SOME FIXED POINT THEOREMS FOR MULTI-VALUED MAPS ON *S*-METRIC SPACE

By

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Abstract
In this paper we introduced a multi-valued contraction in S-metric space and proved some fixed point theorems for multi-valued maps on *S*-metric space. Our results extend and generalize the results of Nadler [8], Sedghi et al. [12] and others. Some examples are also given in support of our results.

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1. Introduction

Metric spaces are very important in various area of Mathematics such as analysis, topology, applied mathematics etc. So, various generalizations of metric space have been studied and many fixed point theorems were obtained by several authors. For example Gahler [4], Dhage [1], and Mustafa-Sim [7] introduced the concept of 2-metric space, *D*-metric space and *G*-metric space respectively as a generalization of metric space.

Recently Sedghi et al. [12] introduced an *S*-metric space, which is different from other spaces and gave some basic properties of *S*-metric space. Many authors have investigated the fixed point property for single-valued maps on *S*-metric space (see for instance [2]-[3], [5]-[6], [9]-[11] and references therein). In this paper, we defined a multi-valued contraction on *S*-metric space and proved some fixed point theorems for multi-valued maps on *S*-metric space.

2. Preliminaries

Throughout the paper, we shall assume that (X,d) be a metric space and N will denote the set of natural numbers. We follow the following notations of Nadler [8].

$CL(X)=\{A: A$ is a nonempty closed subset of $X\}$;

$CB(X)=\{A: A$ is a nonempty closed and bounded subset of $X\}$;

$C(X)=\{A: A$ is a nonempty compact subset of $X\}$.

Definition 2.1. [12]. Let X be a non-empty set. An *S*-metric on X is a function $S: X\times X\times X\to[0,\infty)$ that satisfies the following conditions, for each $x,y,z,a\in X$,

(S_1) $S(x,y,z)\geq 0$,

(S_2) $S(x,y,z)=0$ if and only if $x=y=z$,

(S_3) $S(x,y,z)\leq S(x,x,a)+S(y,y,a)+S(z,z,a)$.

The pair (X,S) is called an *S*-metric space.

Some immediate examples of such *S*-metric spaces are given in [12] as follows:

Example 2.1. Let $X=\mathbb{R}^n$ and $\|.\|$ a norm on X, then $S(x,y,z)=\|y+z-2x\|+\|y-z\|$ is an *S*-metric on X.

Example 2.2. Let $X = \mathbb{R}^n$ and $\|.\|$ a norm on X, then $S(x,y,z) = \|x - z\| + \|y - z\|$ is an S-metric on X.

Example 2.3. Let X be a nonempty set, d is ordinary metric on X, then $S(x,y,z) = d(x,z) + d(y,z)$ is an S-metric on X.

Lemma 2.1. [12]. In an S-metric space, we have $S(x,x,y) = S(y,y,x)$.

Definition 2.2. [12]. Let (X,S) be an S-metric space. For $r > 0$ and $x \in X$, the open ball $B_S(x,r)$ and closed ball $B_S[x,r]$ with a center x and a radius r is defined as follow:

$$B_S(x,r) = \{y \in X : S(y,y,x) < r\},$$
$$B_S[x,r] = \{y \in X : S(y,y,x) \le r\}.$$

Definition 2.3. [9]. Let (X,S) be an S-metric space and A be a nonempty subset of X. The diameter of A is defined by

$$diam\{A\} = \sup\{S(x,x,y) : x,y \in A\}. \tag{2.1}$$

If A is S-bounded, then $diam\{A\} < \infty$.

Definition 2.4. [12]. Let (X,S) be an S-metric space.

(1) If for every $x \in A$ there exist $r > 0$ such that $B_S(x,r) \subset A$, then the subset A is called an open subset of X.

(2) A subset A of X is said to be S-bounded if there exists $r > 0$ such that $S(x,x,y) < r$ for all $x, y \in A$.

(3) A sequence $\{x_n\}$ in X converges to x if and only if $S(x_n, x_n, x) \to 0$ as $n \to \infty$. That is for each $\varepsilon > 0$ there exists $n_0 \in N$ such that for all $n \ge n_0$, $S(x_n, x_n, x) < \varepsilon$ and we denote this by $\lim_{n\to\infty} x_n = x$.

(4) A sequence $\{x_n\}$ in X is called Cauchy sequence if for each $\varepsilon > 0$, there exists $n_0 \in N$ such that for all $n \ge n_0$, $S(x_n, x_n, x) < \varepsilon$.

