2x\frac{d}{dx}H_n(x) + 2nH_n(x) = 0 \tag{4}$$

The $H_n(x)$ is defined by

$$H_n(x) = \begin{cases} (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{x^{-2}}, (the\ operational\ formula) \\ (2x)^n \sum_{r=0}^{\infty} \frac{\Gamma\left(-\frac{n}{2}+r\right)\Gamma\left(\frac{1}{2}-\frac{n}{2}+r\right)}{\Gamma\left(-\frac{n}{2}\right)\Gamma\left(\frac{1}{2}-\frac{n}{2}\right)\Gamma(1+r)} (-x^{-2})^r, (the\ series\ formula) \end{cases} \tag{5}$$

The function Ψ_n also satisfies the recursion formula

$2x\Psi_n = 2n\Psi_{n-1} + \Psi_{n+1}$ (6)

L. C. Evans [6] introduce a fundamental solution $\Theta(x,t) = \begin{cases} \frac{1}{\sqrt{4\pi t}} e^{-\frac{|x|^2}{4t}} (x \in \mathbb{R}, t > 0) \\ 0 \qquad (x \in \mathbb{R}, t < 0) \end{cases}$ for the heat equation $\frac{\partial}{\partial t}\Theta(x,t) - \frac{\partial^2}{\partial x^2}\Theta(x,t) = 0.$ (7)

Some two variable analogue of Hermite polynomials are studied through operational rules [3], [4], [14], in this study, the one variable Hermite polynomials [13] have the relation with two variable Hermite polynomials due to operational techniques.

Motivated by above work, by bi dimensional basis function, we construct a fractional time dependent wave mechanical problem consisting partial in space and time fractional derivatives and then study its properties and application in approximation theory to discuss its various cases in which it is computable.

Now, for constructing a fractional time dependent wave mechanical problem consisting partial in space and time fractional derivatives and to extend the work of Sneddon [17] and Evans [6], and on motivation of the theory due to [3], [4], [12] and [14], we introduce following new bi dimensional basis solution of that problem

$$\Psi_n^\alpha(x,t;\lambda_n) = \begin{cases} e^{-\frac{x^2}{2}} H_n(x) E_\alpha(\lambda_n t^\alpha), 0 < \alpha \le 2, \lambda_n = -(2n+1), x \in (-\infty,\infty), t > 0, & (8, \mathrm{I}) \\ e^{-\frac{x^2}{2}} H_n(x), \qquad x \in (-\infty,\infty), t = 0, & (8, \mathrm{II}) \\ 0, \qquad x \in (-\infty,\infty), t < 0. & (8, \mathrm{III}) \end{cases}$$

Here, in Eqn. (8, I), the function $E_\alpha(\lambda_n t^\alpha)$ is the Mittag – Leffler function [10], satisfying the fractional differential equation (see, Gorenflo and Mainardi [7], Mainardi [8] and Mainardi and Gorenflo [9])

${}_0^C\mathcal{D}_t^\alpha u(t) = \lambda_n u(t), u(t) = E_\alpha(\lambda_n t^\alpha),$ (9)

where, ${}_0^C\mathcal{D}_t^\alpha$, is the Caputo derivative, defined by (see Diethelm [5, p. 49], Samko et al. [16])

$${}_0^C\mathcal{D}_t^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(k-\alpha)} \int_0^t (t-x)^{k-\alpha-1} D_x^k f(x) dx, k-1 < \alpha \le k, \\ 0, \qquad \alpha > k, \end{cases}$$

And $\lim_{\alpha \to k} {}_0^C\mathcal{D}_t^\alpha f(t) = D_t^k f(t) = \frac{d^k}{dt^k} f(t), \forall k \in \mathbb{N} = \{1,2,3 \ldots \ldots\}.$ (10)

1 Wave Mechanical Problem and its Properties

For $t > 0$, consider the function defined in Eqn. (8, I) and make the following operation to get

$$\left(\frac{\partial^2}{\partial x^2} - x^2\right)\Psi_n^\alpha(x,t;\lambda_n) = e^{-\frac{x^2}{2}}\left[\frac{\partial^2 H_n(x)}{\partial x^2} - 2x\frac{\partial}{\partial x}H_n(x) + 2nH_n(x)\right]E_\alpha(\lambda_n t^\alpha)$$
$$-(2n+1)\Psi_n^\alpha(x,t;\lambda_n) \quad (11)$$

Now, use the differential equations (4) and (9) in right hand side of the Eqn. (11) to get the required fractional time dependent wave mechanical problem consisting partial in space and time fractional derivatives in the form

$$\left(\frac{\partial^2}{\partial x^2} - x^2\right)\Psi_n^\alpha(x,t;\lambda_n) = {}_0^C\mathcal{D}_t^\alpha \Psi_n^\alpha(x,t;\lambda_n) \quad (12)$$

