On a new theorem involving a product of some special functions

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Abstract
In this paper, we first establish an interesting theorem by making use of the well-known Orr theorem. Applications of this theorem to obtain certain integral and expansion formula have also been recorded in the elegant manner. Certain special cases of the main theorem are also given.

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1. Introduction

The function ${}_2F_1\,(a,\ b;\ c;\ z)$ is termed or named or known as Gauss function or simply hypergeometric function is represented as

$$ {}_2F_1\,(a, b; c; z) \quad = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n\, n\,!} z^n, \tag{1}$$

where, c is not zero or negative integer. The series in (4.2.3) converges for $|\,z\,| < 1$.

On the boundary $|\,z\,| = 1$ of the region of convergence, a sufficient condition for absolute convergence of the series is Re $(c\text{-}a\text{-}b) > 0$.

In our investigations uses that
If

$$(1-z)^{a+b+c}\ {}_2F_1(\,2a,\ 2b;\ 2c\ ;\ z) = \sum_{n=0}^{\infty} a_n\, z^n \qquad , \tag{2}$$

then

$$ {}_2F_1\left(a, b\ ; c\ +1/2; z\right) {}_2F_1\left(c-a, c-b\ ; c\ +1/2; z\right)$$
$$= \sum_{n=0}^{\infty} \frac{(c)_n a_n}{\left(c+1/2\right)_n} z^n \tag{3}$$

Srivastava ([12], Eqn. (1), p. 1] introduced the general class of polynomials

$$S_n^m[x] \ = \ \sum_{\ell=0}^{[n/m]} \frac{(\,-n)_{m\ell}}{\ell\,!} A_{n,\ell}\ , \qquad \ell = 0,1,2,\ldots \tag{4}$$

where m is an arbitrary positive integer and the coefficients $A_{n,\ell}(n, \ell \geq 0)$ are arbitrary constants, real or complex. On suitably specialization of the coefficients $A_{n,\ell}$, $S_n^m[X]$ yields a number of known polynomials as its special cases. These include, among others, the Hermite polynomials, the Jacobi polynomials, the Lagurre polynomials, the Bessel polynomials, the Gould-Hopper polynomials, the Brafman polynomials and several others ([13], *p*.158-161).
The $\bar{H}$ -function introduced by Inayat-Hussain [5, 6] is a generalization of the well known Fox *H*-function [4].
It is defined and represented in the following manner:

$$\overline{H}_{P,Q}^{M,N}\left[z \,\middle|\, \begin{matrix} (a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P} \\ (b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q} \end{matrix} \right]$$

$$=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \Phi(\xi)\, z^{\xi}\, d\xi\,, \tag{5}$$

where,

$$\Phi(\xi)=\frac{\prod_{j=1}^{M}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{N}\{\Gamma(1-a_j+\alpha_j\xi)\}^{A_j}}{\prod_{j=M+1}^{Q}\{\Gamma(1-b_j+\beta_j\xi)\}^{B_j}\prod_{j=N+1}^{P}\Gamma(a_j-\alpha_j\xi)}, \tag{6}$$

which contains fractional powers of some of the Gamma functions. Here and throughout the paper a_j $(j=1,\ldots,P)$ and $b_j(j = 1,\ldots,Q)$ are complex parameters, $\alpha_j \geq 0(j = 1, \ldots P), \beta_j \geq 0(j = 1, \ldots Q)$ not all zero simultaneously) and the exponent A_j $(j = 1,\ldots,N)$ and B_j $(j=M+1,\ldots,Q)$ can take on non-integer value. The contour in (5) is imaginary axis $Re(\xi) = 0$. It is suitably indented in order to avoid the singularities of the Gamma functions and those singularities on appropriate side. Again, for A_j $(j = 1,\ldots,N)$ not an integer, the poles of Gamma functions of the numerator in (6) are converted to the branch points. However, as long as there is no coincidence of poles from any $\Gamma(b_j-\beta_j\xi)(j = 1,\ldots,M)$ and $\Gamma(1-a_j+\alpha_j\xi)(j = 1,\ldots,N)$ pair, the branch cuts have been chosen so that the path of integration can be distorted in the usual manner. The following sufficient conditions for the absolute convergence of the defining integral for $\overline{H}$ -function given by equation (5) have been given by Buschman and Srivastava [1].

$$T = \sum_{j=1}^{M}\beta_j+\sum_{j=1}^{N}|A_j\alpha_j|-\sum_{j=M+1}^{Q}|B_j\beta_j|-\sum_{j=N+1}^{P}\alpha_j>0, \tag{7}$$

and

$$|\arg z| < \frac{1}{2}T\pi\,. \tag{8}$$

On suitably specialization of the parameters involved in $\overline{H}$-function a large number of special functions have obtained.

