$J\tilde{n}\bar{a}n\bar{a}bha, Vol. 47 (II) (2017), 277-290$

COMMON FIXED POINTS FOR FOUR NON-SELF-MAPPINGS

By
Santosh Kumar[1] and Terentius Rugumisa[2]
[1]Department of Mathematics, College of Natural and Applied Sciences,
University of Dar es Salaam, Tanzania
Email: drsengar2002@gmail.com
[2]Faculty of Science, Technology and Environmental Studies,
The Open University of Tanzania
Email: rterentius@gmail.com
(Received : June 11, 2017 ; Revised : November 03, 2017)

Abstract

In this paper, we formulate a quasi-contraction type non-self mapping on Takahashi convex metric spaces and common fixed point theorems that applies to two pairs of mappings. The result generalizes the fixed point theorems of some previous authors.

2010Mathematics Subject Classification: 47H10, 54H25.
Key words: Common fixed point, metrically convex metric space, non-self mapp*i*ng.

1. Introduction and Preliminaries

Gajić and Rakočević [1] proved a quasi-contraction common fixed point theorem for non-self mappings on Takahashi convex metric spaces for a pair of mappings. In their work they generalized the theorems by Jungck [2], Das and Naik [3], Ćirić et al. [4], Ćirić [5] and Imdad and Kumar [6]. In this study, we extend the theorem by Gajić and Rakoc□ević [1] to apply for two pairs of mappings in metric spaces.

The following are the preliminaries required in this paper.
Given two non-self mappings $f, g: K \to X$ we say that $x \in K$ is *coincidence point* if $fx = gx$. We term the point $y \in X$ as a *point of coincidence* if $y = fx = gx$ where x is a coincidence point. We also say that f and g are *coincidentally commuting* if $fgx = gfx$ whenever x is a coincidence point.

If K is a subset of X, we denote the boundary of K as δK.
Here, we provide the definition of a Takahashi convex metric space which is useful for future discussion.
Definition 1.1. [7]. *Let X be a metric space and $I = [0, 1]$ be the closed unit interval. A mapping $W: X \times X \times I \to X$ is said to be a convex structure on X if for all $x, y \in X$; $\lambda \in I$,*

$$d\big(u, W\,(x, y, \lambda\,)\big) \leq d(u, x) + d(u, y)$$

for every $u \in X$. The metric space (X, d), together with the convex structure is called the Takahashi convex metric space.

If (X, d) is a Takahashi convex metric space, then for every $x, y \in X$, we term

$$\mathrm{seg}[x, y] := \{W(x, y, \lambda) : \lambda \in [0;\ 1]\}.$$

We will use the following property for a Takahashi convex structure in a metric space (X, d).
Lemma 1.2. [7] Let $x, y \in X$ and $z \in \mathrm{seg}[x. y]$, then for all $u \in X$ we have

$$d(u, z) \leq \max\{d(u, x), d(u, y)\}.$$

In Gajić and Rakoc□ević [1], the following theorem was proved:

Theorem 1.3. *Let* (X, d) *be a complete Takahashi convex metric space with convex structure* W *which is continuous in the third variable. Let* C *be a non-empty closed subset of* X *and* δC *be the boundary of* C. *Let* $g, f: C \to X$ *and suppose* $\delta C \neq \emptyset$. *Let us assume that* f *and* g *satisfy the following conditions:*

(i) *For every* $x, y \in C, d(gx, gy) \leq M_\omega(x, y)$ *where*
$M_\omega(x, y) =$
$\max\{[\omega_1[d(fx, fy)], \omega_2[d(fx, gx)], \omega_3[d(fy, gy)], \omega_4[d(fx, gy)], \omega_5[d(gx, fy)]\}$,
$\omega_1: [0, +\infty) \to [0, +\infty), i = 1,2,3,4,5,$ *is a non-decreasing semicontinuous function from the right, such that* $\omega_i(r) < r$ *for* $r > 0$, *and* $lim_{r \to \infty}[r - w_i(r)] = +\infty$,

(ii) $\delta C \subseteq f(C)$,

(iii) $g(C) \cap C \subset f(C)$,

(iv) $fx \in \delta C \Rightarrow gx \in C$ *and*

(v) $f(C)$ *is closed in* X.

Then there exists a coincidence point v *in* C. *Moreover, if* f *and* g *are coincidentally commuting, then* v *remains a unique common fixed point of* f *and* g.

2. Results

This paper seeks to modify Theorem 1.3 to four non-self maps. We seek to prove the following theorem.

Theorem 2.1. *Let* (X, d) *be a complete Takahashi convex metric space with convex structure* W *which is continuous in the third variable. Let* K *be a non-empty closed subset of* X. *Let* δK *be the boundary of* K *with* $\delta K \neq \emptyset$. *Let mappings* $A, B, S, T : K \to X$. *Assume that* A, B, S *and* T *satisfy the following conditions:*

(i) *For every* $x, y \in K, d(Ax, By) \leq M_\omega(x, y)$, *where*
$M_\omega(x, y)$
$= max\{[\omega_1[d(Sx, Ty)], \omega_2[d(Ax, Sx)], \omega_3[d(By, Ty)], \omega_4[d(Ax, Ty)], \omega_5[d(Sx, By)]\}$,
$\omega_1: [0, +\infty) \to [0, +\infty), i = 1, 2, 3, 4, 5$ *is a non-decreasing semicontinuous function from the right, such that* $\omega_i(r) < \frac{1}{2}r$ *for* $r > 0$, *and* $lim_{r \to \infty}[r - 2w_i(r)] = +\infty$,

(ii) $\delta K \subseteq T(K), \delta K \subset S(K)$,

(iii) $Sx \in \delta K \Rightarrow Ax \in K,\ Tx \in \delta K \Rightarrow Bx \in K$,

(iv) $A(K) \cap K \subset T(K), B(K) \cap K \subset S(K)$ *and*

(v) $S(K), T(K)$ *are closed in* X.

Then there exists a coincidence point $z \in K$ *for* A, B, S *and* T. *Moreover, if each of the pairs* $\{A, S\}$ *and* $\{B, T\}$ *is coincidentally commuting, then* z *remains a unique common fixed point of* A, B, S *and* T.

Proof. Commencing with an arbitrary point $w \in \delta K$, we construct a sequence $\{x_n\}$ of points in K as follows:
From assumption (ii), there is a point $x_0 \in K$ such that $Sx_0 = w$. From (iii), $Ax_0 \in K$. According to (iv), we find $x_1 \in K$ such that $Tx_1 = Ax_0$. We locate Bx_1. We consider two scenarios.