Clearly, at stationary position the Eqn. (12) is converted into the Eqn. (1) as setting the transformation $-\lambda_n = 2n+1 = \frac{2T}{h\nu}$ and the function $\Psi_n^\alpha(x,t;\lambda_n)$ has one of the conditions that

$\Psi_n^\alpha(x,t;\lambda_n) \to 0 \text{ as } |x| \to \infty.$ (13)

Again, from Eqn. (12) we may write

$$\left(\frac{\partial^2}{\partial x^2}-x^2\right)\Psi_m^\alpha(x,t;\lambda_m)={}_0^C\mathcal{D}_t^\alpha\Psi_m^\alpha(x,t;\lambda_m) \quad (14)$$

Theorem 1: By the basis function $\Psi_n^\alpha(x,t;\lambda_n)$, defined in Eqn. (8) and satisfying the Eqn. (12) for a fix n, there exists an eigen value problem

${}_0^C\mathcal{D}_t^\alpha U(x,t;\lambda_n)=\lambda_n U(x,t;\lambda_n), where, \lambda_n=-(2n+1).$

Then $U=\Psi_n^\alpha(x,t;\lambda_n)$. (15)

Proof:

We multiply by $\Psi_n^\alpha(x,t;\lambda_n)$ in both sides of Eqn. (14) and that by $\Psi_m^\alpha(x,t;\lambda_m)$ in both sides of Eqn. (12) and again on subtracting them, and then integrating with respect to x both of the sides to get

$$\int_{-\infty}^{\infty}\left[\Psi_m^\alpha(x,t;\lambda_m)\frac{\partial^2}{\partial x^2}\Psi_n^\alpha(x,t;\lambda_n)-\Psi_n^\alpha(x,t;\lambda_n)\frac{\partial^2}{\partial x^2}\Psi_m^\alpha(x,t;\lambda_m)\right]dx$$

$$=\int_{-\infty}^{\infty}\left[\Psi_m^\alpha(x,t;\lambda_m){}_0^C\mathcal{D}_t^\alpha\Psi_n^\alpha(x,t;\lambda_n)-\Psi_n^\alpha(x,t;\lambda_n){}_0^C\mathcal{D}_t^\alpha\Psi_m^\alpha(x,t;\lambda_m)\right]dx \quad (16)$$

An integrating by parts in left hand side of Eqn. (16) shows that

$$\left[\Psi_m^\alpha(x,t;\lambda_m)\frac{\partial}{\partial x}\Psi_n^\alpha(x,t;\lambda_n)-\Psi_n^\alpha(x,t;\lambda_n)\frac{\partial}{\partial x}\Psi_m^\alpha(x,t;\lambda_m)\right]\Bigg|_{x=-\infty}^{x=\infty}-$$

$$\int_{-\infty}^{\infty}\left[\frac{\partial}{\partial x}\Psi_m^\alpha(x,t;\lambda_m)\frac{\partial}{\partial x}\Psi_n^\alpha(x,t;\lambda_n)-\frac{\partial}{\partial x}\Psi_n^\alpha(x,t;\lambda_n)\frac{\partial}{\partial x}\Psi_m^\alpha(x,t;\lambda_m)\right]dx$$

$$=\int_{-\infty}^{\infty}\left[\Psi_m^\alpha(x,t;\lambda_m){}_0^C\mathcal{D}_t^\alpha\Psi_n^\alpha(x,t;\lambda_n)-\Psi_n^\alpha(x,t;\lambda_n){}_0^C\mathcal{D}_t^\alpha\Psi_m^\alpha(x,t;\lambda_m)\right]dx \quad (17)$$

Now, use the condition (13) in first part of left hand side of Eqn. (17) to find the Sturm – Liouville Equation

$$\int_{-\infty}^{\infty}\left[\Psi_m^\alpha(x,t;\lambda_m){}_0^C\mathcal{D}_t^\alpha\Psi_n^\alpha(x,t;\lambda_n)-\Psi_n^\alpha(x,t;\lambda_n){}_0^C\mathcal{D}_t^\alpha\Psi_m^\alpha(x,t;\lambda_m)\right]dx=0 \quad (18)$$

The Eqn. (18) shows that in integrand there exists Wronskian determinant such that

$$\begin{vmatrix}\Psi_m^\alpha(x,t;\lambda_m) & \Psi_n^\alpha(x,t;\lambda_n)\\ {}_0^C\mathcal{D}_t^\alpha\Psi_m^\alpha(x,t;\lambda_m) & {}_0^C\mathcal{D}_t^\alpha\Psi_n^\alpha(x,t;\lambda_n)\end{vmatrix}=0$$

Therefore, due to Sturm – Liouville Equation (18) (see R. V. Churchill [2]), for a fix n, there exists an eigen value problem

${}_0^C\mathcal{D}_t^\alpha\Psi_n^\alpha(x,t;\lambda_n)=\lambda_n\Psi_n^\alpha(x,t;\lambda_n), where, \lambda_n=-(2n+1)$, which on comparing with the Eqn. (15) gives us $U=\Psi_n^\alpha(x,t;\lambda_n)$. Hence the theorem has proved.

