SHEFFER AND BRENKE POLYNOMIALS ASSOCIATED WITH GENERALIZED BELL NUMBERS

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(Received : November 15, 2017)

Abstract
In a recent paper, the first two authors have studied the integer number sequences generated by the higher order Bell polynomials. In this article we introduce new sets of Sheffer polynomial sequences based on the higher order Bell numbers, and we study their basic properties. Furthermore, we introduce new sequences of associated Sheffer and Brenke polynomial sequences.

AMS 2010 Mathematics Subject Classification: 05A10, 11P81, 33C99.
Keywords and phrases: Higher order Bell polynomials and numbers, Sheffer and associated Sheffer polynomials, Brenke polynomials, combinatorial analysis.

1. Introduction

The Bell polynomials [1], first appear as a mathematical tool for representing the nth derivative of a composite function.

Being related to partitions, the Bell polynomials are often used in Combinatorial Analysis [28]. They have been also applied in many different frameworks, such as: the Blissard problem (see [28, p. 46]); the representation of Lucas polynomials of the first and second kind [9, 14]; the construction of representation formulas for the Newton sum rules for the zeros of polynomials [19, 20]; the recurrence relations for a class of Freud-type polynomials [4], the representation formulas for the symmetric functions of a countable set of numbers, generalizing the classical algebraic Newton-Girard formulas. Consequently, as it was recognized [10], it is possible to find reduction formulas for the *orthogonal invariants* of a strictly Positive Compact Operator (shortly PCO), deriving in a simple way the so called Robert formulas [29].

Some generalized forms of Bell polynomials already appeared in literature (see e.g. [27, 17]).
Generalizations of the Bell polynomials suitable for the differentiation of multivariable composite functions have been also defined (see [26, 5]).
Recently the Adomian polynomials [21], have been used in order to derive relations between the Bell polynomials [22].

Higher order Bell polynomials (i.e. Bell polynomials for representing the derivatives of several nested functions) were already defined by the first two authors [23], and the introduction of higher order multivariable Bell polynomials was also achieved [5], and this subject has been considered in a forthcoming article by Chinese mathematicians [16], who are also able to show an application to white noise distribution theory.

In a recent article [25] the first two authors have studied the integer number sequences generated by the higher order Bell polynomials. In this article we introduce new sets of Sheffer polynomial sequences based on the higher order Bell numbers, and we study, in some particular case, their basic properties. See also [36].
Furthermore, we introduce new sequences of associated Sheffer and Brenke polynomial sequence, associated with the same number sequences.

2. Recalling the Bell polynomials

We recall that the Bell polynomials are a classical mathematical tool for representing the *n*th derivative of a composite function. In fact by considering the composite function $\Phi(t) := f(g(t))$ of functions $x = g(t)$ and $y = f(x)$ defined in suitable intervals of the real axis and n times differentiable with respect to the relevant independent variables and by using the following notation

$$\Phi_h := D_t^h \Phi(t), f_h := D_x^h f(x)\Big|_{x = g(t)}, g_h := D_t^h g(t) \tag{2.1}$$

and

$$([f, g]_n) := (f_1, g_1; f_2, g_2; \dots; f_n, g_n) \tag{2.2}$$

They are defined as follows:

$$Y_n([f, g]_n) := \Phi_n \tag{2.3}$$

Inductively, by using the notation
$[g]_n := (g_1, g_2, \dots, g_n)$,

we can write

$$Y_n([f, g]_n) = \sum_{k=1}^n A_{n,k}([g]_n) f_k, \tag{2.4}$$

where, the coefficient $A_{n,k}$, for any $k = 1, \dots, n$, is a polynomial in $g_1, g_2, \dots, g_n$, homogeneous of degree k and *isobaric* of weight n (i.e., it is a linear combination of monomials ${g_1}^{k_1} {g_2}^{k_2} \dots {g_n}^{k_n}$ whose weight is constantly given by $k_1 + 2k_2 + \dots + nk_n = n$).

Proposition 2.1. *The Bell polynomials satisfy the recurrence relation*

$$\begin{cases} Y_0([f, g]_0) := f_1 \\ Y_{n+1}([f, g]_{n+1}) = \sum_{k=0}^n \binom{n}{k} Y_{n-k}([f_1, g]_{n-k}) g_{k+1} \end{cases} \tag{2.5}$$

where, $([f_1, g]_{n-k}) := (f_2, g_1; f_3, g_2; \dots; f_{n-k+1}, g_{n-k})$.

An explicit expression for the Bell polynomials is given by the Fa`a di Bruno formula [15, 30, 32], however, as this formula makes use of partitions, it is not useful for computing higher order Bell polynomials, whereas this can be done in a easy way by means of the following recursion formula for the coefficients $A_{n,k}$ in equation (2.4) which are known as partial Bell polynomials.