(1) If $Bx_1 \in K$, then, using (iv), we can locate $x_2 \in K$ such that $Bx_1 = Sx_2$. We then find Ax_2. If it happens $Ax_2 \in K$, then, from (iv), we can find $x_3 \in K$ such that $Ax_2 = Tx_3$. If however $Ax_2 \notin K$, because W is continuous in the third variable, there is $\lambda_{22} \in (0,1)$ such that $W\ (Sx_2, Ax_2, \lambda_{22}) \in \text{seg}[Sx_2, Ax_2] \cap \delta K$. As $W(Sx_2, Ax_2, \lambda_{22}) \in \delta K$, by (ii), there is $x_3 \in K$ such that $Tx_3 = W(Sx_2, Ax_2, \lambda_{22}) \in \delta K$.

(2) (2) In the case where $Bx_1 \notin K$, because W is continuous in the third variable, there is $\lambda_{11} \in (0,1)$ such that $W(Tx_1, Bx_1, \lambda_{11}) \in \text{seg}[Tx_1, Bx_1] \cap K$. As $W(Tx_1, Bx_1, \lambda_{11}) \in \delta K$, by (ii), there is $x_2 \in K$ such that $Sx_2 = W(Tx_1, Bx_1, \lambda_{11}) \in \delta K$.

In general, we construct the rest of the sequence by proceeding inductively using the following procedure. If $Ax_{2n} \in K$, then, by (iv), we choose $x_{2n+1} \in K$ such that $Tx_{2n+1} = Ax_{2n}$. Similarly if $Bx_{2n+1} \in K$, then, by (iv), we choose $x_{2n+2} \in K$ such that $Sx_{2n+2} = Bx_{2n+1}$.

If however $Ax_{2n} \notin K$, it means, by (iii), there is $\lambda_{2n,2n} \in (0,1)$ and we can choose $x_{2n+1} \in K$ such that $Tx_{2n+1} = W\left(Sx_{2n}, Ax_{2n}, \lambda_{2n,2n}\right) \in K$.
Similarly if $Bx_{2n+1} \notin K$, it means there is $\lambda_{2n+1,2n+1} \in (0,1)$ and we can choose $x_{2n+2} \in K$ such that $Sx_{2n+2} = W\left(Tx_{2n+1}, Bx_{2n+1}, \lambda_{2n+1,2n+1}\right) \in \delta K$.
Now we first prove that

$$Ax_{2n} \neq Tx_{2n+1} \Rightarrow Bx_{2n-1} = Sx_n \tag{2.1}$$

Suppose we have $Bx_{2n-1} \neq Sx_{2n}$. Then we have $Sx_{2n} \in \delta K$, which by (iii) means $Ax_{2n} \in K$. By (iv), this implies that $Ax_{2n} = Tx_{2n+1}$, which is a contradiction. Using a similar argument we have

$$Bx_{2n+1} \neq Sx_{2n+2} \Rightarrow Ax_{2n} = Tx_{2n+1} \tag{2.2}$$

We now prove that the sequences $\{Sx_{2n}\}, \{Ax_{2n}\}, \{Bx_{2n+1}\}$ and $\{Tx_{2n+1}\}$ are bounded. For each $n \geq 1$, let

$$D_n = \left(\bigcup_{i=0}^{n-1} \{Ax_{2i}\}\right) \cup \left(\bigcup_{i=0}^{n-1} \{Bx_{2i+1}\}\right) \cup \left(\bigcup_{i=0}^{n-1} \{Sx_{2i}\}\right) \cup \left(\bigcup_{i=0}^{n-1} \{Tx_{2i+1}\}\right).$$

Let $\alpha_n = \text{diam}(D_n)$. We want to show that

$$\alpha_n \leq \max\{d\left(Sx_0, Ax_{2j}\right), d\left(Sx_0, Bx_{2j+1}\right\}, \ 0 \leq j \leq n-1 \tag{2.3}$$

Let us consider the case where $\alpha_n = 0, n \geq 1$.
If $\alpha_n = 0$, we have $Sx_0 = Ax_0 = Bx_1 = Tx_1$. We shall show that Sx_0 is a common fixed point of A and S. As the mappings A and S are coincidentally commuting at the coincidence point x_0, we have

$$Sx_0 = Ax_0 \Rightarrow SSx_0 = SAx_0 = ASx_0 \tag{2.4}$$

From (i), we have for some $t = 1, 4$ or 5,

$$\begin{aligned}
d(SSx_0, Sx_0) = d(ASx_0, Bx_1) &\leq M_\omega(Sx_0, x_1) \\
&= \max\{\omega_1[d(SSx_0, Tx_1)], \omega_2[d(ASx_0, SSx_0)], \omega_3[d(Bx_1, Tx_1)], \\
&\qquad \omega_4[d(ASx_0, Tx_1)], \omega_5[d(SSx_0, Bx_1)]\} \\
&= \max\{\omega_1[d(SSx_0, Sx_0)], \omega_2[d(SSx_0, SSx_0)], \omega_3[d(Sx_0, Sx_0)], \\
&\qquad \omega_4[d(SSx_0, Sx_0)], \omega_5[d(SSx_0, Sx_0)]\} \\
&= \omega_t[d(SSx_0, Sx_0)] \\
&< \frac{1}{2} d(SSx_0, Sx_0) \text{ for } d(SSx_0, Sx_0) > 0 \\
&\Rightarrow \ d(SSx_0, Sx_0) = 0 \\
&\Rightarrow \ SSx_0 = Sx_0.
\end{aligned}$$

Hence Sx_0 is a fixed point of S. From (2.4), we have $SSx_0 = ASx_0$, which implies $d(ASx_0, Sx_0) = 0$, making Sx_0 a fixed point of A too.

Using a similar argument we have $Tx_1 = Sx_0$ being a common fixed point of T and B. Hence, $z = Sx_0$ is a common fixed point of all four mappings A, B, S and T.

To show the uniqueness of the fixed point, let z' be also a fixed point of A, B, S and T. Then for some $i = 1, 4$ or 5,

$$\begin{aligned} d(z,z') &= d(Az,Bz') \\ &\leq \max\{\omega_1[d(Sz,Tz')], \omega_2[d(Az,Sz)], \omega_3[d(Bz',Tz')], \\ &\qquad \omega_4[d(Az,Tz')], \omega_5[d(Sz,Bz')],\} \\ &= \max\{\omega_i[d(z,z')]\} \\ &< \frac{1}{2}d(z,z') \text{ for } d(z,z') > 0 \\ &\Rightarrow d(z,z') = 0 \\ &\Rightarrow z = z'. \end{aligned}$$

Hence when $\alpha_n = 0, z = Sx_0$ is the unique common fixed point of A, B, S and T.