The series representation of $\overline{H}$ -function [7] is as follows:

$$\overline{H}_{P,Q}^{M,N}[z] = \overline{H}_{P,Q}^{M,N}\left[z \,\middle|\, \begin{matrix} (a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P} \\ (b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+!,Q} \end{matrix} \right]$$

$$=\sum_{g=1}^{M}\sum_{k=0}^{\infty}\frac{(-1)^k\,\overline{\varphi}(\xi_{g,k})z^{\xi_{g,k}}}{k!\;\beta_g}, \tag{9}$$

where

$$\overline{\varphi}(\xi_{g,k})=\frac{\prod_{\substack{j=1\\ j\neq g}}^{M}\Gamma(b_j-\beta_j\xi_{g,k})\prod_{j=1}^{N}\{\Gamma\Gamma 1-a_j+\alpha_j\xi_{g,k})\}^{A_j}}{\prod_{j=M+1}^{Q}\{\Gamma\Gamma 1-b_j+\beta_j\xi_{g,k})\}^{B_j}\prod_{j=N+1}^{P}\Gamma(a_j-\alpha_j\xi_{g,k})}, \tag{10}$$

and $\xi_{g,k} = \dfrac{b_g + k}{\beta_g}$. (11)

The *M*-series introduced by Sharma ([8], p.188, eqn. (3)) is defined as follows

$$ {}_P\overset{\alpha}{M}_Q\, [y] = {}_P\overset{\alpha}{M}_Q \left(\begin{matrix} a_1,\ldots,a_P,\alpha\, ; \\ b_1,\ldots,b_Q\, ; \end{matrix}\; y \right) $$

$$ = \sum_{k=0}^{\infty} \frac{(a_1)_k \ldots (a_P)_k}{(b_1)_k \ldots (b_Q)_k} \frac{y^k}{\Gamma(\alpha k + a)}, \tag{12} $$

here $\alpha \in C, \mathrm{Re}(\alpha) > 0$ *and* $(a_j)_k, (b_j)_k$ are the Pochhammer symbols. The series in (12) is defined when none of the parameters $b_j's$, j = 1,…,*Q*, is a negative integer or zero. If any numerator parameter a_j is a negative integer or zero, then the series terminates to a polynomial in *y*. From the ratio that it is evident that the series in (12) is convergent for all *z* if $P \le Q$, it is convergent for if $P = Q + 1$ and divergent, if $P > Q+1$. When $P = Q + 1$ and $|z| = 1$, then series can converge in some cases. Let $\tau = \sum_{j=1}^{P} a_j - \sum_{j=1}^{Q} b_j$. It can be shown that $P = Q + 1$ the series is absolutely convergent for $|z| = 1$, if $\mathrm{Re}(\tau) < 0$, conditionally convergent for $z = -1$, if $0 \le \mathrm{Re}(\tau) < 1$, and divergent for $|z| = 1$ if $1 \le \mathrm{Re}(\tau)$.
The following result will also be required in this sequel

$$ \int_0^1 {}_2F_1(\sigma, \delta; \gamma + 1/2; \mathrm{x})\, {}_2F_1(\gamma - \alpha, \gamma - \delta; \gamma + 1/2; \mathrm{x}) $$

$$ S_M^N [x^\rho]\; {}_P\overset{\alpha}{M}_Q \left[z, x^v \middle| \begin{matrix} (a_1),\ldots,(a_P) \\ (b_1),\ldots,(b_Q) \end{matrix} \right] $$

$$.\,\overline{H}_{p',q'}^{m',n'} \left[z_3 x^\omega \middle| \begin{matrix} (c'_j,\gamma'_j;C'_j)_{1,n'}\, ,\; (c'_j,\gamma'_j)_{n'+1,p'} \\ (d'_j,\delta'_j)_{1,m'}\, ,\; (d'_j,\delta'_j;D'_j)_{m'+1,q'} \end{matrix} \right] \overline{H}_{p,q}^{m,n} \left[z_2 x^u \middle| \begin{matrix} (c_j,\gamma_j;C_j)_{1,n}\, ,\; (c_j,\gamma_j)_{n+1,p} \\ (d_j,\delta_j)_{1,m}\, ,\; (d_j,\delta_j;D_j)_{m+1,q} \end{matrix} \right] dx $$

$$ = \sum_{s=0}^{[N/M]} \sum_{h=1}^{m'} \sum_{r,t,k=0}^{\infty} \frac{a_t(\gamma)_t\, A'_{N,s}(-N)_{Ms}}{(\gamma + 1/2)_t\, s!\, t\,!} $$

$$.\,\frac{\prod_{\substack{j=1 \\ j \ne h}}^{m'} \Gamma(d'_j - \delta'_j \xi_{h,r}) \prod_{j=1}^{n'} \{\Gamma(1 - c'_j + \gamma'_j \xi_{h,r})\}^{C'_j} (-1)^r z_3^{\xi_{h,r}} (a_1)_k \ldots (a_P)_k z_1^k}{\prod_{j=m'=1}^{q'} \{(\Gamma(1 - d'_j + \delta'_j \xi_{h,r})\}^{D'_j} \prod_{j=n'+1}^{p'} \Gamma(c'_j - \gamma' \xi_{h,r})\, r\,!\, \delta_h (b_1)_k \ldots (b_Q)_k \Gamma(\alpha k + 1)} $$