Theorem 2.2. *We have, for all integer* n,
$A_{n+1,1} = g_{n+1}, A_{n+1,n+1} = g_1^{n+1}$. (2.6)
Furthermore, for all $k = 1, 2, \ldots, n-1$, *the* $A_{n,k}$ *coefficients can be computed by the recurrence relation*
$A_{n+1,k+1}([g]_{n+1}) = \sum_{h=0}^{n-k} \binom{n}{h} A_{n-h,k}([g]_{n-h}) g_{h+1}$. (2.7)

Definition 2.1. *The complete Bell polynomials, considered in literature, are defined by*
$B_n([g]_n) = Y_n(1, g_1; 1, g_2; \ldots; 1, g_n) = \sum_{k=1}^{n} A_{n,k}([g]_n)$, (2.8)

and the Bell numbers by

$b_n = Y_n(1{,}1; 1{,}1; \ldots; 1{,}1) = \sum_{k=1}^{n} A_{n,k}(1{,}1, \ldots, 1)$. (2.9)

3. Bell polynomials of order r

In [5] the following extension of the classical Bell polynomials was achieved.

Consider $\Phi(t) := f(\varphi^{(1)}(\varphi^{(2)}(\cdots(\varphi^{(r)}(t)))))$, i.e., the composition of functions $x^{(r)} = \varphi^{(r)}(t), \ldots, x^{(2)} = \varphi^{(2)}(x^{(3)})$, $x^{(1)} = \varphi^{(1)}(x^{(2)})$ $y = f(x^{(1)})$ defined in suitable intervals of the real axis, and suppose that the functions $\varphi^{(r)}, \ldots, \varphi^{(2)}, \varphi^{(1)}, f$ are n times differentiable with respect to the relevant independent variables so that, by using the chain rule, $\Phi(t)$ can be differentiated n times with respect to t.
We use the following notation
$\Phi_h := D_t^h \Phi(t)$,
$f_h := D_{x^{(1)}}^h f \Big|_{x^{(1)} = \varphi^{(1)}\left(\ldots\left(\varphi^{(r)}(t)\right)\right)}$,
$\varphi_h^{(1)} := D_{x^{(2)}}^h \varphi^{(1)} \Big|_{x^{(2)} = \varphi^{(2)}\left(\ldots\left(\varphi^{(r)}(t)\right)\right)}$,
$\varphi_h^{(r)} := D_t^h \varphi^{(r)}(t)$, and
$\left([f, \varphi^{(1)}, \ldots, \varphi^{(r)}]_n\right) := \left(f_1, \varphi_1^{(1)}, \ldots, \varphi_1^{(r)}; \ldots; f_n, \varphi_n^{(1)}, \ldots, \varphi_n^{(r)}\right)$. (3.1)

Then the nth derivative of the function Φ allows us to define the one-dimensional Bell polynomials of order r, $Y_n^{[r]}$ as follows:
$Y_n^{[r]}\left([f, \varphi^{(1)}, \ldots, \varphi^{(r)}]_n\right) := \Phi_n$. (3.2)
For $r = 1$ we obtain the ordinary Bell polynomials
$Y_n^{[1]}\left([f, \varphi^{(1)}]_n\right) = Y_n\left([f, \varphi^{(1)}]_n\right)$.
Note that we are considering here the one-dimensional case, while in [5] even the multi-dimensional Bell polynomials were introduced.
The first polynomials have the following explicit expressions
$Y_1^{[r]}\left([f, \varphi^{(1)}, \ldots, \varphi^{(r)}]_1\right) = f_1 \varphi_1^{(1)} \ldots \varphi_1^{(r)}$,
$Y_2^{[r]}\left([f, \varphi^{(1)}, \ldots, \varphi^{(r)}]_2\right) = f_2\left(\varphi_1^{(1)} \ldots \varphi_1^{(r)}\right)^2 + f_1 \varphi_2^{(1)}\left(\varphi_1^{(2)} \ldots \varphi_1^{(r)}\right)^2$
$+ f_1 \varphi_1^{(1)} \varphi_2^{(2)}\left(\varphi_1^{(3)} \ldots \varphi_1^{(r)}\right)^2 + f_1 \varphi_1^{(1)} \varphi_1^{(2)} \ldots \varphi_1^{(r-1)} \varphi_2^{(r)}$. (3.3)

In general, we have

$$Y_n^{[r]}\left(\left[f,\varphi^{(1)},\dots,\varphi^{(r)}\right]_n\right)=\sum_{k=1}^{n}A_{n,k}^{[r]}\left(\left[\varphi^{(1)},\dots,\varphi^{(r)}\right]_n\right)f_k. \tag{3.4}$$

A generalized form of the Fa`a di Bruno formula, a recurrence relation and further results relevant to this extension can be found in [5].

The recurrence relation (2.6) - (2.7) can be generalized as follows:

Theorem 3.1. *For all integer* n*, we have*

$$A_{n+1,1}^{[r]}=Y_{n+1}^{[r-1]}\left(\left[\varphi^{(1)},\dots,\varphi^{(r)}\right]_{n+1}\right), A_{n+1,n+1}^{[r]}=\left(Y_1^{[r-1]}\left(\left[\varphi^{(1)},\dots,\varphi^{(r)}\right]_1\right)\right)^{n+1}=\left(\varphi_1^{(1)},\dots,\varphi_1^{(r)}\right)^{n+1}. \tag{3.5}$$