(5) The S-metric space (X,S) is said to be complete if every Cauchy sequence is convergent.

Lemma 2.2. [12]. Let (X,S) be an S-metric space. If there exist sequences $\{x_n\}$ and $\{y_n\}$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$ then $\lim_{n\to\infty} S(x_n, x_n, y_n) = S(x,x,y)$.

The relationship between a metric and an S-metric was shown in [6] as follow:

Lemma 2.3. Let (X,d) be a metric space. Then the following properties are satisfied:

(1) $S_d(x,y,z) = d(x,z) + d(y,z)$ for all $x,y,z \in X$ is an S-metric on X.

(2) $x_n \to x$ in (X,d) if and only if $x_n \to x$ in (X, S_d).

(3) $\{x_n\}$ is Cauchy in (X,d) if and only if $\{x_n\}$ is Cauchy in (X, S_d).

(4) (X,d) is complete if and only if (X, S_d) is complete.

The metric S_d is called as the S-metric generated by d (see [9]).

Definition 2.5. [10]. Let (X,S) be an S-metric space, $x \in X$ and $A, B \subset X$, then the distance of the point x to the set A is defined as

$$S(x,x,A) = \inf\{S(x,x,y) : y \in A\}.$$

It is clear by the definition of $S(x,x,A)$, that $S(x,x,A) = 0 \Leftrightarrow x \in \bar{A}$.

Now we define Hausdorff S-distance on $CB(X)$ as follow:

$$H_S(A,A,B) = \max\left\{\sup_{x\in A} S(x,x,B),\ \sup_{y\in B} S(y,y,A)\right\},$$

where we recall $S(x,x,B) = \inf\{S(x,x,y):\ y \in B\}$. Then H_S is called Hausdorff S-distance on $CB(X)$ induced by S-metric.

3. Fixed Point Theorems

In this section we start with the following lemmas. The proof of these lemmas are simple consequence of definition of the Hausdorff S-distance $H_S(A,A,B)$.

Lemma 3.1. Let (X,S) be an S-metric space and $A,B \in CB(X)$. Then for each $a \in A$, we have

$$S(a,a,B) \le H_S(A,A,B).$$

Lemma 3.2. If $A,B \in CB(X)$ and $a \in A$, then for each $\varepsilon > 0$ there exists $b \in B$ such that

$$S(a,a,b) \le H_S(A,A,B) + \varepsilon.$$

Now, first we define a multi-valued contraction on S-metric space and then state our main results.

Definition 3.1. Let (X,S) be an S-metric space. A function $T: X \to CB(X)$ is said to be a multi-valued contraction on X if there exists a constant α, with $0 \le \alpha < 1$, such that

$$H_S(Tx,Tx,Ty) \le \alpha S(x,x,y) \text{ for all } x,y \in X. \tag{3.1}$$

Theorem 3.1. Let (X,S) be a complete S-metric space. If $T: X \to CB(X)$ is a multi-valued contraction on X then T has a fixed point.

Proof. Let x_0 be an arbitrary point in X and choose $x_1 \in Tx_0$. If $x_1 = x_0$ then x_0 is a fixed point of T and the proof is complete. Assume that $x_1 \ne x_0$. Then by Lemma 3.2, there exists $x_2 \in Tx_1$ such that

$$S(x_1,x_1,x_2) \le H_S(Tx_0,Tx_0,Tx_1) + \alpha.$$

Also we get there exist $x_3 \in Tx_2$ such that

$$S(x_2,x_2,x_3) \le H_S(Tx_1,Tx_1,Tx_2) + \alpha^2.$$

Continuing in this fashion, we construct a sequence $\{x_n\}$ in X such that

$$x_{n+1} \in Tx_n$$

and

$$S(x_n,x_n,x_{n+1}) \le H_S(Tx_{n-1},Tx_{n-1},Tx_n) + \alpha^n, \text{ for all } n \ge 1.$$

Then from (3.1)

$$\begin{aligned} S(x_n,x_n,x_{n+1}) &\le H_S(Tx_{n-1},Tx_{n-1},Tx_n) + \alpha^n \\ &\le \alpha\left(H_S\left(Tx_{n-2},Tx_{n-2},Tx_{n-1}\right) + \alpha^{n-1}\right) + \alpha^n \\ &\le \alpha^2 S(x_{n-2},x_{n-2},x_{n-1}) + 2\alpha^n \\ &\ \ \vdots \\ &\le \alpha^n S(x_0,x_0,x_1) + n\alpha^n, \text{ for all } n \ge 1. \end{aligned}$$