Theorem 2: If $\Psi_n^\alpha(x,t;\lambda_n)$ satisfies the Eqn. (12), then, for $0<\alpha\le 2$ there exists a quadrature formula with weight function e^{-t} of the determinant due to molecular vibrations

$$B_{n,n-1}^\alpha=2^n n!\sqrt{\pi}\sum_{k=0}^{\infty}P_k^\alpha\left(\left(\frac{(2n-1)}{(2n+1)}\right)\right)\{-(2n+1)\}^k,$$

$$P_k^\alpha\left(\left(\frac{(2n-1)}{(2n+1)}\right)\right)=\sum_{l=0}^{k}\frac{\Gamma(\alpha k+1)\left(\frac{(2n-1)}{(2n+1)}\right)^l}{\Gamma(\alpha l+1)\Gamma(\alpha(k-l)+1)}.$$

Proof: Apply the Eqn. (15) in the Eqn. (18) to get

$$\int_{-\infty}^{\infty}\left[\Psi_m^\alpha(x,t;\lambda_m){}_0^C\mathcal{D}_t^\alpha\Psi_n^\alpha(x,t;\lambda_n)-\Psi_n^\alpha(x,t;\lambda_n){}_0^C\mathcal{D}_t^\alpha\Psi_m^\alpha(x,t;\lambda_m)\right]dx$$

$$=2(m-n)\int_{-\infty}^{\infty}[\Psi_m^\alpha(x,t;\lambda_m)\Psi_n^\alpha(x,t;\lambda_n)-\Psi_n^\alpha(x,t;\lambda_n)\Psi_m^\alpha(x,t;\lambda_m)]dx=0, when\ m\neq n. \quad (19)$$

Hence if we *let* $\varphi_{m,n}^\alpha(t)=\int_{-\infty}^{\infty}\Psi_m^\alpha(x,t;\lambda_m)\Psi_n^\alpha(x,t;\lambda_n)\,dx$, *in Eqn. (19), and then*

$\varphi_{m,n}^{\alpha}(t) = \int_{-\infty}^{\infty} \Psi_m^{\alpha}(x,t;\lambda_m)\Psi_n^{\alpha}(x,t;\lambda_n)dx = \int_{-\infty}^{\infty} \Psi_n^{\alpha}(x,t;\lambda_n)\Psi_m^{\alpha}(x,t;\lambda_m)dx, \, when\ m \neq n.$ (20)

This Eqn. (20) shows that the product $\Psi_m^{\alpha}(x,t;\lambda_m)\Psi_n^{\alpha}(x,t;\lambda_n)$ is commutative with respect to m and n.

In particular, the Eqn. (20) follows that

$$\varphi_{n-1,n+1}^{\alpha}(t) = 0 \Longrightarrow \varphi_{n-1,n+1}^{\alpha}(t) = \int_{-\infty}^{\infty} \Psi_{n-1}^{\alpha}(x,t;\lambda_{n-1})\Psi_{n+1}^{\alpha}(x,t;\lambda_{n+1})dx = 0. \quad (21)$$

Again, from Eqns. (8) and (20), we find that

$$\varphi_{m,n}^{\alpha}(t) = \{E_{\alpha}(\lambda_n t^{\alpha})E_{\alpha}(\lambda_m t^{\alpha})\}\int_{-\infty}^{\infty} e^{-x^2}H_n(x)H_m(x)dx \quad (22)$$

Eqn. (22) gives us

$$\varphi_{m,n}^{\alpha}(t) = \{E_{\alpha}(\lambda_n t^{\alpha})E_{\alpha}(\lambda_m t^{\alpha})\}2^n n!\sqrt{\pi}\delta_{mn} \quad (23)$$

Again, by Eqn. (6), we may write the relation

$$2x\Psi_n^{\alpha}(x,t;\lambda_n) = -\frac{E_{\alpha}(\lambda_n t^{\alpha})}{E_{\alpha}(\lambda_{n-1}t^{\alpha})}\,{}_0^C\mathcal{D}_t^{\alpha}\Psi_{n-1}^{\alpha}(x,t;\lambda_{n-1}) + \frac{E_{\alpha}(\lambda_n t^{\alpha})}{E_{\alpha}(\lambda_{n-1}t^{\alpha})}\Psi_{n-1}^{\alpha}(x,t;\lambda_{n-1}) + \frac{E_{\alpha}(\lambda_n t^{\alpha})}{E_{\alpha}(\lambda_{n+1}t^{\alpha})}\Psi_{n+1}^{\alpha}(x,t;\lambda_{n+1}) \quad (24)$$

Then, on applying the Eqns. (21), (23) and (24) and Theorem 1, for $t \in (0,\infty)$ we get the determinant (this type of determinant has used to study molecular vibrations ([1] and [19]))

$$A_{n,n-1}^{\alpha}(t) = \int_{-\infty}^{\infty} 2x\,\Psi_n^{\alpha}(x,t;\lambda_n)\Psi_{n-1}^{\alpha}(x,t;\lambda_{n-1})dx = E_{\alpha}(\lambda_{n-1}t^{\alpha})E_{\alpha}(\lambda_n t^{\alpha})2^n n!\sqrt{\pi} \quad (25)$$