Furthermore, for all $k=1,2,\dots,n-1$*, the* rth *order partial Bell polynomials* $A_{n,k}^{[r]}$ *satisfy the recursion*

$$A_{n+1,n+1}^{[r]}\left(\left[\varphi^{(1)},\dots,\varphi^{(r)}\right]_{n+1}\right)=\sum_{h=0}^{n-k}\binom{n}{h}A_{n-h,k}^{[r]}\left(\left[\varphi^{(1)},\dots,\varphi^{(r)}\right]_{n-h}\right)Y_{h+1}^{[r-1]}\left(\left[\varphi^{(1)},\dots,\varphi^{(r)}\right]_{h+1}\right). \tag{3.6}$$

Definition 3.1. *The complete Bell polynomials of order* $r, B_n^{[r]}$*, are defined by the equation*

$$B_n^{[r]}\left(\left[\varphi^{(1)},\dots,\varphi^{(r)}\right]_n\right)=Y_n^{[r]}\left(1,\varphi_1^{(1)},\dots,\varphi_1^{(r)};\dots;1,\varphi_n^{(1)},\dots,\varphi_n^{(r)}\right)$$
$$=\sum_{k=1}^{n}A_{n,k}^{[r]}\left(\left[\varphi^{(1)},\dots,\varphi^{(r)}\right]_n\right).$$

and the rth *order Bell numbers by*

$$b_n^{[r]}=Y_n^{[r]}(1,1,1;\dots;1,1,1)=\sum_{k=1}^{n}A_{n,k}^{[r]}(1,1;\dots;1,1).$$

4. Higher order Bell numbers, for $r=2,3,4,5$

It is worth to note that the sequences of higher order Bell numbers which are presented here appear in the Encyclopedia of Integer Sequences [34] under the $A144150$, arising from a problem of Combinatorial Analysis and even as the McLaurin coefficients of the functions [2, 18]

$$\begin{gathered} exp(exp(exp(x)-1)-1),\\ exp(exp(exp(exp(x)-1)-1)-1),\\ exp(exp(exp(exp(exp(x)-1)-1)-1)-1),\\ exp(exp(exp(exp(exp(exp(x)-1)-1)-1)-1)-1), \end{gathered} \tag{4.1}$$

for the cases $r=2, r=3, r=4, r=5$, respectively, and so on for the subsequent values of r. Whereas in our approach they assume a more general meaning, as they are independent of the functions $f,\varphi^{(1)},\dots,\varphi^{(r)}$.

Let $b_n^{[1]}:=b_n$ (namely, the classical Bell numbers). According to the above reference we have found, using the recurrence relation (3.5) - (3.6) and by means of the computer algebra program

Mathematica©, the following sequences for the higher order Bell numbers $b_n^{[2]}, b_n^{[3]}, b_n^{[4]}, b_n^{[5]}$, $(n = 1,2,3, \dots, 10)$:

n	$b_n^{[1]}$	$b_n^{[2]}$	$b_n^{[3]}$	$b_n^{[4]}$	$b_n^{[5]}$
1	1	1	1	1	1
2	2	3	4	5	6
3	5	12	22	35	51
4	15	60	154	315	561
5	52	358	1304	3455	7556
6	203	2471	12915	44590	120196
7	877	19302	146115	660665	2201856
8	4140	167894	1855570	11035095	45592666
9	21147	1606137	26097835	204904830	1051951026
10	115975	16733779	402215465	4183174520	26740775306

Table 1: Bell and higher order Bell numbers for $n = 1, 2, \dots, 10$.

Of course, the above table can be extended up to the desired order, as the Mathematica© program runs efficiently.

5. Sheffer polynomials

The Sheffer polynomials $\{s_n(x)\}$ are introduced [33] by means of the exponential generating function [35] of the type:

$$A(t) exp\big(xH(t)\big) = \sum_{n=0}^{\infty} s_n(x) \frac{t^n}{n!} \tag{5.1}$$

where, $A(t) = \sum_{n=0}^{\infty} a_n \frac{t^n}{n!} (a_0 \neq 0)$, and $H(t) = \sum_{n=0}^{\infty} h_n \frac{t^n}{n!} (h_0 = 0)$. (5.2)

According to a different characterization (see [31, p. 18]), the same polynomial sequence can be defined by means of the pair $(g(t), f(t))$, where $g(t)$ is an invertible series and $f(t)$ is a delta series:

$$g(t) = \sum_{n=0}^{\infty} g_n \frac{t^n}{n!} (g_0 \neq 0), \text{ and } f(t) = \sum_{n=0}^{\infty} f_n \frac{t^n}{n!} (f_0 = 0, f_1 \neq 0). \tag{5.3}$$

Denoting by $f^{-1}(t)$ the compositional inverse of $f(t)$ (i.e. such that $f(f^{-1}(t)) = f^{-1}(f(t)) = t$), the exponential generating function of the sequence $\{s_n(x)\}$ is given by

$$\frac{1}{g[f^{-1}(t)]} exp(xf^{-1}(t)) = \sum_{n=0}^{\infty} s_n(x) \frac{t^n}{n!}, \tag{5.4}$$

so that $A(t) = \frac{1}{g[f^{-1}(t)]}, H(t) = f^{-1}(t)$. (5.5)