We now consider the cases when $\alpha_n > 0$.

Case 1: Consider the case where $\alpha_n = d(Sx_{2i}, Ax_{2j})$ for some $0 \leq i, j \leq n-1$..
Subcase (1. i): If $i \geq 1$ and $Sx_{2i} = Bx_{2i-1}$ we have for some $s \in \{1, 2, \ldots, 5\}$

$$\begin{aligned} \alpha_n = d(Sx_{2i}, Ax_{2j}) &= d(Ax_{2j}, Bx_{2i-1}) \\ &\leq M_\omega(x_{2j}, x_{2i-1}) \\ &\leq \omega_s(\alpha_n) \\ &< \tfrac{1}{2}\alpha_n, \end{aligned}$$

which is a contradiction. Hence $i = 0$.

Subcase (1.ii): If however $i \geq 1$ and $Sx_{2i} \neq Bx_{2i-1}$, it implies $Sx_{2i} \in \text{seg}[Ax_{2i-2}, Bx_{2i-1}]$ and hence by Lemma 1.2 we have

$$\alpha_n = d(Sx_{2i}, Ax_{2j}) \leq \max\{d(Ax_{2j}, Bx_{2i-1}), d(Ax_{2i-2}, Ax_{2j})\}.$$

Subcase (1.ii.1): If $d(Ax_{2j}, Bx_{2i-1}) \geq d(Ax_{2i-2}, Ax_{2j})$, we have

$$\alpha_n = d(Sx_{2i}, Ax_{2j}) \leq d(Ax_{2j}, Bx_{2i-1}),$$

which leads to the contradiction in Subcase (1.i).

Subcase (1.ii.2): Otherwise if $d(Ax_{2j}, Bx_{2i-1}) < d(Ax_{2i-2}, Ax_{2j})$, then for

k: $2i - 2 < 2k + 1 < 2j$, and for some $s, t \in \{1,2,\ldots,5\}$, we have

$$\begin{aligned} \alpha_n = d(Sx_{2i}, Ax_{2j}) &\leq d(Ax_{2i-2}, Ax_{2j}) \\ &\leq d(Ax_{2i-2}, Bx_{2k+1}) + d(Ax_{2j}, Bx_{2k+1}) \\ &\leq M_\omega(x_{2i-2}, x_{2k+1}) + M_\omega(x_{2j}, x_{2k+1}) \\ &\leq \omega_s(\alpha_n) + \omega_t(\alpha_n) \end{aligned}$$

$$< \frac{1}{2}\alpha_n + \frac{1}{2}\alpha_n$$
$$= \alpha_n,$$

which is a contradiction. Hence $i = 0$.

Case 2: The case where $\alpha_n = d(Ax_{2i}, Bx_{2j+1})$ leads to a contradiction by Subcase (1.i).

Case 3: The case where $\alpha_n = d(Ax_{2i}, Ax_{2j})$ leads to a contradiction by Subcase (1.ii.2).

Case 4: If $\alpha_n = d(Bx_{2i+1}, Bx_{2j+1})$ then for $k : 2i + 1 < 2k < 2j + 1$, and for some $s, t \in \{1,2, \dots ,5\}$, we have

$$\begin{aligned}\alpha_n &= d(Bx_{2i+1}, Bx_{2j+1}) \\ &\leq d(Ax_{2k}, Bx_{2i+1}) + d(Ax_{2k}, Bx_{2j+1}) \\ &\leq M_\omega(x_{2k}, x_{2i+1}) + M_\omega(x_{2k}, x_{2j+1}) \\ &\leq \omega_{s(\alpha_n)} + \omega_t(\alpha_n) \\ &< \frac{1}{2}\alpha_n + \frac{1}{2}\alpha_n \\ &= \alpha_n,\end{aligned}$$

which is a contradiction.

Case 5: If $\alpha_n = d(Tx_{2i+1}, Bx_{2j+1})$ for some $0 \leq i, j \leq n - 1$, then:
Subcase (5.i): If $Tx_{2i+1} = Ax_{2i}$, we have $\alpha_n = d(Tx_{2i+1}, Bx_{2j+1}) = d(Ax_{2i}, Bx_{2j+1})$, which is a contradiction by Subcase (1.i).

Subcase (5.ii): Otherwise if $Tx_{2i+1} \neq Ax_{2i}$ then $Tx_{2i+1} \in \text{seg}[Bx_{2i-1}, Ax_{2i}]$ and hence by Lemma 1.2,

$$\alpha_n = d(Tx_{2i+1}, Bx_{2j+1}) \leq \max\{d(Bx_{2i-1}, Bx_{2j+1}), d(Ax_{2i}, Bx_{2j+1})\}.$$

This means we have either $d(Tx_{2i+1}, Bx_{2j+1}) \leq d(Bx_{2i-1}, Bx_{2j+1})$, which is a contradiction by Case 4 or $d(Tx_{2i+1}, Bx_{2j+1}) \leq d(Ax_{2i}, Bx_{2j+1})$, which is a contradiction by Subcase (1.i).

Case 6: If $\alpha_n = d(Tx_{2i+1}, Ax_{2j})$ for some $0 \leq i, j \leq n - 1$, then:

Subcase (6.i): If $Tx_{2i+1} = Ax_{2i}$ we have $\alpha_n = d(Tx_{2i+1}, Ax_{2j}) = d(Ax_{2i}, Ax_{2j})$, which is not possible by Subcase (1.ii.2)

Subcase (6.ii): Otherwise if $Tx_{2i+1} \neq Ax_{2i}$ then $Tx_{2i+1} \in \text{seg}[Bx_{2i-1}, Ax_{2i}]$ and hence $\alpha_n = d(Tx_{2i+1}, Ax_{2j}) \leq \max\{d(Ax_{2j}, Bx_{2i-1}), d(Ax_{2i}, Ax_{2j})\}$. This implies we have either $d(Tx_{2i+1}, Ax_{2j}) \leq d(Ax_{2j}, Bx_{2i-1})$ which is a contradiction by Subcase (1.i) or else we have $d(Tx_{2i+1}, Ax_{2j}) \leq d(Ax_{2i}, Ax_{2j})$ which is a contradiction by Subcase (1.ii.2).