$$.\,\overline{H}_{p+1,q+1}^{m,\,n+1} \left[z_2 \middle| \begin{matrix} (-t-\rho s - vk - \omega\xi_{h,r}, u, 1), (c_j,\gamma_j;C_j)_{1,n}, (c_j,\gamma_j)_{n+1,p} \\ (d_j,\delta_j)_{1,m}, (d_j,\delta_j;D_j)_{m+1,q}, (-1-t-\rho s - vk - \omega\xi_{h,r}, u; 1) \end{matrix} \right], \tag{13} $$

provided that $\rho > 0, u > 0, v > 0, -\frac{1}{2} < \gamma - \sigma - \delta) < \frac{1}{2}, \mathrm{Re}\left[1 + u\frac{d_j}{\delta_j} + \omega\frac{d'_{j'}}{\delta'_{j'}}\right] > 0,$

$j = 1,\ldots, m$; $|\arg z_2|<\frac{1}{2}\pi T$, $T > 0$, $|\arg z_3|<\frac{1}{2}\pi T'$, $T' > 0$, T, T' are given in (7), M is an arbitrary positive integer and the coefficients $A'_{N,s}(N,s\geq 0)$ are arbitrary, real or complex, $P \leq Q$, $|z_1|<1$ and $\xi_{h,r}=(d'h+r)/\delta'_h$.

2. The Main Theorem

Theorem 1. With T defined by (7), $\rho > 0$, $u > 0$, $v > 0$, $w > 0$, $-\frac{1}{2}<(\gamma-\sigma-\delta)<\frac{1}{2}$, $T > 0$,

(i) $\quad \mathrm{Re}\left(1+u\,\frac{d_j}{\delta_j}+\omega\frac{d'_{j'}}{\delta'_{j'}}\right)>0,\ j = 1,\ldots,m\ ;\ j'=1,\ldots,m',$

(ii) $\quad |\arg z_1| < \frac{1}{2}T\pi$,

(iii) $\quad P \leq Q$, $|z_1|<1$,

(iv) M is an arbitrary positive integer and the coefficients $A'_{N,s}(N,s \geq 0)$ are arbitrary, real or complex,

and if

$$(1-x)^{\sigma+\delta-\gamma}\ {}_2F_1(2\sigma,2\delta;2\gamma;x) = \sum_{t=0}^{\infty} a_t\, x^t \tag{14}$$

then

$$\int_0^1 {}_2F_1(\sigma,\delta;\gamma+1/2;x)\ {}_2F_1(\gamma-\sigma,\gamma-\delta;\gamma+1/2;x) S_N^M[x^\rho]\ {}_P\overset{\alpha}{M}{}_Q\left[{}^{a_1,\ldots,a_P,\alpha}_{b_1,\ldots,b_Q};z\,x^v\right]$$

$$.\overline{H}^{m',n'}_{p',q'}\left[z_3x^{\omega}\left|\begin{matrix}(c'_j,\gamma'_j;C'_j)_{1,n'},\ (c'_j,\gamma'_j)_{n'+1,p'}\\(d'_j,\delta'_j)_{1,m'},\ (d'_j,\delta'_j;D'_j)_{m'+1,q'}\end{matrix}\right.\right]\overline{H}^{m,n}_{p,q}\left[z_2x^{u}\left|\begin{matrix}(c_j,\gamma_j;C_j)_{1,n},\ (c_j,\gamma_j)_{n+1,p}\\(d_j,\delta_j)_{1,m},\ (d_j,\delta_j;D_j)_{m+1,q}\end{matrix}\right.\right]dx$$

$$=\sum_{s=0}^{[N/M]}\sum_{h=1}^{m'}\sum_{r,t,k=0}^{\infty}\frac{a_t(\gamma)_t\,A'_{N,s}(-N)_{Ms}}{(\gamma+1/2)_t\ s!\quad t!}$$

$$.\frac{\prod_{\substack{j=1\\j\neq h}}^{m'}\Gamma(d'_j-\delta'_j\xi_{h,r})\prod_{j=1}^{n'}\{\Gamma(1-c'_j+\gamma'_j\xi_{h,r})\}^{C'_j}(-1)^r(z_2)^{\xi_{h,r}}(a_1)_k\ldots(a_P)_k(z_1)^k}{\prod_{j=m'+1}^{q'}\{\Gamma(1-d'_j+\delta'_j\xi_{h,r})\}^{D'_j}\prod_{j=n'+1}^{p'}\Gamma(c'_j-\gamma'\xi_{h,r})r!\ \delta_h(b_1)_k\ldots(b_Q)_k\Gamma(\alpha z_1+1)}$$

$$.\ \overline{H}^{m,n+1}_{p+1,q+1}\left[z_2\left|\begin{matrix}(-t-\rho s-vk-\omega\xi_{h,r},u;1),\ (c_j,\gamma_j;C_j)_{1,n},(c_j,\gamma_j)_{n+1,p}\\(d_j,\delta_j)_{1,m},(d_j,\delta_j,D_j)_{m+1,q},(-1-t-\rho s-\omega\xi_{h,r},u;1)\end{matrix}\right.\right]. \tag{15}$$