When $g(t) \equiv 1$, the Sheffer sequence corresponding to the pair $(1, f(t))$ is called the associated Sheffer sequence $\{\sigma_n(x)\}$ for $f(t)$, whose exponential generating function is

$$exp(xf^{-1}(t)) = \sum_{n=0}^{\infty} \sigma_n(x) \frac{t^n}{n!}. \tag{5.6}$$

A list of known Sheffer polynomial sequences and associated ones can be found in [7]. In particular, assuming

$$A(t) = exp(\beta t), H(t) = 1 - exp(t), \tag{5.7}$$

the corresponding pair $(g\ (t),\ f\ (t))$ is given by
$g\ (t)\ =\ (1\ -\ t)^{-\beta}\ ,\ f\ (t)\ =\ log\ (1\ -\ t)\ ,$ (5.8)

and the exponential generating function of the Sheffer sequence is
$exp\big(\beta t + x(1-e^t)\big) = \sum_{n=0}^{\infty} \alpha_n^{(\beta)}(x)\frac{t^n}{n!}$ (5.9)

or, in equivalent form:

$exp\big(\beta t + x(e^t-1)\big) = \sum_{n=0}^{\infty} \alpha_n^{(\beta)}(-x)\frac{t^n}{n!}.$ (5.10)
The polynomials $\{\alpha_n^{(\beta)}(x)\}$ are called in [7] "Actuarial polynomials".

6. New Sheffer polynomial sequences

In this section, we introduce new Sheffer sequences generalizing the above mentioned Actuarial polynomials. This can be done by assuming the pair

$A(t)\ =\ exp(\beta t)\ , H_1(t)\ =\ f_1^{-1}(t)\ =\ exp(exp(t)\ -\ 1) - 1,$ (6.1)

so that the coefficients of the Taylor expansion of $H_1(t)$ are given by the Bell numbers $b_1\ = b_n^{[1]}$:

$H_1(t)\ = \sum_{n=0}^{\infty} b_n^{(1)}\frac{t^n}{n!},$ (6.2)
and consequently
$f_1(t) =\ log\ (log(t\ +\ 1) +\ 1), g_1(t) =\ [log(t\ +\ 1) +\ 1]^{-\beta}.$ (6.3)

Therefore, the exponential generating function of the corresponding Sheffer sequence $\left\{\alpha_n^{([1],\beta)}(x)\right\}$ is given by
$exp(\beta t + xH_1(t)) = \sum_{n=0}^{\infty} \alpha_n^{([1],\beta)}(x)\frac{t^n}{n!}$ (6.4)

The above method can be iterated by letting
$A(t)\ =\ exp\ (\beta t)\ ,$
$H_2\ (t)\ =\ f_2^{-1}(t)\ =\ exp(exp(exp(t)\ -\ 1)\ -\ 1)\ -\ 1\ ,$ (6.5)

so that the coefficients of the Taylor expansion of $H_2(t)$ are given by the 2nd order Bell numbers $b_n^{[2]}$:
$H_2(t)\ \ = \sum_{n=0}^{\infty} b_n^{(2)}\frac{t^n}{n!}$ (6.6)
and consequently
$f_2(t) =\ log\ (log\ (log(t\ +\ 1) +\ 1), g_2(t) =\ [log(log(t\ +\ 1) +\ 1]^{-\beta}$ (6.7)
Therefore, the exponential generating function of the corresponding Sheffer sequence $\left\{\alpha_n^{([2],\beta)}(x)\right\}$ is given by:
$exp(\beta t + xH_2(t)) = \sum_{n=0}^{\infty} \alpha_n^{([2],\beta)}(x)\frac{t^n}{n!}\ .$ (6.8)

In general, by letting
$A(t) = exp(\beta t)$,
$H_r\,(t) = exp(\ldots exp(exp(t) - 1) - 1) \ldots - 1$, ($r$ times exp) (6.9)
so that the coefficients of the Taylor expansion of $H_r(t)$ are given by the Bell numbers $b_n = b_n^{[r]}$:
$$H_r(t) = \sum_{n=0}^{\infty} b_n^{(r)} \frac{t^n}{n!} \tag{6.10}$$
and consequently
$f_r(t) = log\,(log(\ldots\,(log(t + 1) + 1) \ldots) + 1)$, ($r$ times log)
$g_r(t) = [log(\ldots(log(t + 1) + 1) \ldots) + 1]^{-\beta}$, $\big((r - 1)$ times $log\big)$. (6.11)
Therefore, the exponential generating function of the corresponding Sheffer sequence $\left\{\alpha_n^{([r],\beta)}(x)\right\}$ is given by

$$exp(\beta t + xH_r(t)) = \sum_{n=0}^{\infty} \alpha_n^{([r],\beta)}(x) \frac{t^n}{n!}. \tag{6.12}$$

7. The associated Sheffer polynomials

The associated Sheffer polynomials relevant to the pair $(1, f(t))$, where
$f(t) = log(t + 1)\,, f^{-1}(t) = H(t) = e^t - 1\,,$ (7.1)
have been introduced by E.T. Bell [1] and are known as "Exponential polynomials" $\Phi_n(x)$. Their exponential generating function is given by

$$exp\big(x(\exp(t) - 1)\big) = \sum_{n=0}^{\infty} \Phi_n(x) \frac{t^n}{n!}. \tag{7.2}$$