Case 7: If $\alpha_n = d(Tx_{2i+1}, Tx_{2j+1})$ for some $0 \leq i, j \leq n - 1$, then:

Subcase (7.i): If $Tx_{2j+1} = Ax_{2j}$, we have $\alpha_n = d(Tx_{2i+1}, Tx_{2j+1}) = d(Tx_{2i+1}, Ax_{2j})$ which a contradiction by Case 6.

Subcase (7.ii): Otherwise if $Tx_{2j+1} \neq Ax_{2j}$, then $Tx_{2j+1} \in \text{seg}[Bx_{2j-1},\ Ax_{2j}]$ and hence $\alpha_n = d(Tx_{2i+1}, Tx_{2j+1}) \leq \max\{d(Tx_{2i+1}, Bx_{2j-1}), d(Tx_{2i+1}, Ax_{2j})\}$.

This implies we have either $d(Tx_{2i+1}, Tx_{2j+1}) \leq d(Tx_{2i+1}, Bx_{2j-1})$, which results in a contradiction by Case 5 or else we have $d(Tx_{2i+1}, Tx_{2j+1}) \leq d(Tx_{2i+1}, Ax_{2j})$, which is a contradiction by Case 6.

Case 8: If $\alpha_n = d(Sx_{2i}, Bx_{2j+1})$ for some $0 \leq i, j \leq n-1$, then:

Subcase (8.i): If $i \geq 1$ and $Sx_{2i} = Bx_{2i-1}$ then $\alpha_n = d(Sx_{2i}, Bx_{2j+1}) = d(Bx_{2i-1}, Bx_{2j+1})$, which is not possible as per Case 4. Hence $i = 0$.

Subcase (8.ii): If however $i \geq 1$ and $Sx_{2i} \neq Bx_{2i-1}$, it means that $Sx_{2i} \in$ $\text{seg}[Ax_{2i-2}, Bx_{2i-1}]$. This implies that
$\alpha_n = d(Sx_{2i}, Bx_{2j+1}) \leq \max\{d(Ax_{2i-1}Bx_{2j+1}), d(Bx_{2i-1}, Bx_{2j+1})\}$. This leads to a contradiction by Subcase (1.i) when
$d(Sx_{2i}, Bx_{2j+1}) \leq d(Ax_{2i-2}, Bx_{2j+1})$ and a contradiction by Case 4 when it happens that $d(Sx_{2i}, Bx_{2j+1}) \leq d(Bx_{2i-1}, Bx_{2j+1})$. Hence $i = 0$.

Case 9: If $\alpha_n = d(Sx_{2i}, Sx_{2j})$ for some $0 < i < j < n-1$, then:

Subcase (9.i): If $i \geq 1$ and $Sx_{2j} = Bx_{2j-1}$, then we have
$\alpha_n = d(Sx_{2i}, Sx_{2j}) = d(Sx_{2i}, Bx_{2j-1})$, which leads to a contradiction according to Case 8. Hence $i = 0$.

Subcase (9.ii): If $i \geq 1$ and $Sx_{2j} \neq Bx_{2j-1}$, it implies that $Sx_{2j} \in \text{seg}[Ax_{2j-2}, Bx_{2j-1}]$ and $d(Sx_{2i}, Sx_{2j}) \leq \max\{d(Sx_{2i}, Ax_{2j-2}),\ d(Sx_{2i}, Bx_{2j-1})\}$.
If it happens $d(Sx_{2i}, Sx_{2j}) \leq d(Sx_{2i}, Ax_{2j-2})$, we get a contradiction by Case 1. However if it happens that $d(Sx_{2i}, Sx_{2j}) \leq d(Sx_{2i}, Bx_{2j-1})$, then we get a contradiction by Case 8. Hence $i = 0$.

Case 10: If $\alpha_n = d(Sx_{2i},\ Tx_{2j+1})$, for some $0 \leq i, j \leq n-1$, we have
Subcase (10.i): If $i \geq 1$ and $Sx_{2i} = Bx_{2i-1}$ then we have
$\alpha_n = d(Sx_{2i},\ Tx_{2j+1}) = d(Tx_{2j+1}, Bx_{2i-1})$, which is not possible as per Case 8. Hence $i - 0$.
Subcase (10.ii): If however $i \geq 1$ and $Sx_{2i} \neq Bx_{2i-1}$ it implies that
$Sx_{2i} \in \text{seg}[Ax_{2i-2}, Bx_{2i-1}]$ and
$d(Sx_{2i}, Tx_{2j+1}) \leq \max\{d(Tx_{2j+1}, Ax_{2i-2}), d(Tx_{2j+1}, Bx_{2i-1})\}$. This leads to contradictions by Case 6 and Case 5. Hence $i = 0$.
We have considered 10 possible cases for α_n and conclude that

$$\alpha_n \in \{d(Sx_0, Sx_{2j}), d(Sx_0, Ax_{2j}), d(Sx_0, Bx_{2j+1}), d(Sx_0, Tx_{2j+1})\},$$

for some $0 \leq j \leq n-1$. By the construction of the sequences, we have
$d(Sx_0, Sx_{2j}) \leq \max\{d(Sx_0, Ax_{2j-2}), d(Sx_0, Bx_{2j-1})\}$ and
$(Sx_0, Tx_{2j+1}) \leq \max\{d(Sx_0, Ax_{2j}), d(Sx_0, Bx_{2j-1})\}$. Thus we have now proved (2.3) that is,

$$\alpha_n \leq \max\{d(Sx_0, Ax_{2j}), d(Sx_0, Bx_{2j+1})\}, 0 \leq j \leq 1.$$

Consider the case where $\max\{d(Sx_0, Ax_{2j})\} \leq \max\{d(Sx_0, Bx_{2j+1})\}$, $0 \leq j \leq n-1$. Then we have for some $0 \leq j \leq n-1$, and for some $u \in \{1,2,\dots,5\}$

$$\begin{aligned} \alpha_n &\leq d(Sx_0, Bx_{2j+1}) \\ &\leq d(Sx_0, Ax_0) + d(Ax_0, Bx_{2j+1}) \\ &\leq d(Sx_0, Ax_0) + \omega_u[\alpha_n] \\ &\leq d(Sx_0, Ax_0) + 2\omega_u[\alpha_n] \end{aligned}$$

$\Rightarrow \alpha_n - 2\omega_u[\alpha_n] \leq d(Sx_0, Ax_0)$.