Thus from the Eqn. (25), we find the quadrature formula with weight e^{-t} of the determinant $A_{n,n-1}^{\alpha}(t)$, given by

$$B_{n,n-1}^{\alpha} = \int_0^{\infty} e^{-t}A_{n,n-1}^{\alpha}(t)dt$$

$$= \int_0^{\infty}\int_{-\infty}^{\infty} 2x\,e^{-t}\,\Psi_n^{\alpha}(x,t;\lambda_n)\Psi_{n-1}^{\alpha}(x,t;\lambda_{n-1})dxdt = 2^n n!\sqrt{\pi}\sum_{k=0}^{\infty} P_k^{\alpha}\left(\left(\frac{(2n-1)}{(2n+1)}\right)\right)\{-(2n+1)\}^k \quad (26)$$

where, $P_k^{\alpha}\left(\left(\frac{(2n-1)}{(2n+1)}\right)\right) = \sum_{l=0}^{k}\frac{\Gamma(\alpha k+1)\left(\frac{(2n-1)}{(2n+1)}\right)^l}{\Gamma(\alpha l+1)\Gamma(\alpha(k-l)+1)} = \sum_{l=0}^{k}\binom{\alpha k}{\alpha k-\alpha l}\left(\frac{(2n-1)}{(2n+1)}\right)^l.$ (27)

Hence the theorem has proved.

Remark1. By the polynomial given in Eqn. (27), we may find the Riordon arrays [15] and this result may also used it in combinatorial theory.

Corollary1. There exists the quadrature formula of the determinant

$$lim_{\alpha\to 1}\, B_{n,n-1}^{\alpha} = 2^n n!\sqrt{\pi}\frac{1}{(1+4n)}, \quad (28)$$

and $lim_{\alpha\to 2}\, B_{n,n-1}^{\alpha} = 2^{n-1}n!\,\pi F_4\left[\begin{matrix}1,\frac{1}{2};\\ \frac{1}{2},\frac{1}{2};\end{matrix} -(2n+1), -(2n-1)\right]$, provided that $n > 0$. (29)

In Eqns. (26) and (27) put the limit $\alpha \to 1$, we have

$$B_{n,n-1}^{1} = \int_0^{\infty}\int_{-\infty}^{\infty} 2x\,e^{-t}\,\Psi_n^{\alpha}(x,t;\lambda_n)\Psi_{n-1}^{\alpha}(x,t;\lambda_{n-1})dxdt$$

$$= 2^n n!\sqrt{\pi}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{\Gamma(k+l+1)\left(-(2n-1)\right)^l\left(-(2n+1)\right)^k}{\Gamma(l+1)\Gamma(k+1)} \quad (30)$$

Now use the Srivastava's formula (see Srivastava and Manocha [18, *p.* 52]) in Eqn. (30), finally, we find the relation (28).

Now, set the limit $\alpha \to 2$, in Eqns. (26) and (27), we have

$B_{n,n-1}^2 = \int_0^\infty \int_{-\infty}^\infty 2x\, e^{-t}\, \Psi_n^\alpha(x,t;\lambda_n)\Psi_{n-1}^\alpha(x,t;\lambda_{n-1})dxdt$

$= 2^{n-1}n!\,\pi \sum_{k=0}^\infty \sum_{l=0}^\infty \frac{(1)_{k+l}\left(\frac{1}{2}\right)_{k+l}}{\left(\frac{1}{2}\right)_l\left(\frac{1}{2}\right)_k} \frac{\left(-(2n+1)\right)^k}{k!} \frac{\left(-(2n-1)\right)^l}{l!}$

$= 2^{n-1}n!\,\pi F_4\left[\begin{matrix}1,\frac{1}{2}; \\ \frac{1}{2},\frac{1}{2};\end{matrix} -(2n+1),-(2n-1)\right],$

which is valid for $\sqrt{|2n-1|} + \sqrt{|2n+1|} < 1.$ (31)

The Eqn. (31) follows that

$|2n+1| < \left(1-\sqrt{|2n-1|}\right)^2 = |1+2n| - 1 - 2\sqrt{|2n-1|}$ which implies that

$2\sqrt{|2n-1|} < -1, \Longrightarrow \sqrt{|2n-1|} < -\frac{1}{2}, or\ |2n-1| > \frac{1}{4}, or\ 2n-1 > \frac{1}{4}, \Rightarrow\ n > \frac{5}{8} > 0.$

Hence the result (29) has followed.