8. New sets of associated Sheffer polynomials

Results similar to that obtained in Section 6 can be achieved by introducing the associated Sheffer polynomials relevant to the pair $(1, f_1(t))$, where $f_1(t) = log(log(t + 1) + 1)\,, H_1(t) = f_1^{-1}(t) = exp(exp(t) - 1) - 1\,,$ (8.1)
so that the exponential generating function of the associated Sheffer polynomials is given by
$$exp(xH_1(t)) = \sum_{n=0}^{\infty} (\mathrm{x})_{\mathrm{n}}^{[1]} \frac{t^n}{n!}, \tag{8.2}$$

and in general, assuming:

$$f_r(t) = log\,(log\,(\ldots(log(t + 1) + 1)\ldots) + 1)\,, (r\ times\ log)\,, \tag{8.3}$$

$$H_r(t) = f_r^{-1}(t) = exp(\ldots exp(exp(t) - 1) - 1) \cdots - 1\,, (r\ times\ exp)\,, \tag{8.4}$$
we find the associated Sheffer polynomials defined by the exponential generating function
$$exp(xH_r(t)) = \sum_{n=0}^{\infty} (\mathrm{x})_{\mathrm{n}}^{[\mathrm{r}]} \frac{t^n}{n!}. \tag{8.5}$$

9. The Sheffer polynomials $\left\{\boldsymbol{\alpha}_{\boldsymbol{n}}^{([r],\boldsymbol{\beta})}(\boldsymbol{x})\right\}$

9.1 Explicit expression

The explicit expression of these polynomials can be easily done only by using the Taylor expansion of the associated Sheffer polynomials $(\mathrm{x})_{\mathrm{n}}^{[\mathrm{r}]}$, namely

Theorem 9.1. *We have, for all integer n,*

$$\left\{\alpha_n^{([r],\beta)}(x)\right\} = \sum_{k=0}^{n} \binom{n}{k} \beta^{n-k}. (x)_k^{[r]} \tag{9.1}$$

Proof. - It is a trivial consequence of equation (6.12) by using the Cauchy product of $exp(\beta t)$ times the series expansion (8.5) of $exp\ (xH_r(t))$.

Note that in order to find the explicit expression of the above Sheffer polynomials $(x)_k^{[r]}$ it is necessary to evaluate the coefficients of the Taylor expansion of the function $exp\ (xH_r(t))$ (as a function of t, because x is just a parameter), and this can be done by using the Bell polynomials of order r [23], recalled in Sect. 3, as the considered function is a higher order composite function of the type $exp(xy_1(y_2(\dots(y_r(t))\dots)$
where $y_r(t) = exp(t) - 1, y_{r-1} = exp(y_r(t)) - 1, \dots, y_1 = exp(y_2(t)) - 1.$

For completeness, in order to depict the polynomials $\left\{\alpha_n^{([r],\beta)}(x)\right\}$, here we report the explicit expression of the associated Sheffer polynomials $(x)_k^{[r]}$for $r = 1$ up to order $n = 6$:

$(x)_0^{[1]} = 1;$
$(x)_1^{[1]} = x;$
$(x)_2^{[1]} = x(x + 2);$
$(x)_3^{[1]} = x(x^2 + 6x + 5);$
$(x)_4^{[1]} = x(x^3 + 12x^2 + 32x + 15);$
$(x)_5^{[1]} = x(x^4 + 20x^3 + 110x^2 + 175x + 52);$
$(x)_6^{[1]} = x(x^5 + 30x^4 + 280x^3 + 945x^2 + 1012x + 203),$

$and\ r = 2\ up\ to\ order\ n = 6$:
$(x)_0^{[2]} = 1;$
$(x)_1^{[2]} = x;$
$(x)_2^{[2]} = x(x + 3);$
$(x)_3^{[2]} = x(x^2 + 9x + 12);$
$(x)_4^{[2]} = x(x^3 + 18x^2 + 75x + 60);$
$(x)_5^{[2]} = x(x^4 + 30x^3 + 255x^2 + 660x + 358);$
$(x)_6^{[2]} = x(x^5 + 45x^4 + 645x^3 + 3465x^2 + 6288x + 2471).$

Now, using the expression of the associated Sheffer polynomials above, we can depict the Sheffer polynomials $\left\{\alpha_n^{([r],\beta)}(x)\right\}$ for any β for the cases $r = 1, r = 2$. As an example, we report in Figures $1, 2, 3, 4$, the graphs of the $\left\{\alpha_n^{([1],\beta)}(x)\right\}$ for different values of β.

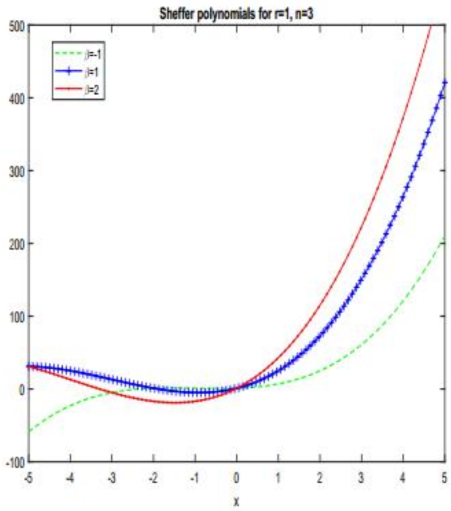

Figure 1: Graph of the Sheffer polynomials $\left\{\alpha_n^{([1],\beta)}(x)\right\}$ of degree $n = 3$, for $\beta = -1, 1, 2$.