Alternatively, if $\max\{d(Sx_0, Ax_{2j})\} > \max\{d(Sx_0, Bx_{2j+1})\}$, $0 \leq j \leq n-1$, then for some $0 \leq j \leq n-1$ and for some $v \in \{1,2,\dots,5\}$ and Subcase (1.ii.2) we have

$$\begin{aligned} \alpha_n &\leq d(Sx_0, Ax_{2j}) \\ &\leq d(Sx_0, Ax_0) + d(Ax_0, Ax_{2j}) \\ &\leq d(Sx_0, Ax_0) + 2\omega_v[\alpha_n] \end{aligned}$$

$\Rightarrow \alpha_n - 2\omega_v[\alpha_n] \leq d(Sx_0, Ax_0)$.

Thus in both cases we have for some $s \in \{1, 2, 3, 4, 5\}$

$$\alpha_n - 2\omega_s[\alpha_n] \leq d(Sx_0, Ax_0) \tag{2.5}$$

By assumption (i), there is $r_0 \in [0, +\infty)$ such that for each $s \in \{1,2,\dots,5\}$, we have $r - 2\omega_s[r] > d(Sx_0, Ax_0)$ for $r > r_0$. Thus, there is a subsequence $\{a_n\}$ of $\{\alpha_n\}$ and $s \in \{1,2,\dots,5\}$ such that for each n we have

$$a_n - 2\omega_s[a_n] \leq d(Sx_0, Ax_0).$$

Thus by (2.5), $a_n \leq r_0$, $n = 1,2,\dots$, and also

$$a := \lim_{n\to+\infty} a_n = diam(D) \leq r_0.$$

We have hence proved that $\{Sx_{2n}\}, \{Tx_{2n+1}\}$, $\{Ax_{2n}\}$ and $\{Bx_{2n+1}\}$ are bounded sequences.

To prove that $\{Sx_{2n}\}, \{Tx_{2n+1}\}$, $\{Ax_{2n}\}$ and $\{Bx_{2n+1}\}$ converge in K, we reflect on the set

$$E_n = \left(\bigcup_{i=n}^{+\infty}\{Ax_{2i}\}\right) \cup \left(\bigcup_{i=n}^{+\infty}\{Bx_{2i+1}\}\right) \cup \left(\bigcup_{i=n}^{+\infty}\{Sx_{2i}\}\right) \cup \left(\bigcup_{i=n}^{+\infty}\{Tx_{2i+1}\}\right),$$

$n = 2, 3, \dots$.

By (2.3) we have

$$e_n := \mathrm{diam}(E_n) \leq \sup_{j\geq n}\{d(Sx_{2n}, Ax_{2j}), d(Sx_{2n}, Bx_{2j+1})\}, n = 2, 3, \dots$$

If $Sx_{2n} = Bx_{2n-1}$, we have, as in Case 1 and Case 8, for each $j \geq n$, and for some $u \in \{1,2,\dots,5\}$

$$\begin{aligned} e_n &\leq \sup_{j\geq n}\{d(Ax_{2j}, Bx_{2n-1}), d(Bx_{2j+1}, Bx_{2n-1}), d(Ax_{2j}, Bx_{2n-1}),\}, n = 2,3,\dots \\ &\leq 2\omega_u[e_{n-1}]. \end{aligned} \tag{2.6}$$

If however $Sx_{2n} \neq Bx_{2n-1}$, it implies $Sx_{2n} \in \mathrm{seg}[Ax_{2n-2}, Bx_{2n-1}]$. Hence, as in Case 1 and Case 8, for each $j \geq n$ and for some $u \in \{1,2,\dots,5\}$, we have,

$$e_n \leq \sup_{j\geq n}\{d(Ax_{2n-2}, Ax_{2j}), d(Bx_{2n-1}, Ax_{2j}), d(Ax_{2n-2}, Bx_{2j+1}), d(Bx_{2n-1}, Bx_2$$

$\leq 2\omega_v(e_{n-2}).$ (2.7)

By (2.6) and (2.7), there is a subsequence $\{\varepsilon_n\}$ of $\{e_n\}$ and some $s \in \{1,2,\dots,5\}$ such that for each n, we have
$\varepsilon_n \leq 2\omega_s[\varepsilon_{n-2}], n = 2,3,\dots$
$\leq \varepsilon_{n-2}.$ (2.8)
We note that $e_n \geq e_{n+1}$ for every n. Let $\lim_{n\to\infty} e_n = \lim_{n\to\infty} \varepsilon_n = e$. We claim that $e = 0$. If $e > 0$, then by (2.8) and assumption (i) we have

$$\lim_{n\to\infty} \varepsilon_n < \lim_{n\to\infty} \varepsilon_{n-2} \Rightarrow e < e,$$

which is a contradiction. Hence $e = 0$.

This means that the sequences $\{Sx_{2n}\}, \{Tx_{2n+1}\}, \{Ax_{2n}\}$ and $\{Bx_{2n+1}\}$ converge to a point z. Since $\{Sx_{2n}\}, \{Tx_{2n+1}\} \in K$ and $S(K), T(K)$ are closed in the complete metric space X, we conclude that

$$\lim_{n\to\infty} Sx_{2n} = \lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} Ax_{2n} = \lim_{n\to\infty} Bx_{2n+1} = z \in S(K) \cap T(K). \quad (2.9)$$

As $z \in S(K)$, there is a point $u \in K$ such that $Su = z$. We show that u is a coincidence point of A, B and S.

$$d(Au, Bx_{2n+1}) \leq \max\{\omega_1[d(Su, Tx_{2n+1})]; \omega_2[d(Au, Su)], \omega_3[d(Bx_{2n+1}, Tx_{2n+1})], \omega_4[d(Au, Tx_{2n+1})], \omega_5[d(Su, Bx_{2n+1})]\}$$
$$= \max\{\omega_1[d(z, Tx_{2n+1})]; \omega_2[d(Au, z)], \omega_3[d(Bx_{2n+1}, Tx_{2n+1})], \omega_4[d(Au, Tx_{2n+1})], \omega_5[d(z, Bx_{2n+1})]\}.$$

Taking $n \to +\infty$ and applying (2.9) we get
$d(Au, z) \leq \max\{\omega_1[d(z, z)], \omega_2[d(Au, z)], \omega_3[d(z, z], \omega_4[d(Au, z)], \omega_5[d(z, z)]\}$
$\leq \omega_i[d(Au, z)]$ for some $i \in \{2,4\}$
$< d(Au, z)$ for $d(Au, z) > 0$
$\Rightarrow d(Au, z) = 0$
$\Rightarrow Au = z.$

Using a similar procedure, when we expand $d(Ax_{2n}, Bu)$, we get $Bu = z$ making u a coincidence point of A, B and S. By the coincidental commutativity of S and A we have

$$SAu = ASu \Rightarrow Sz = Az.$$

From (2.9), $z \in T(K)$ means there is $v \in K$, such that $Tv = z$. We show that $Bv = z$.