3. Application for evaluation of the Solution of given Problem

On making an appeal to the Eqn. (18), the wave mechanical problem (12) in the form $\left(\frac{\partial^2}{\partial x^2} - x^2\right)\Phi = {}_0^C\mathcal{D}_t^\alpha \Phi$, has the solution $\Phi = K\Psi_n^\alpha(x,t;\lambda_n)$,

where, $-\lambda_n = 2n+1.$ (32)

Hence due to Eqn. (32), we may write

$|\Phi|^2 = K^2\Psi_n^\alpha(x,t;\lambda_n)\Psi_n^\alpha(x,t;\lambda_n)$ and thus we have

$\int_0^\infty \int_{-\infty}^\infty e^{-t}\,|\Phi|^2 dxdt = 2^n n!\sqrt{\pi}K^2 \int_0^\infty e^{-t}\,\{E_\alpha(\lambda_n t^\alpha)E_\alpha(\lambda_n t^\alpha)\}dt$

$= 2^n n!\sqrt{\pi}K^2 \sum_{k,l=0}^\infty \frac{\Gamma(\alpha(k+l)+1)(\lambda_n)^{k+l}}{\Gamma(\alpha k+1)\Gamma(\alpha l+1)}$ (33)

Therefore, by Eqn. (33), we obtain

$K = \left\{\frac{\int_0^\infty \int_{-\infty}^\infty e^{-t}|\Phi|^2 dxdt}{2^n n!\sqrt{\pi}\sum_{k,l=0}^\infty \frac{\Gamma(\alpha(k+l)+1)(\lambda_n)^{k+l}}{\Gamma(\alpha k+1)\Gamma(\alpha l+1)}}\right\}^{1/2}, \lambda_n = -(2n+1)$ that use in Eqn. (32), we obtain

$\Phi = \left\{\frac{M}{2^n n!\sqrt{\pi}\sum_{k,l=0}^\infty \frac{\Gamma(\alpha(k+l)+1)(\lambda_n)^{k+l}}{\Gamma(\alpha k+1)\Gamma(\alpha l+1)}}\right\}^{1/2} \Psi_n^\alpha(x,t;\lambda_n),$

where, let $M = \int_0^\infty \int_{-\infty}^\infty e^{-t}\,|\Phi|^2 dxdt > 0.$ (34)

4. Different Shapes of the Solution of the Problem (32) in Various Conditions

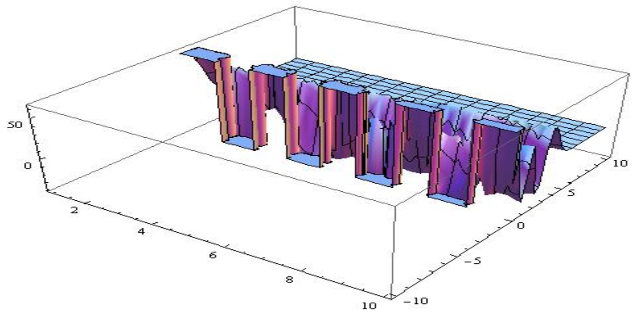
Set $M = \frac{1}{2}$ in Eqn. (34), and apply the formula (5), we find that

$$\Phi = \left\{\frac{1}{2^{n+1}\Gamma(n+1)\sqrt{\pi}\sum_{k,m=0}^\infty \frac{\Gamma(\alpha(k+m)+1)\left(-(2n+1)\right)^{k+m}}{\Gamma(\alpha k+1)\Gamma(\alpha m+1)}}\right\}^{1/2}$$

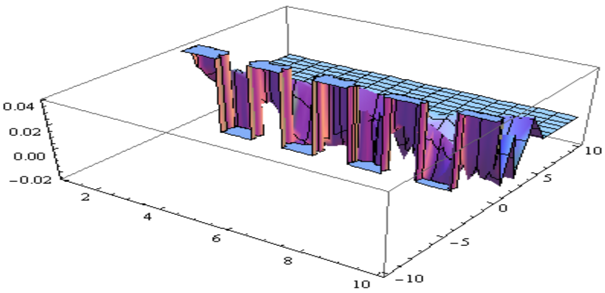
$\times e^{-\frac{x^2}{2}}(2x)^n \sum_{r=0}^\infty \frac{\Gamma\left(-\frac{n}{2}+r\right)\Gamma\left(\frac{1}{2}-\frac{n}{2}+r\right)}{\Gamma\left(-\frac{n}{2}\right)\Gamma\left(\frac{1}{2}-\frac{n}{2}\right)\Gamma(1+r)}(-x^{-2})^r \sum_{s=0}^\infty \frac{(-(2n+1)t^\alpha)^s}{\Gamma(1+\alpha s)}, \forall x \in (-\infty,\infty), t > 0.$ (35)

Now set different conditions of the parameters and variables in Eqn. (35) to find many shapes of the function Φ as:

Case1. When $\alpha = .5, t$ varies from 1 to 10, and the horizontal dimensions are $n \in \mathbb{N} = \{1,2,3,...\}$ and $x \in (-\infty,\infty)$, and Φ fluctuates vertically up and down, then shape of the graph due to the function defined in Eqn. (8, I) is