Proof. We have ([10], *p*.75)

$$_2F_1(\sigma,\delta;\gamma+1/2;x)\,{}_2F_1(\gamma-\sigma,\gamma-\delta;\gamma+1/2;x)$$

$$=\sum_{t=0}^{\infty}\frac{(\gamma)_t}{(\gamma+1/2)_t}a_t\,x^t, \tag{16}$$

where a_t is given by (14).

Multiplying both sides of (16) by

$$S_N^M[x^\rho]\ {}_P\overset{\alpha}{M}_Q\left[\begin{matrix}a_1,\dots,a_{P,\alpha}\\ b_1,\dots,b_Q\end{matrix};z_1x^v\right]\overline{H}_{p',q'}^{m',n'}\left[z_3x^\omega\left|\begin{matrix}(c_j',\gamma_j';C_j')_{1,n'},(c_j',\gamma_j')_{n'+1,p'}\\(d_j',\delta_j')_{1,m'},(d_j',\delta_j';D_j')_{m'+1,q'}\end{matrix}\right.\right]$$

$$.\overline{H}_{p,q}^{m,n}\left[z_2x^u\left|\begin{matrix}(c_j,\gamma_j;C_j)_{1,n}\ ,\ (c_j,\gamma_j)_{n+1,p}\\(d_j,\delta_j)_{1,m}\ ,\ (d_j,\delta_j;D_j)_{m+1,q}\end{matrix}\right.\right],$$

expressing the functions by their respective series form with the help of (4), (12) and (10) respectively with making use of Mellin-Barnes contour integral definition of the $\overline{H}$ -function as given in (5), and then interchanging the order of integrations and summations which is permissible under the conditions stated with (15), and evaluating the remaining simple integral with the help of (13), with a little simplification and adjustment of parameters, we arrive at the required result.

3. Application

(A) Taking σ = γ in the main theorem, the values of a_t in (14) comes out to be equal to $(\delta)_t$, the result in (15) reduces to the following theorem

Theorem 1 (A)

(a) With T defined by (7), ρ > 0, u > 0, v > 0, T > 0, $,\omega>0,-\frac{1}{2}<(\gamma-\sigma-\delta)<\frac{1}{2}$,

(i) $\text{Re}\left(1+u\frac{d_j}{\delta_j}+\omega\frac{d_{j'}'}{\delta_{j'}'}\right)>0,\ j=1,\dots,m\ ;\ j'=1,\dots,m'$,

(ii) $|\arg z_1|<\frac{1}{2}T\pi$,

(iii) $P\le Q,\ |z_1|<1$,

(iv) M is an arbitrary positive integer and the coefficients $A_{N,s}'(N,s\ \ge 0)$ are arbitrary, real or complex and $\xi_{h,r}=\frac{d_h'+r}{\delta_h'}$, then

$$\int_0^1 {}_2F_1(\sigma,\delta;\gamma+1/2;x)\ S_N^M[x^\rho]\ {}_P\overset{\alpha}{M}_Q[z_1x^v]$$

$$.\overline{H}_{p',q'}^{m',n'}\left[z_3x^\omega\left|\begin{matrix}(c_j',\gamma_j';C_j')_{1,n'}\ ,\ (c_j',\gamma_j')_{n'+1,p'}\\(d_j',\delta_j')_{1,m'},\ (d_j',\delta_j';D_j')_{m'+1,q'}\end{matrix}\right.\right]\overline{H}_{p,q}^{m,n}\left[z_2x^u\left|\begin{matrix}(c_j,\gamma_j;C_j)_{1,n}\ ,\ (c_j,\gamma_j)_{n+1,p}\\(d_j,\delta_j)_{1,m}\ ,\ (d_j,\delta_j;D_j)_{m+1,q}\end{matrix}\right.\right]dx$$

$$=\sum_{s=0}^{[N/M]}\sum_{h=1}^{m'}\sum_{r,t,k=0}^{\infty}\frac{A_{N,s}'(-N)_{Ms}\,\Phi(k)\,(\sigma)_t(\delta)_t}{s!\ t!\,(\sigma+1/2)_t}$$

$$.\frac{\prod_{\substack{j=1\\j\neq h}}^{m'} \Gamma(d'_j-\delta'_j\xi_{h,r})\prod_{j=1}^{n'}\{\Gamma(1-c'_j+\gamma'_j\xi_{h,r})\}^{C'_j}(-1)^r(z_3)^{\xi_{h,r}}(a_1)_k...(a_P)_k(z_1)^k}{\prod_{j=m'+1}^{q'}\{\Gamma(1-d'_j+\delta'_j\xi_{h,r})\}^{D'_j}\prod_{j=n'+1}^{p'}\Gamma(c'_j-\gamma'\xi_{h,r})r!\ \delta_h(b_1)_k...(b_Q)_k\Gamma(\alpha k_1+1)}$$