$d(z, Bv) = d(Au, Bv)$
$\leq \max\{\omega_1[d(Su, Tv)], \omega_2[d(Su, Au)], \omega_3[d(Tv, Bv)], \omega_4[d(Au, Tv)], \omega_5[d(Su, Bv)]\}$
$= \max\{\omega_1[d(z, z)], \omega_2[d(z, z)], \omega_3[d(z, Bv)], \omega_4[d(z, z)], \omega_5[d(z, Bv)]\}$
$\leq \omega_j[d(z, Bv)]$ for $j = 3$ or 5,
$< d(z, Bv)$ for $d(z, Bv) > 0$
$\Rightarrow Bv = z.$
Thus v is a coincidence point of B and T. By the coincidental commutativity property, we have $BTv = TBv \Rightarrow Bz = Tz$.
$d(Az, Bz) \leq \max\{\omega_1[d(Sz, Tz)], \omega_2[d(Sz, Az)], \omega_3[d(Tz, Bz)], \omega_4[d(Az, Tz)], \omega_5[d(Sz, Bz)]\}$
$= \max\{\omega_1[d(Az, Bz)], \omega_2[d(Az, Az)], \omega_3[d(Bz, Bz)],$

$$\omega_4[d(Az,Bz)],\omega_5[d(Az,Bz)]\}$$
$$\leq \omega_i[d(Az,Bz)] \text{ for } i \in \{1,4,5\}$$
$$< d(Az,Bz) \text{ for } d(Az,Bz) > 0$$
$$\Rightarrow Az = Bz.$$

Hence we have

$$Az = Bz = Sz = Tz. \tag{2.10}$$

Now we consider the following:

$$\begin{aligned} d(z,Bz) &= d(Au,Bz) \\ &\leq \max\{\omega_1[d(Su,Tz)],\omega_2[d(Au,Su)],\omega_3[d(Bz,Tz)], \\ &\qquad \omega_4[d(Au,Tz)],\omega_5[d(Su,Bz)]\} \\ &\leq \max\{\omega_1[d(z,Bz)],\omega_2[d(z,z)],\omega_3[d(Bz,Bz)], \\ &\qquad \omega_4[d(z,Bz)],\omega_5[d(z,Bz)]\} \\ &\leq \omega_j[d(z,Bz)] \text{ for } j \in \{1,4,5\} \\ &< d(z,Bz) \text{ for } d(z,Bz) > 0 \\ \Rightarrow d(z,Bz) &= 0 \\ \Rightarrow Bz &= z. \end{aligned}$$

From (2.10) we conclude that

$$Az = Bz = Sz = Tz = z.$$

This means that z is a common fixed point of A,B,S and T.

We now show that z is unique. Suppose z' is also a common fixed point of A,B,S and T. We get

$$\begin{aligned} d(z,z') &= d(Az,Bz') \\ &\leq \max\{\omega_1[d(Sz,Tz')],\ \omega_2[d(Az,Sz)],\ \omega_3[d(Bz',Tz')], \\ &\qquad \omega_4[d(Az,Tz')],\omega_5[d(Sz,Bz')]\} \\ &\leq \max\{\omega_1[d(z,z')],\ \omega_2[d(z,z)],\ \omega_3[d(z',z')],\omega_4[d(z,z')],\omega_5[d(z,z')]\} \\ &\leq \omega_k[d(z,z')] \text{ for } k \in \{1,4,5\} \\ &< d(z,z') \text{ for } d(z,z') > 0 \\ &\Rightarrow d(z,z') = 0 \\ &\Rightarrow z = z'. \end{aligned}$$

This proves that the common fixed point of A,B,S and T is unique.

If we define $\omega_t[r] = hr$ for $0 \leq 2h < 1$, we get the following corollary:

Corollary 2.2. *Let (X,d) be a complete Takahashi convex metric space with convex structure W which is continuous in the third variable. Let K be a non-empty closed subset of X and δK be the boundary of K, with $\delta K \neq \emptyset$. Let the mappings $A,B,S,T: K \to X$. Suppose that A,B,S and T satisfy the following conditions:*

(i) *For every $x,y \in K$ we have $d(Ax,By) \leq hM(x,y)$, where $0 \leq 2h < 1$ and* $M(x,y) = max\{d(Sx,Ty),d(Ax,Sx),d(By,Ty),d(Ax,T y),d(Sx,By)\}$,

(ii) $\delta K \subseteq T(K), \delta K \subseteq S(K)$,

(iii) $Sx \in \delta K \Rightarrow Ax \in K, Tx \in \delta K \Rightarrow Bx \in K$ *and*

(iv) $S(K),T(K)$ *are closed in* X.

Then there exists a coincidence point z *for* A, B, S *and* T *in* K. *Moreover, if each of the pairs* $\{A, S\}$ *and* $\{B, T\}$ *is coincidentally commuting, then* z *remains a unique common fixed point of* A, B, S *and* T.

We deduce another corollary by letting $A = B$. When this is the situation, in the proof for Theorem 2.1, Case 4 is identical to Subcase (1.i). Moreover, Subcase (1.i) enables us to change the property in Theorem 2.1(i) from $\omega_1[r] < r/2$ for $r > 0$ to $\omega_1[r] < r$ for $r > 0$.

Corollary 2.3. *Let* (X, d) *be a complete Takahashi convex metric space with convex structure* W *which is continuous in the third variable. Let* K *be a non-empty closed subset of* X *and* δK *be the boundary of* K, *with* $\delta K \neq \emptyset$. *Let the mappings* $A, S, T: K \to X$. *Suppose that* A, S *and* T *satisfy the following conditions:*

(i) *For every* $x, y \in K, d(Ax, Ay) \leq M_\omega(x, y)$ *where*
$M_\omega(x, y) =$
$max\{\omega_1[d(Sx, Ty)], \omega_2[d(Ax, Sx)], \omega_3[d(Ay, Ty)], \omega_4[d(Ax, Ty)], \omega_5[d(Sx, Ay)]\}$ *where* $\omega_1: [0, +\infty) \to [0, +\infty), i = 1,2,3,4,5,$ *is a non-decreasing semicontinuous function from the right, such that* $\omega_i(r) < r$ *for* $r > 0$, *and* $lim_{r\to\infty}[r - w_i(r)] = +\infty$,
(ii) $\delta K \subseteq T(K), \delta K \subset S(K)$,
(iii) $Sx \in \delta K \Rightarrow Ax \in K,\ Tx \in \delta K \Rightarrow Ax \in K$,
(iv) $A(K) \cap K \subset T(K), A(K) \cap K \subset S(K)$ *and*
(v) $S(K), T(K)$ *is closed in* X.
Then there exists a coincidence point $z \in K$ *for* A, S *and* T. *Moreover, if each of the pairs* $\{A, S\}$ *and* $\{A, T\}$ *is coincidentally commuting, then* z *remains a unique common fixed point of* A, S *and* T.
Remark 1: If we set $S = T$ in Corollary 2.3, we get Theorem 1.3 by Gajić and Rakočević [1].