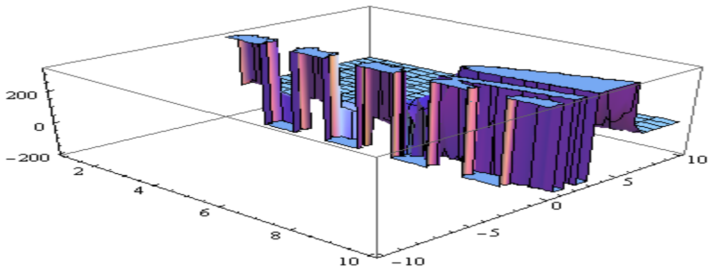


Case2. Again in above put**,** $\alpha = 1.5, t$ varies from 1 to 10, and the horizontal dimensions are $n \in \mathbb{N} = \{1,2,3, \ldots\}$ and $x \in (-\infty, \infty)$, and Φ fluctuates vertically up and down, then shape of the graph is

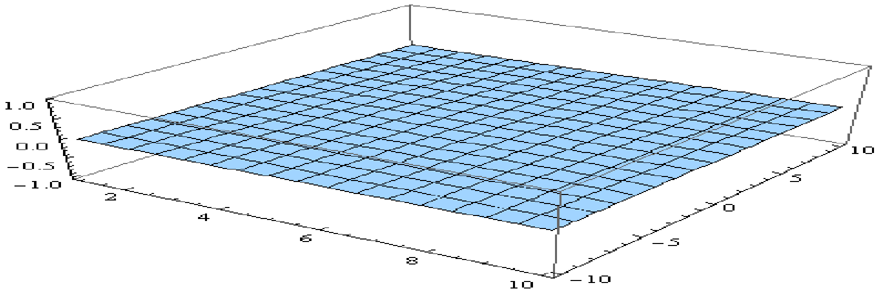


Remark2. It is remarkable that in above cases 1 and 2, if we take the horizontal dimensions $t \in \mathbb{N} = \{1,2,3, \ldots\}$ and $x \in (-\infty, \infty)$ in place of $n \in \mathbb{N} = \{1,2,3, \ldots\}$ and $x \in (-\infty, \infty)$, then the solution becomes indeterminate and not computable.

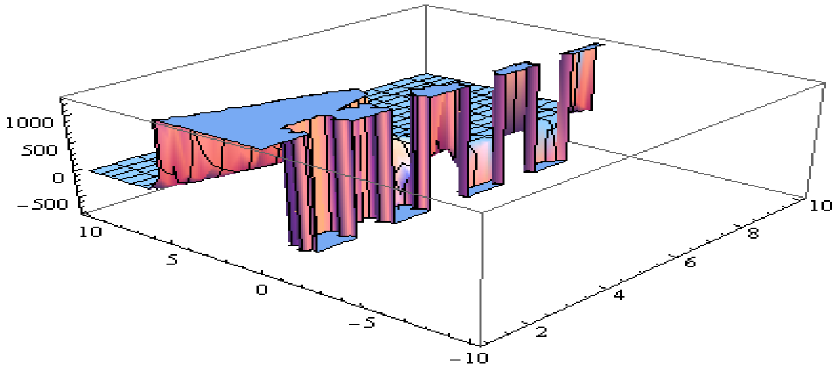
Case3. When $t = 0$ and the horizontal dimensions are $n \in \mathbb{N} = \{1,2,3, \ldots\}$ and $x \in (-\infty, \infty)$, and Φ fluctuates vertically up and down, then shape of the graph due to the function defined in Eqn. (8, II) is



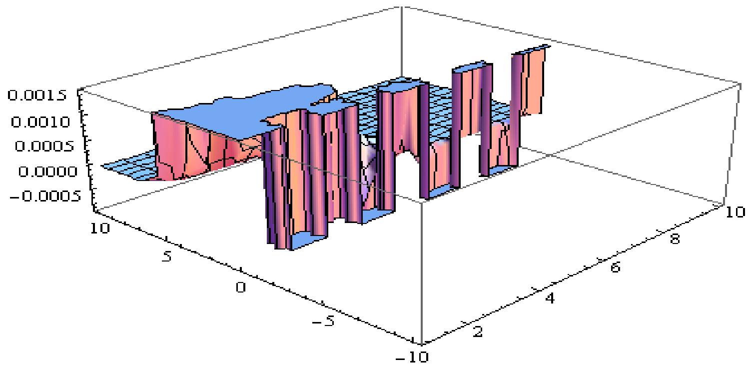
Case4. When $t < 0$ and the horizontal dimensions are $n \in \mathbb{N} = \{1,2,3, \ldots\}$ and $x \in (-\infty, \infty)$, then the function Φ is always zero, see the following graph (as we also taken in our definition (8,III)