$$.\bar{H}^{m,n+1}_{p+1,q+1}\left[z_2\left|\begin{matrix}(-t-\rho s-\omega\xi_{h,r}-vk,u;1),(c_j,\gamma_j;C_j)_{1,n},(c_j,\gamma_j)_{n+1,p}\\(d_j,\delta_j)_{1,m},(d_j,\delta_j,D_j)_{m+1,q},(-1-t-\rho s-\omega\xi_{h,k}-vk,u;1)\end{matrix}\right.\right]. \quad (17)$$

(B) Theorem 1 (B)

(b) With T defined by (7), $\rho > 0, u > 0, v > 0, \quad T > 0, \ \omega > 0$ *and* $\xi_{h,r} = d'_h + r/\delta'_h$,

(i) $\mathrm{Re}\left(1+u\dfrac{d_j}{\delta_j}+\omega\dfrac{d'_{j'}}{\delta'_{j'}}\right) > 0, j = 1,...,m\ ; j' = 1,...,m',$

(ii) $|\arg z_1| < \dfrac{1}{2}T\pi,$

(iii) $P \le Q, |z_1|<1$,

(iv) M is an arbitrary positive integer and the coefficients $A'_{N,s}(N,s \ge 0)$ are arbitrary, real or complex and

$$\int_0^1 {}_2F_1(-\mu;x)\ S_N^M[x^\rho]\ {}_PM^{\alpha}_Q\left[\begin{matrix}a_1,...,a_P,\alpha;\\b_1,...,b_Q;\end{matrix}\ z_1x^v\right]$$

$$.\bar{H}^{m',n'}_{p',q'}\left[z_3x^\omega\left|\begin{matrix}(c'_j,\gamma'_j;C'_j)_{1,n'},\ (c'_j,\gamma'_j)_{n'+1,p'}\\(d'_j,\delta'_j)_{1,m'},\ (d'_j,\delta'_j;D'_j)_{m'+1,q'}\end{matrix}\right.\right]\bar{H}^{m,n}_{p,q}\left[z_2x^u\left|\begin{matrix}(c_j,\gamma_j;C_j)_{1,n},\ (c_j,\gamma_j)_{n+1,p}\\(d_j,\delta_j)_{1,m},\ (d_j,\delta_j;D_j)_{m+1,q}\end{matrix}\right.\right]dx$$

$$=\sum_{s=0}^{[N/M]}\sum_{h=1}^{\mu'}\sum_{r,t,k=0}^{\infty}\frac{A'_{N,s}(-N)_{Ms}(-\mu)_t}{s!\ t!}$$

$$.\frac{\prod_{\substack{j=1\\j\neq h}}^{m'} \Gamma(d'_j-\delta'_j\xi_{h,r})\prod_{j=1}^{n'}\{\Gamma(1-c'_j+\gamma'_j\xi_{h,r})\}^{C'_j}(-1)^r(z_2)^{\xi_{h,r}}(a_1)_k...(a_P)_k(z_1)^k}{\prod_{j=m'+1}^{q'}\{\Gamma(1-d'_j+\delta'_j\xi_{h,r})\}^{D'_j}\prod_{j=n'+1}^{p'}\Gamma(c'_j-\gamma'\xi_{h,r})r!\ \delta_h(b_1)_k...(b_Q)_k\Gamma(\alpha k_1+1)}$$

$$.\bar{H}^{m,n+1}_{p+1,q+1}\left[z_2\left|\begin{matrix}(-t-\rho s-\omega\xi_{h,r}-vk,u;1),(c_j,\gamma_j;C_j)_{1,n},(c_j,\gamma_j)_{n+1,p}\\(d_j,\delta_j)_{1,m},(d_j,\delta_j,D_j)_{m+1,q},(-1-t-\rho s-\omega\xi_{h,r}-vk,u;1)\end{matrix}\right.\right]. \quad (18)$$

4. Expansion Formula

On evaluating on the left hand side of (18) with the help of a known result [2], we have the following interesting expansion formula