Remark 2: If we set $S = T = I$ in Corollary 2.3, we get the theorem as proved by Ćirić [5].

We form the following corollary by setting $A = B = I$ in Theorem 2.1, that is, setting $A = I$ in Corollary 2.3.

Corollary 2.4. *Let* (X, d) *be a complete Takahashi convex metric space with convex structure* W *which is continuous in the third variable. Let* K *be a non-empty closed subset of* X *and* δK *be the boundary of* K, *with* $\delta K \neq \emptyset$. *Let the mappings* $S, T: K \to X$. *Suppose that* S *and* T *satisfy the following conditions:*
(i) *For every* $x, y \in K, d(x, y) \leq M_\omega(x, y)$ *where*
$M_\omega(x, y) = \max\{\omega_1[d(Sx, Ty)], \omega_2[d(x, Sx)], \omega_3[d(y, Ty)], \omega_4[d(x, Ty)], \omega_5[d(Sx, y)]\}$ and $\omega_1: [0, +\infty) \to [0, +\infty), i = 1, 2, 3, 4, 5,$ *is a non-decreasing semicontinuous function from the right, such that* $\omega_i(r) < r$ *for* $r > 0$, *and* $lim_{r\to\infty}[r - w_i(r)] = +\infty$,
(ii) $Sx \in \delta K \Rightarrow x \in K,\ Tx \in \delta K \Rightarrow x \in K$,
(iii) $K \subset T(K), K \subset S(K)$ *and*
(iv) $S(K), T(K)$ *is closed in* X.
Then there exists a unique common fixed point of S *and* T.

We form yet another corollary from Corollary 2.4 by setting $S = T$.

Corollary 2.5. *Let* (X, d) *be a complete Takahashi convex metric space with convex structure* W *which is continuous in the third variable. Let* K *be a non-empty closed subset of* X *and* δK

be the boundary of K*, with* $\delta K \neq \emptyset$*. Let the mapping* $S: K \to X$*. Suppose that* S *satisfies the following conditions:*

(i) *For every* $x, y \in K, d(x, y) \leq M_\omega(x, y)$ *where*
$M_\omega(x, y) = \max\{\omega_1[d(Sx, Sy)], \omega_2[d(x, Sx)], \omega_3[d(y, Sy)], \omega_4[d(x, Sy)], \omega_5[d(Sx, y)]\}$ and $\omega_1: [0, +\infty) \to [0, +\infty), i = 1, 2, 3, 4, 5,$ *is a non-decreasing semicontinuous function from the right, such that* $\omega_i(r) < r$ *for* $r > 0$*, and* $lim_{r\to\infty}[r - w_i(r)] = +\infty$*,*

(ii) $Sx \in \delta K \Rightarrow x \in K$*,*

(iii) $K \subset S(K)$ *and*

(iv) $S(K)$ *is closed in* X*.*

Then there exists a unique fixed point of S*.*

If we let $B = I$ in Theorem 2.1, to get the following corollary:

Corollary 2.6.
Let (X, d) *be a complete Takahashi convex metric space with convex structure* W *which is continuous in the third variable. Let* K *be a non-empty closed subset of* X*. Let* δK *be the boundary of* K *with* $\delta K \neq \emptyset$*. Let mappings* $A, S, T : K \to X$*. Assume that* A, S *and* T *satisfy the following conditions:*

(vi) *For every* $x, y \in K, d(Ax, y) \leq M_\omega(x, y)$*, where*
$M_\omega(x, y)$
$= max\{[\omega_1[d(Sx, Ty)], \omega_2[d(Ax, Sx)], \omega_3[d(y, Ty)], \omega_4[d(Ax, Ty)], \omega_5[d(Sx, y)]\}$,
$\omega_1: [0, +\infty) \to [0, +\infty), i = 1, 2, 3, 4, 5$ *is a non-decreasing semicontinuous function from the right, such that* $\omega_i(r) < \frac{1}{2}r$ *for* $r > 0$*, and* $lim_{r\to\infty}[r - 2w_i(r)] = +\infty$*,*

(vii) $\delta K \subseteq T(K)$*,*

(viii) $Sx \in \delta K \Rightarrow Ax \in K,\ Tx \in \delta K \Rightarrow Bx \in K$*,*

(ix) $A(K) \cap K \subset T(K), K \subset S(K)$ *and*

(x) $S(K), T(K)$ *are closed in* X*.*

Then there exists a coincidence point $z \in K$ *for* A, S *and* T*. Moreover, if the pair* $\{A, S\}$ *is coincidentally commuting, then* z *remains a unique common fixed point of* A, S *and* T*.*

When we set S = T in Theorem 2.1, we get the following corollary:

Corollary 2.7. *Let* (X, d) *be a complete Takahashi convex metric space with convex structure* W *which is continuous in the third variable. Let* K *be a non-empty closed subset of* X*. Let* δK *be the boundary of* K *with* $\delta K \neq \emptyset$*. Let mappings* $A, B, S : K \to X$*. Assume that* A, B *and* S *satisfy the following conditions:*

(xi) *For every* $x, y \in K, d(Ax, By) \leq M_\omega(x, y)$*, where*
$M_\omega(x, y)$
$= max\{[\omega_1[d(Sx, Sy)], \omega_2[d(Ax, Sx)], \omega_3[d(By, Sy)], \omega_4[d(Ax, Sy)], \omega_5[d(Sx, By)]\}$,
$\omega_1: [0, +\infty) \to [0, +\infty), i = 1, 2, 3, 4, 5,$ *is a non-decreasing semicontinuous function from the right, such that* $\omega_i(r) < \frac{1}{2}r$ *for* $r > 0$*, and* $lim_{r\to\infty}[r - 2w_i(r)] = +\infty$*.*

(xii) $\delta K \subset S(K)$*,*

(xiii) $Sx \in \delta K \Rightarrow Ax, Bx \in K$*,*

(xiv) $A(K) \cap K \subset S(K), B(K) \cap K \subset S(K)$ *and*

(xv) $S(K)$*, is closed in* X*.*

Then there exists a coincidence point $z \in K$ *for* A, B *and* S*. Moreover, if each of the pairs* $\{S, A\}$ *and* $\{S, B\}$ *is coincidentally commuting, then* z *remains a unique common fixed point of* A, B *and* S*.*

We form another corollary by setting $x = y$ in Theorem 2.1.