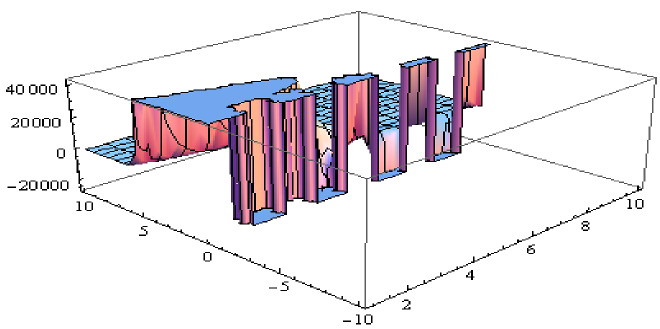
Case5. When $\alpha = .5, t$ is fixed as $t = 5$ and the horizontal dimensions are $n \in \mathbb{N} = \{1,2,3,...\}$ and $x \in (-\infty, \infty)$, and Φ fluctuates vertically up and down, then shape of the graph due to the function defined in Eqn. (8, I) is



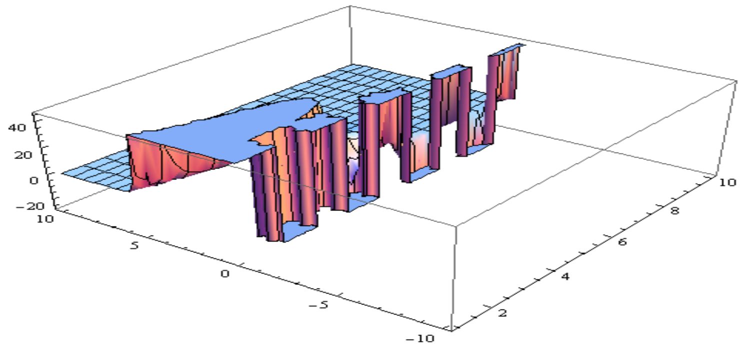
Case6. When $\alpha = 1.5, t$ is fixed as $t = 5$ and the horizontal dimensions are $n \in \mathbb{N} = \{1,2,3,...\}$ and $x \in (-\infty, \infty)$, and Φ fluctuates vertically up and down, then shape of the graph due to the function defined in Eqn. (8, I) is



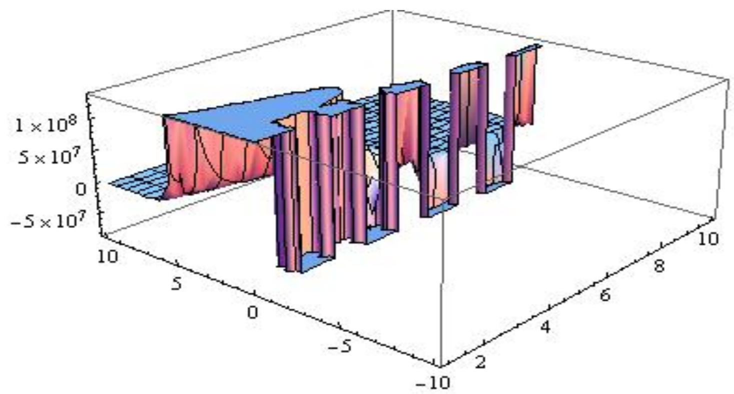
Case7. When $\alpha = .5, t$ is fixed as $t = 10$ and the horizontal dimensions are $n \in \mathbb{N} = \{1,2,3,...\}$ and $x \in (-\infty, \infty)$, and Φ fluctuates vertically up and down, then shape of the graph due to the function defined in Eqn. (8, I) is



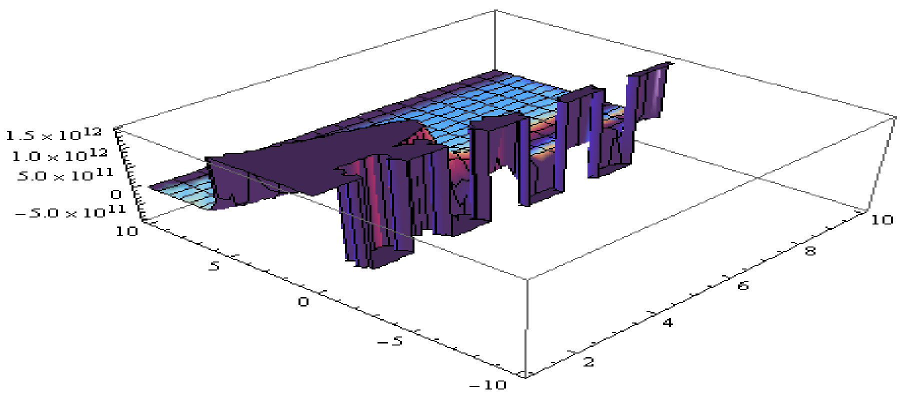
Case8. When $\alpha = 1.5, t$ is fixed as $t = 10$ and the horizontal dimensions are $n \in \mathbb{N} = \{1,2,3,...\}$ and $x \in (-\infty, \infty)$, and Φ fluctuates vertically up and down, then shape of the graph due to the function defined in Eqn. (8, I) is