Theorem 1(C)

$$\sum_{s=0}^{[N/M]}\sum_{h=1}^{m'}\sum_{t=0}^{\mu}\sum_{r,k=0}^{\infty}\frac{A'_{N,s}(-N)_{Ms}(-\mu)_t}{s!\ t!}$$

$$\cdot\frac{\prod_{\substack{j=1\\j\neq h}}^{m'}\Gamma(d_j'-\delta_j'\xi_{h,r})\prod_{j=1}^{n'}\{\Gamma(1-c_j'+\gamma_j'\xi_{h,r})\}^{C_j'}(-1)^r(z_2)^{\xi_{h,r}}(a_1)_k\ldots(a_P)_k(z_1)^k}{\prod_{j=m'+1}^{q'}\{\Gamma(1-d_j'+\delta_j'\xi_{h,r})\}^{D_j'}\prod_{j=n'+1}^{p'}\Gamma(c_j'-\gamma'\xi_{h,r})r!\ \delta_h(b_1)_k\ldots(b_Q)_k\Gamma(\alpha k_1+1)}$$

$$.\overline{H}^{m,n+1}_{p+1,q+1}\left[z_2\left|\begin{matrix}(-t-\rho s-\omega\xi_{h,r}-vk,u;1),(c_j,\gamma_j;C_j)_{1,n},(c_j,\gamma_j)_{n+1,p}\\(d_j,\delta_j)_{1,m},(d_j,\delta_j,D_j)_{m+1,q},(-1-t-\rho s-\omega\xi_{h,k}-vk,u;1)\end{matrix}\right.\right]$$

$$=\sum_{s=0}^{[N/M]}\sum_{h=1}^{m'}\sum_{r,t,k=0}^{\infty}\frac{A_{N,s}'(-N)_{Ms}\,\Gamma(\mu+\rho s+\omega\xi_{h,r}+vk+1)}{s!}$$

$$\cdot\frac{\prod_{\substack{j=1\\j\neq h}}^{m'}\Gamma(d_j'-\delta_j'\xi_{h,r})\prod_{j=1}^{n'}\{\Gamma(1-c_j'+\gamma_j'\xi_{h,r})\}^{C_j'}(-1)^r(z_2)^{\xi_{h,r}}(a_1)_k\ldots(a_P)_k(z_1)^k}{\prod_{j=m'+1}^{q'}\{\Gamma(1-d_j'+\delta_j'\xi_{h,r})\}^{D_j'}\prod_{j=n'+1}^{p'}\Gamma(c_j'-\gamma'\xi_{h,r})r!\ \delta_h(b_1)_k\ldots(b_Q)_k\Gamma(\alpha k_1+1)}$$

$$.\overline{H}^{m,n+1}_{p+1,q+1}\left[z_2\left|\begin{matrix}(0,u;1),(c_j,\gamma_j;C_j)_{1,n},(c_j,\gamma_j)_{n+1,p}(\ \ldots\ldots..)\\(d_j,\delta_j)_{1,m},(d_j,\delta_j,D_j)_{m+1,q},(-1-\mu-\rho s-\omega\xi_{h,r}-vk,u;1)\end{matrix}\right.\right], \tag{19}$$

provided that both the sides exists.

5. Special Cases

(i) Letting $m = 1$, $n = 2 = p = q$, and replacing z by $-z$ in (14), and using

$$g(\theta(\theta,\sigma',\tau,z) = \frac{R_{\rho'-1}\Gamma(\varphi+1)\Gamma((1+\tau)/2)}{(-1)^{\varphi}2^{2-\varphi}\,\Gamma(\theta)\Gamma(\theta-\tau/2)\sqrt{\pi}}$$

$$\overline{H}^{1,3}_{3,3}\left[-z\left|\begin{matrix}(1-\theta,1;1),\left(1-\theta+\frac{\tau}{2},1;1\right),(1-\sigma',1;\varphi+1)\\(0,1),\left(-\frac{\tau}{2},1;1\right),(-\sigma',1;\varphi+1)\end{matrix}\right.\right],$$

where

$$R_{\rho'-1}=\frac{2^{1-\rho'}\pi^{-\rho'/2}}{\Gamma(\sigma'/2)}, \tag{20}$$

the function in (20) is connected with a certain class of Feynman integrals, we get the following result

$$\int_0^1 {}_2F_1(\sigma,\delta;\gamma+1/2;x)\,{}_2F_1(\gamma-\sigma,\gamma-\delta;\gamma+1/2;x)\quad S_N^M[x^\rho]$$

$$.\ {}_P\overset{\alpha}{M}{}_Q\left[\begin{matrix}(a_1,\ldots,a_P,\alpha;\\b_1,\ldots b_Q;\end{matrix}\ x^v\right]g(\theta,\sigma',\tau,\varphi;z_3x^u)$$

$$.\overline{H}^{m',n'}_{p',q'}\left[z_3x^{\omega}\left|\begin{matrix}(c_j',\gamma_j';C_j')_{1,n'},(c_j',\gamma_j')_{n'+1,p'}\\(d_j',\delta_j')_{1,m'},(d_j',\delta_j';D_j')_{m'+1,q'}\end{matrix}\right.\right]dx$$

$$=\sum_{s=0}^{[N/M]}\sum_{h=1}^{m'}\sum_{r,k,t=0}^{\infty}\frac{A_{N,s}'(-N)_{Ms}}{s!\qquad t!}$$