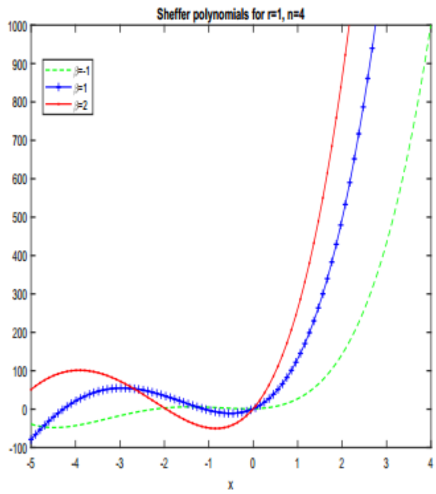

Figure 2: Graph of the Sheffer polynomials $\left\{\alpha_n^{([1],\beta)}(x)\right\}$ of degree $n = 4$, for $\beta = -1, 1, 2$.

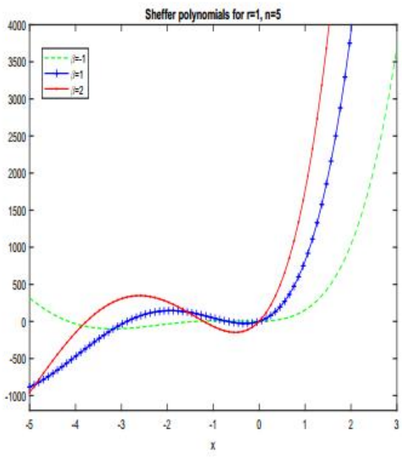

Figure 3: Graph of the Sheffer polynomials $\left\{\alpha_n^{([1],\beta)}(x)\right\}$ of degree $n = 5$, for $\beta = -1, 1, 2$.

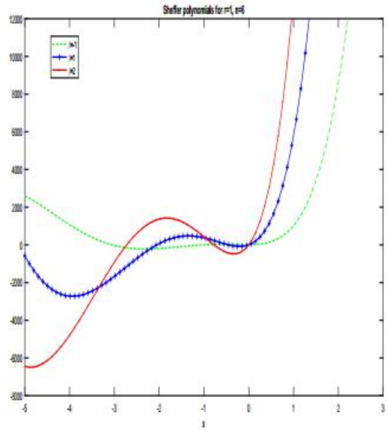

Figure 4: Graph of the Sheffer polynomials $\left\{\alpha_n^{([1],\beta)}(x)\right\}$ of degree $n = 6$, for $\beta = -1, 1, 2$.

10. Brenke polynomials

The Brenke polynomials $\{\Upsilon_n(x)\}$ are introduced [8] by means of the exponential generating function of the type:

$$A(t)\psi(xt) = \sum_{n=0}^{\infty} \Upsilon_n(x)\frac{t^n}{n!}, \tag{10.1}$$

where

$$A(t) = \sum_{n=0}^{\infty} a_n \frac{t^n}{n!}, (a_0 \neq 0),$$

$$\psi(t) = \sum_{n=0}^{\infty} \gamma_n \frac{t^n}{n!}, (\gamma_0 \neq 0). \tag{10.2}$$

They are a particular case of a more general family of polynomials, namely, the BoasBuck polynomials [6, 7], which are defined by the exponential generating function

$$A(t)\psi(xH(t) = \sum_{n=0}^{\infty} p_n(x)\frac{t^n}{n!}, \tag{10.3}$$

$where$

$$A(t) = \sum_{n=0}^{\infty} a_n \frac{t^n}{n!}, (a_0 \neq 0),$$

$$\psi(t) = \sum_{n=0}^{\infty} \gamma_n \frac{t^n}{n!}, (\gamma_n \neq 0 \quad \forall n), \tag{10.4}$$

with $\psi(t)$ not a polynomial, and lastly

$$H(t) = \sum_{n=0}^{\infty} h_n \frac{t^n}{n!}, (h_1 \neq 0). \tag{10.5}$$

The Boas-Buck polynomials include also the Humbert polynomials (see [35]), and, as a particular case, the Gegenbauer polynomials.

11. New Brenke polynomial sequences

In this section we introduce new Brenke sequences.