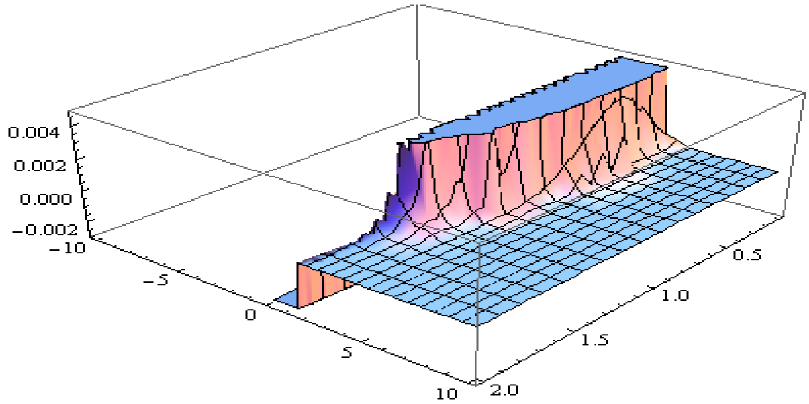
Case9. When $\alpha = .5, t$ is fixed as $t = 50$ and the horizontal dimensions are $n \in \mathbb{N} = \{1,2,3, \dots\}$ and $x \in (-\infty, \infty)$, and Φ fluctuates vertically up and down, then shape of the graph due to the function defined in Eqn. (8, I) is



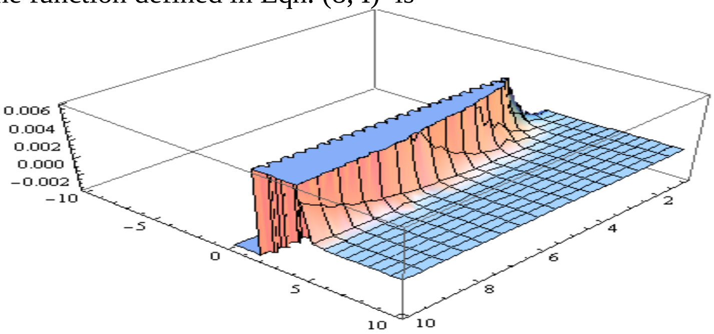
Case10. When $\alpha = 1.5, t$ is fixed as $t = 50$ and the horizontal dimensions are $n \in \mathbb{N} = \{1,2,3, \dots\}$ and $x \in (-\infty, \infty)$, and Φ fluctuates vertically up and down then shape of the graph due to the function defined in Eqn. (8, I) is



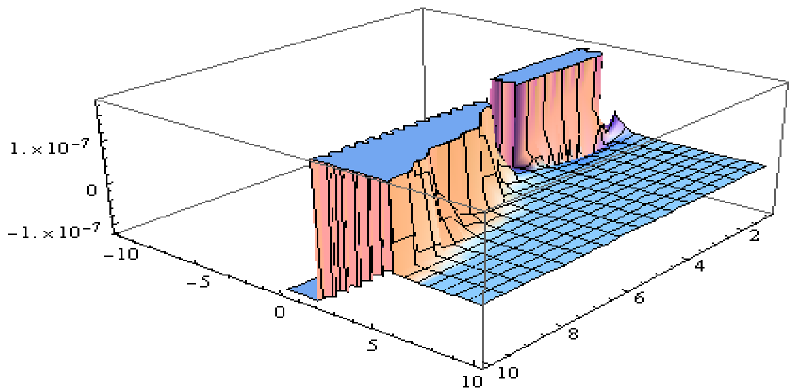
Case11. When t is fixed as $t = 10$, replace n by $\frac{1}{n}$, $n \in \mathbb{N}$ and the horizontal dimensions are $\alpha \in \{0.1, \dots, 2.0\}$ and $x \in (-\infty, \infty)$, and Φ fluctuates vertically up and down, then shape of the graph due to the function defined in Eqn. (8, I) is



Case12. When α is fixed as $\alpha = .5$, replace n by $\frac{1}{n}$, $n \in \mathbb{N}$ and the horizontal dimensions are $t \in \mathbb{N} = \{1,2,3, \dots\}$ and $x \in (-\infty, \infty)$, and Φ fluctuates vertically up and down, then shape of the graph due to the function defined in Eqn. (8, I) is



Case13. When α is fixed as $\alpha = 1.5$, replace n by $\frac{1}{n}$, $n \in \mathbb{N}$ and the horizontal dimensions are $t \in \mathbb{N} = \{1,2,3, \dots\}$ and $x \in (-\infty, \infty)$, and Φ fluctuates vertically up and down, then shape of the graph due to the function defined in Eqn. (8, I) is



Conclusion

In previous section, we claim 13 cases and by MATHEMATICA, we find that the solution (35) of the fractional wave problem (32) has many values and shapes of the wave in different dimensions and different values of α and t. Therefore, our introduced bi dimensional basis solution, involving Hermite and Mittag – Leffler functions, of fractional wave problem is a magic function which has various shapes in different conditions.

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