Corollary 2.8. *Let (X, d) be a complete Takahashi convex metric space with convex structure W which is continuous in the third variable. Let K be a non-empty closed subset of X. Let δK be the boundary of K with $\delta K \neq \emptyset$. Let mappings $S, T, A, B : K \to X$. Assume that S, T, A and B satisfy the following conditions:*

(i) *For every $x, y \in K, d(Ax, Bx) \leq M_\omega(x)$, where*
$M_\omega(x)$
$= max\{[\omega_1[d(Sx, Tx)], \omega_2[d(Ax, Sx)], \omega_3[d(Bx, Tx)], \omega_4[d(Ax, Tx)], \omega_5[d(Sx, Bx)]\}$,
$\omega_1: [0, +\infty) \to [0, +\infty), i = 1, 2, 3, 4, 5$, *is a non-decreasing semicontinuous function from the right, such that $\omega_i(r) < \frac{1}{2}r$ for $r > 0$, and $lim_{r\to\infty}[r - 2w_i(r)] = +\infty$.*
(ii) $\delta K \subseteq T(K), \delta K \subset S(K)$,
(iii) $Sx \in \delta K \Rightarrow Ax \in K,\ Tx \in \delta K \Rightarrow Bx \in K$,
(iv) $A(K) \cap K \subset T(K), B(K) \cap K \subset S(K)$ *and*
(v) $S(K), T(K)$ *is closed in* X.
Then there exists a coincidence point $z \in K$ for A, B, S and T. Moreover, if each of the pairs $\{S, A\}$ and $\{T, B\}$ is coincidentally commuting, then z remains a unique common fixed point of A, B, S and T.

Here we give an example on the use of our result (Theorem 2.1).

Example 2.1: Let $X = [0, +\infty), K = [1, 3]$ and $d(x, y) = |x - y|$.
Let $\omega_i[r] = \frac{1}{3}r$ for $i \in \{1, 2, 3, 4, 5\}$. We note that $\omega_i[r] < \frac{1}{2}r$. Define $A, B, S, T: K \to X$ by

$$Sx = \begin{cases} 2x^4 - 1 & for\ x \in [1,2] \\ 7 & for\ x \in (2,3] \end{cases}, \qquad Ax = \begin{cases} x^2 & for\ x \in [1,2] \\ 1 & for\ x \in (2,3] \end{cases}$$

$$Sx = \begin{cases} 2x^6 - 1 & for\ x \in [1,2] \\ 7 & for\ x \in (2,3] \end{cases}, \qquad Ax = \begin{cases} x^3 & for\ x \in [1,2] \\ 2 & for\ x \in (2,3] \end{cases}.$$

We have $S(K) = [1, 31], T(K) = [0, 127]$ both of which are closed. We also have $\delta K = \{1,3\} \subseteq S(K), T(K)$.
We find out that $\{x \in K: Sx \in \delta K\} = \{1, 2^{1/4}\}$ and $A(\{1, 2^{1/4}\}) = \{1, \sqrt{2}\} \in K$. Similarly $\{x \in K: Tx \in \delta K\} = \{1, 2^{1/6}\}$ and $B(\{1, 2^{1/6}\}) = \{1, \sqrt{2}\} \in K$.
We note that $\{A, S\}$ and $\{B, T\}$ are both coincidentally commuting at $x = 1$, that is, $SA(1) = AS(1) = 1$ and $TB(1) = BT(1) = 1$. We also note that all four mappings are discontinuous at $x = 2$. Without loss of generality let $y \geq x$.

Consider $x, y \in (2, 3]$. Then we have

$$d(Ax, By) = d(2,2) = 0 \leq \frac{1}{3}d(Sx, Ty).$$

For $x, y \in [1,2]$, we have

$$d(Ax, By) = |x^2 - y^3|$$
$$= |x^2 - y^3| \times \frac{|x^2 + y^3|}{|x^2 + y^3|}$$
$$= \frac{|x^4 - y^6|}{|x^2 + y^3|}$$
$$\leq \frac{1}{2} \times \frac{1}{2} |2x^4 - 2x^6|$$
$$= \frac{1}{4} d(Sx, Ty)$$
$$\leq \frac{1}{3} d(Sx, Ty).$$

Finally, for $x \in [1, 2]$, $y \in (2, 3]$, we get

$$d(Ax, By) = |x^2 - 2|$$
$$= |x^2 - 2| \frac{|x^2 + 2|}{|x^2 + 2|}$$
$$= \frac{|x^4 - 4|}{|x^2 + 2|}$$
$$< \frac{1}{3} \times \frac{1}{2} |2x^4 - 8|$$
$$= \frac{1}{6} d(Sx, Ty)$$
$$\leq \frac{1}{3} d(Sx, Ty).$$

Thus in all cases, for every $x, y \in K$, we have

$$d(Ax, By) \leq \max\{ \omega_1[d(Sx, Ty)], \omega_2[d(Ax, Sx)], \omega_3[d(By, Ty)], \omega_4[d(Ax, Ty)], \omega_5[d(Sx, By)]\}.$$

Thus all the conditions of Theorem 2.1 are satisfied and 1 is the unique common fixed point of A, B, S and T.

References

[1] L. Gajić and V. Rakočević, Pair of non-self mappings and common fixed points, *Appl. Math. Comp.* **187** (2007), 999-1006.

[2] G. Jungck, Commuting maps and fixed points, *Amer. Math. Monthly* **83** (1976), 261-263.

[3] K. Das and K. Naik, Common fixed point theorems for commuting maps on metric space, *Proc. Amer. Math. Soc.* **77** (1979), 369-373.

[4] L. Ćirić, J. Ume, M. Khan and H. Pathak, On some non-self mappings, *Math. Nach.* **251** (2003), 28-33.

[5] L. Ćirić, Contractive type non-self mappings on metric spaces of hyperbolic type, *J. Math. Anal. Appl.* **317** (2006), 28-42.

[6] M. Imdad and S. Kumar, Rhoades-type fixed-point theorems for a pair of non-self mappings, *Comput. Math. Appl.* **46** (2003), 919-927.

[7] W. Takahashi, A convexity in metric space and non-expansive mappings, *I. Kodai Math. Sem. Pep.* **22** (1970), 142-149.