Hence for all $m,n \ge 1$, we have

$$S(x_n, x_n, x_m) \le 2\sum_{r=n}^{m-2} S(x_r, x_r, x_{r+1}) + S(x_{m-1}, x_{m-1}, x_m)$$

$$\le 2\sum_{r=n}^{m-2} \alpha^r S(x_0, x_0, x_1) + 2\sum_{r=n}^{m-2} r\alpha^r + \alpha^{m-1} S(x_0, x_0, x_1) + (m-1)\alpha^{m-1}$$
$$\le 2\sum_{r=n}^{m-1} \alpha^r S(x_0, x_0, x_1) + 2\sum_{r=n}^{m-1} r\alpha^r.$$

It follows that the sequence $\{x_n\}$ is a Cauchy. Since (X, S) is a complete S-metric space, the sequence $\{x_n\}$ converges to same point $x_0 \in X$. Therefore the sequence $\{Tx_n\}$ converges to Tx_0 and since $x_{n+1} \in Tx_n$ for all $n \ge 1$, it follows that $x_0 \in Tx_0$. This completes the proof of the theorem.

Example 3.1. Let $X = [0, \infty)$ be endowed with S-metric $S(x, y, z) = |x - z| + |x + z - 2y|$ for all $x, y \in X$. Define $T : X \to CB(X)$ by $Tx = [0, x/2]$ for all $x \in X$.

Then for all $x, y \in X$, we have

$$S(x, x, y) = |x - y| + |x + y - 2x| = 2|x - y|$$

and

$$H_S(Tx, Tx, Ty) = |x - y|.$$

Thus

$$H_S(Tx, Tx, Ty) \le \alpha S(x, x, y) \text{ for } \alpha = 1/2$$

that is, T is a multi-valued contraction on X with $\alpha = 1/2$ and T has a fixed point at $x = 0$ in X.

Theorem 3.2. Let (X, S) be a compact S-metric space with $T : X \to CB(X)$ satisfying

$$H_S(Tx, Tx, Ty) < S(x, x, y) \text{ for all } x, y \in X \text{ and } x \ne y. \quad (3.2)$$

Then T has a fixed point in X.

Proof. Let us suppose that T does not have any fixed point in X that is, $x \notin Tx$ for all $x \in X$. Then the function $f : X \to [0, \infty)$ given by $f(x) = S(x, x, Tx) = \inf\{S(x, x, z) : z \in Tx\}$ for all $x \in X$, is continuous on compact set X therefore it attains its minimum value at some point (say) a in X. Thus

$$S(a, a, Ta) \le S(x, x, Tx) \text{ for all } x \in X.$$

Since $S(a, a, Ta) = \inf\{S(a, a, z) : z \in Ta\}$ therefore for particular $z = z_0 \in Ta$, we have $S(a, a, Ta) = S(a, a, z_0)$ and

$$S(a, a, z_0) \le S(x, x, Tx) \text{ for all } x \in X.$$

Now using (3.2) and Lemma 3.1, we have

$$S(a, a, z_0) \le S(z_0, z_0, Tz_0) \le H_S(Ta, Ta, Tz_0) < S(a, a, z_0),$$

which gives $S(a, a, z_0) = 0$ and $z_0 = a \in Ta$. Hence T has a fixed point on a compact S-metric space X.

Example 3.2. Let $X=[0,1]$ be endowed with usual S-metric space on X defined in [11] as follows

$$S(x,y,z)=|x-y|+|y-z| \text{ for all } x,y\in X.$$

We define $T:X\to CB(X)$ by $Tx=\left[0,\frac{1}{1+x}\right]$ for all $x\in X$. Then for all $x,y\in X$ and $x\neq y$, we have

$$S(x,x,y)=|x-y|$$

and

$$H_S(Tx,Tx,Ty)=\left|\frac{1}{1+x},-\frac{1}{1+y}\right|=\frac{|x-y|}{(1+x)(1+y)}<|x-y|.$$

Hence T satisfies condition (3.2) of Theorem 3.2 and the set of fixed points of T is $\left[0,\frac{\sqrt{5}-1}{2}\right]$.When x,y are sufficient close to 0, then $\frac{H_S(Tx,Tx,Ty)}{S(x,x,y)}=\frac{1}{(1+x)(1+y)}$ obtained its value arbitrary closed to 1. Therefore T is not a multi-valued contraction on [0,1] with usual *S*-metric.

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