We start by assuming

$$A(t) = e^t,$$

$$\psi_1(t) = exp(exp(t) - 1), \tag{11.1}$$

so that the coefficients of the Taylor expansion of $\psi(t)$ are given by the Bell numbers $b_n = b_n^{[1]}$:

$$\psi_1(t) = \sum_{n=0}^{\infty} b_n^{[1]} \frac{t^n}{n!}. \tag{11.2}$$

Therefore, the exponential generating function of the corresponding Brenke sequence $\left\{\Upsilon_n^{[1]}(x)\right\}$ is given by

$$A(t)\psi_1(xt) = e^t \sum_{n=0}^{\infty} b_n^{[1]}(x) \frac{(xt)^n}{n!} = \sum_{n=0}^{\infty} \Upsilon_n^{[1]}(x) \frac{t^n}{n!}. \tag{11.3}$$

The above method can be iterated by letting

$$A(t) = e^t ,$$
$$\psi_2(t) = exp(exp(exp(t) - 1) - 1) , \tag{11.4}$$

so that the coefficients of the Taylor expansion of $\psi(t)$ are given by the 2nd order Bell numbers $b_n^{[2]}$:

$$\psi_2(t) = \sum_{n=0}^{\infty} b_n^{[2]} \frac{t^n}{n!} \tag{11.5}$$

Therefore, the exponential generating function of the corresponding Brenke sequence $\left\{\Upsilon_n^{[2]}(x)\right\}$ is given by

$$A(t)\psi_2(xt) = e^t \sum_{n=0}^{\infty} b_n^{[2]}(x) \frac{(xt)^n}{n!} = \sum_{n=0}^{\infty} \Upsilon_n^{[2]}(x) \frac{t^n}{n!}. \tag{11.6}$$

In general, by letting

$$A(t) = e^t ,$$
$$\psi_r(t) = exp(\dots exp(exp(t) - 1) \cdots - 1) , (r\ times\ exp) \tag{11.7}$$

so that the coefficients of the Taylor expansion of $\psi_r(t)$ are given by the rth order Bell numbers $b_n^{[r]}$:

$$\psi_r(t) = \sum_{n=0}^{\infty} b_n^{[r]} \frac{t^n}{n!}, \tag{11.8}$$

we find the corresponding Brenke sequence $\left\{\Upsilon_n^{[r]}(x)\right\}$ whose exponential generating function is given by

$$A(t)\psi_r(xt) = e^t \sum_{n=0}^{\infty} b_n^{[r]}(x) \frac{(xt)^n}{n!} = \sum_{n=0}^{\infty} \Upsilon_n^{[r]}(x) \frac{t^n}{n!}. \tag{11.9}$$

12. Main properties of the polynomials $\Upsilon_n^{[r]}(x)$

First of all, we derive the explicit expression of the $\Upsilon_n^{[r]}(x)$.

12.1 Explicit expression

We assume, as it is usual (see [34]), for every integer $r, b_0^{[r]} = 0$.

Theorem 12.1. *We have, for every integer n,*

$$\Upsilon_n^{[r]}(x) = \sum_{k=0}^{n} \binom{n}{k} b_k^{[r]} x^k \tag{12.1}$$

Proof. - It is a trivial consequence of equation (11.9) by using the Cauchy product of $exp(t)$ times the series expansion of $\psi_r(t)$.

For completeness, we report some graphs of the Brenke polynomials in the next Figures 5, 6, 7.

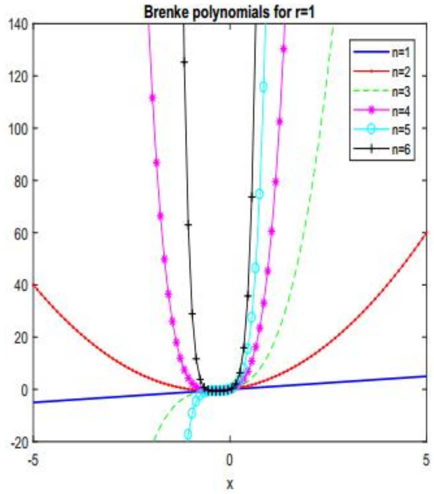

Figure 5: Graph of the Brenke polynomials $Y_n^{[1]}(x)$ up to $n = 6$.

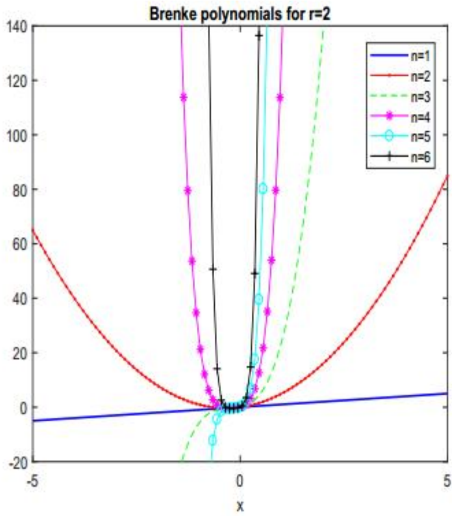

Figure 6: Graph of the Brenke polynomials $Y_n^{[2]}(x)$ up to $n = 6$.

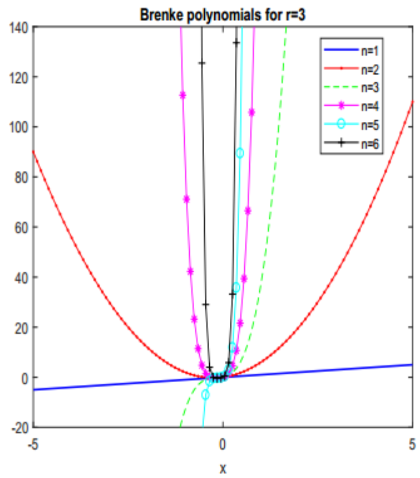

Figure 7: Graph of the Brenke polynomials $\Upsilon_n^{[3]}(x)$ up to $n = 6$.