$$\cdot \frac{\prod_{\substack{j=1\\ j\neq h}}^{m'} \Gamma(d_j' - \delta_j' \xi_{h,r}) \prod_{j=1}^{n'} \{\Gamma(1-c_j' + \gamma_j' \xi_{h,r})\}^{C_j'} (-1)^r (z_3)^{\xi_{h,r}} (a_1)_k \ldots (a_P)_k (z_1)^k}{\prod_{j=m'+1}^{q'} \{\Gamma(1-d_j' + \delta_j' \xi_{h,r})\}^{D_j'} \prod_{j=n'+1}^{p'} \Gamma(c_j' - \gamma' \xi_{h,r}) r!\ \delta_h (b_1)_k \ldots (b_Q)_k \Gamma(\alpha k_1 + 1)}$$

$$\cdot \frac{R_{\rho'-1} \Gamma(\varphi+1) \Gamma\left((1+\tau)/2\right)(\gamma)_t\ a_t}{(-1)^{\varphi} 2^{2+\varphi}\ \Gamma(\theta) \Gamma\left(\theta - \tau/2\right) \sqrt{\pi} \left(\gamma + 1/2\right)_t}$$

$$\cdot \overline{H}_{4,4}^{1,4}\left[-z_2 \left| \begin{matrix} (-t-\rho s - \omega \xi_{h,r} - vk, u; 1),\ (1-\theta, 1; 1), \left(1-\theta+\frac{\tau}{2}, 1; 1\right), (1-\sigma', 1; \varphi+1) \\ (0,1), \left(-\frac{\tau}{2}, 1; 1\right), (-\sigma', 1; \varphi+1), (-1-t-\rho s - \omega \xi_{h,r} - vk,\ u; 1) \end{matrix} \right. \right], \qquad (21)$$

valid under the conditions surrounding (14).

(ii) Setting $m = 1$, $n = p$, $q = q+1$ and replacing z_2 by $-z_2$ in the equation (14), and making use of the following transformation

$${}_p\overline{\psi}_q\left[z_2 \left| \begin{matrix} (c_j, \gamma_j; C_j)_{1,p} \\ (d_j, \delta_j; D_j)_{1,q} \end{matrix} \right. \right] = \overline{H}_{p,q+1}^{1,p}\left[-z_2 \left| \begin{matrix} (1-c_j, \gamma_j; C_j)_{1,p} \\ (0,1), (1-d_j, \delta_j; D_j)_{1,q} \end{matrix} \right. \right], \qquad (22)$$

we have

$$\int_0^1 {}_2F_1\left(\sigma, \delta; \gamma+1/2; x\right) {}_2F_1\left(\gamma-\sigma, \gamma-\delta; \gamma+1/2; x\right)\ S_N^M [x^{\rho}]$$

$$\cdot {}_P\overset{\alpha}{M}{}_Q\left[z_1 x^v \left| \begin{matrix} a_1, \ldots, a_P, \alpha\alpha \\ b_1, \ldots, b_Q; \end{matrix} \right. \right] {}_p\overline{\psi}_q\left[z_2 x^u \left| \begin{matrix} (c_j, \gamma_j; C_j)_{1,p} \\ (d_j, \delta_j; D_j)_{1,q} \end{matrix} \right. \right]$$

$$\cdot\ \overline{H}_{p',q'}^{m',n'}\left[z_3 x^{\omega} \left| \begin{matrix} (c_j', \gamma_j'; C_j')_{1,n'}, (c_j', \gamma_j')_{n'+1,p'} \\ (d_j', \delta_j')_{1,m'}, (d_j', \delta_j'; D_j')_{m'+1,q'} \end{matrix} \right. \right] dx$$

$$= \sum_{s=0}^{[N/M]} \sum_{h=1}^{m'} \sum_{r,k,t=0}^{\infty} \frac{A_{N,s}'(-N)_{Ms}\, \psi(h,r) \Phi(k) (\gamma)_t}{s!\quad t!\ \left(\gamma+1/2\right)_t}$$

$$= \sum_{s=0}^{[N/M]} \sum_{h=1}^{m'} \sum_{r,k,t=0}^{\infty} \frac{A_{N,s}'(-N)_{Ms} (\gamma)_t}{s!\quad t!\left(\gamma+1/2\right)_t}$$

$$\cdot \frac{\prod_{\substack{j=1\\ j\neq h}}^{m'} \Gamma(d_j' - \delta_j' \xi_{h,r}) \prod_{j=1}^{n'} \{\Gamma(1-c_j' + \gamma_j' \xi_{h,r})\}^{C_j'} (-1)^r (z_3)^{\xi_{h,r}} (a_1)_k \ldots (a_P)_k (z_1)^k}{\prod_{j=m'+1}^{q'} \{\Gamma(1-d_j' + \delta_j' \xi_{h,r})\}^{D_j'} \prod_{j=n'+1}^{p'} \Gamma(c_j' - \gamma' \xi_{h,r}) r!\ \delta_h (b_1)_k \ldots (b_Q)_k \Gamma(\alpha k_1 + 1)}$$

$$\cdot \overline{H}_{p+1,q+2}^{1,p+1}\left[-z_2 \left| \begin{matrix} (-t-\rho s - \omega \xi_{h,r} - vk, u; 1),\ (1-c_j, \gamma_j; C_j)_{1,p} \\ (0,1), (1-d_j, \delta_j; D_j)_{1,q}, (-1-t-\rho s - \omega \xi_{h,r} - vk,\ u; 1) \end{matrix} \right. \right], \qquad (23)$$

valid under the conditions as obtainable from (14).