.12.2. Shift operators

We recall that a polynomial set $\{p_n(x)\}$ is called quasi-monomial if and only if there exist two operators $\acute{P}$ and $\acute{M}$ such that

$$\acute{P}\,(p_n\,(x)) = np_{n-1}\,(x),\ \acute{M}(p_n\,(x)) = p_{n+1}(x),\ (n = 1,2,\ldots). \tag{12.2}$$

$\acute{P}$ is called the *derivative* operator and $\acute{M}$ the *multiplication* operator, as they act in the same way of classical operators on monomials.

This definition traces back to a paper by G. Dattoli [11], and was used in many articles and was the source of several application.

Y. Ben Cheikh [3] proved that every polynomial set is quasi-monomial under the action of suitable derivative and multiplication operators. In particular, in the same article (Corollary 3.2), the following result is proved

Theorem 12.2. *Let $\left(p_n(x)\right)$ a Boas-Buck polynomial set generated by the generating function*

$$A(t)\psi(xH(t) = \sum_{n=0}^{\infty} p_n(x)\frac{t^n}{n!}, \tag{12.3}$$

where

$$A(t) = \sum_{n=0}^{\infty} \tilde{a}_n t^n\,, (\tilde{a}_0 \neq 0),$$

$$\psi(t) = \sum_{n=0}^{\infty} \tilde{\gamma}_n t^n\,, (\tilde{\gamma}_n \neq 0 \quad \forall n), \tag{12.4}$$

with $\psi(t)$ not a polynomial, and lastly

$$H(t) = \sum_{n=0}^{\infty} \tilde{h}_n t^{n+1}\,, \left(\tilde{h}_0 \neq 0\right). \tag{12.5}$$

Let $\sigma \in \Lambda^{(-)}$ the lowering operator defined by

$$\sigma(1) = 0\,, \sigma(x^n) = \frac{\tilde{\gamma}_{n-1}}{\tilde{\gamma}_n} x^{n-1}, (n = 1,2,\ldots). \tag{12.6}$$

Put

$$\sigma^{-1}(x^n) = \frac{\tilde{\gamma}_{n+1}}{\tilde{\gamma}_n} x^{n+1}, (n = 0,1,2,\ldots). \tag{12.7}$$

Denoting, as before, by $f(t)$ the compositional inverse of $H(t)$, the Boas-Buck polynomial set $\{p_n(x)\}$ is quasi-monomial under the action of the operators

$$\acute{P} = f(\sigma),\ \acute{M} = \frac{A'(\sigma)}{A(\sigma)} + xD_x H'(\sigma)\sigma^{-1}, \tag{12.8}$$

where prime *denotes the ordinary derivatives with respect to t.*

For the Brenke polynomials we are considering here, as $A(t) = e^t$ and $H(t) = t$ (so that $f(t) = t$), taking into account the coefficients of ψ, this result reduces to

Theorem 12. 3. *Let $(\Upsilon_n^{[r]}(x))$ the polynomial set defined by the generating function*

$$e^t \sum_{n=0}^{\infty} b_n^{[r]}(x) \frac{(xt)^n}{n!} = \sum_{n=0}^{\infty} \Upsilon_n^{[r]}(x) \frac{t^n}{n!}. \tag{12.9}$$

Let $\sigma \in \Lambda^{(-)}$ the lowering operator defined by

$$\sigma(1) = 0\,, \sigma(x^n) = n\frac{b_{n-1}^{[r]}}{b_n^{[r]}} x^{n-1}, (n = 1,2,\ldots). \tag{12.10}$$

Put

$$\sigma^{-1}(x^n) = \frac{1}{n+1}\frac{b_{n+1}^{[r]}}{b_n^{[r]}} x^{n+1}, (n = 0,1,2,\ldots). \tag{12.11}$$

The Brenke polynomial set $(\Upsilon_n^{[r]}(x))$ is quasi-monomial under the action of the operators

$$\acute{P} = \sigma\,,\ \acute{M} = Id + xD_x\sigma^{-1}\,, \tag{12.12}$$

where Id denotes the identity operator.

12.3. Differential equation

According to the results of monomiality principle [11], the quasi-monomial polynomials $\{p_n(x)\}$ satisfy the differential equation

$$\acute{M}\acute{P}p_n(x) = np_n(x) \tag{12.13}$$

In the present case, we have

Theorem 12.4. *The Brenke polynomials $(\Upsilon_n^{[r]}(x))$ satisfy the operational-differential equation*

$$(xD_x + \sigma)\Upsilon_n^{[r]}(x) = n\Upsilon_n^{[r]}(x)\,, \tag{12.14}$$

where σ is the operator defined by equation (12.10).

Conclusion

We have shown that the higher order Bell numbers enter in a natural way in the definition of new sets of Sheffer, associated Sheffer and Brenke polynomials. This could be used in order to understand the combinatorial character of the higher order Bell numbers.

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