(iii) Taking $m = 1$, $n = p = q+1$, $\gamma_j = 1 = \delta_1$ and replacing z_2 by $-z_2$ in the equation (14) and making use of the following relation

$$
{}_p\overline{F}_q\left[z_2\left|\begin{matrix}(c_j,1;C_j)_{1,p}\\(d_j,1;D_j)_{1,q}\end{matrix}\right.\right]
=\frac{\prod_{j=1}^{q}\{\Gamma(d_j)\}^{D_j}}{\prod_{j=1}^{p}\{\Gamma(c_j)\}^{C_j}}\ \overline{H}^{1,p}_{p,q+1}\left[-z_2\left|\begin{matrix}(1-c_j,1;C_j)_{1,p}\\(0,1),(1-d_j,1;D_j)_{1,q}\end{matrix}\right.\right], \tag{24}
$$

we have

$$
\int_0^1 {}_2F_1(\sigma,\delta;\gamma+1/2;x)\,{}_2F_1(\gamma-\sigma,\gamma-\delta;\gamma+1/2;x)\ S_N^M[x^\rho]
$$

$$
.\,{}_P\overset{\alpha}{M}_Q\left[z_1x^v\left|\begin{matrix}a_1,\dots,a_P,\alpha\\b_1,\dots,b_Q\end{matrix}\right.\right]{}_p\overline{F}_q\left[z_2x^u\left|\begin{matrix}(c_j,1;C_j)_{1,p}\\(d_j,1;D_j)_{1,q}\end{matrix}\right.\right]
$$

$$
.\ \overline{H}^{m',n'}_{p',q'}\left[z_3x^\omega\left|\begin{matrix}(c'_j,\gamma'_j;C'_j)_{1,n'},(c'_j,\gamma'_j)_{n'+1,p'}\\(d'_j,\delta'_j)_{1,m'},(d'_j,\delta'_j;D'_j)_{m'+1,q'}\end{matrix}\right.\right]dx
$$

$$
=\sum_{s=0}^{[N/M]}\sum_{h=1}^{m'}\sum_{r,k,t=0}^{\infty}\frac{A'_{N,s}(-N)_{Ms}(\gamma)_t\prod_{j=1}^{q}\{\Gamma(d_j)\}^{D_j}}{s!\,t!\,(\gamma+1/2)_t\prod_{j=1}^{p}\{\Gamma(c_j)\}^{C_j}}
$$

$$
.\frac{\prod_{j=1,\,j\neq h}^{m'}\Gamma(d'_j-\delta'_j\xi_{h,r})\prod_{j=1}^{n'}\{\Gamma(1-c'_j+\gamma'_j\xi_{h,r})\}^{C'_j}(-1)^r(z_3)^{\xi_{h,r}}(a_1)_k\dots(a_P)_k(z_1)^k}{\prod_{j=m'+1}^{q'}\{\Gamma(1-d'_j+\delta'_j\xi_{h,r})\}^{D'_j}\prod_{j=n'+1}^{p'}\Gamma(c'_j-\gamma'\xi_{h,r})\,r!\ \delta_h(b_1)_k\dots(b_Q)_k\Gamma(\alpha k_1+1)}
$$

$$
.\,\overline{H}^{1,p+1}_{p+1,q+2}\left[-z_2\left|\begin{matrix}(-t-\rho s-\omega\xi_{h,r}-vk,u;1),(1-c_j,\gamma_j;C_j)_{1,p}\\(0,1),(1-d_j,\delta_j;D_j)_{1,q},(-1-t-\rho s-\omega\xi_{h,r}-vk,u;1)\end{matrix}\right.\right], \tag{25}
$$

valid under the conditions as obtainable from (14).

Known Results

(i) Setting $v\to 0$ in (15), we find a known result due to Chaurasia and Soni ([3], p.1715, eqn. (2.2)).

(ii) Taking $v\to 0$, $N\to 0$, $w\to 0$, with $C_j = 1 = D_j$ in (15), we arrive at the known result derived by Chaurasia ([2], p.185, eqn. (6.5.2)).

(iii) Putting $\gamma_j=\delta_j=1=C_j=D_j$ for all j, $N\to 0$, $v\to 0$, the results (17) through (19) reduce to the known results obtained by Srivastava ([11], p.236, eqn. (1.2)).

(iv) Letting $w\to 0, v\to 0, \gamma_j=\delta_j=1=C_j=D_j$, for all j, the results (17) through (25) reduce to the known results due to Sharma [9].

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