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ON KAEHLERIAN CONHARMONIC* BI-RECURRENT SPACES

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Abstract
In this paper, we define and study Kaehlerian conharmonic* bi-recurrent spaces. Also, we establish the necessary and sufficient conditions for a Kaehlerian conharmonic* bi-recurrent space to be Kaehlerian bi-recurrent and is that the space be Kaehlerian bi-ricci-recurrent, Kaehlerian projective bi-recurrent. Several other theorems are also derived.

2010 Mathematics Subject Classification: 53C55, 53B35.
Keywords: Riemannian space, Kaehlerian space, conharmonic* curvature, projective curvature, Bochner curvature, tensor, bi-recurrent.

1. Introduction

Recently, Negi [3] defined and studied on almost product and decomposable spaces. In 2009, Negi and Rawat [4] have obtained the relations on almost Kaehlerian spaces with recurrent and symmetric projective curvature tensors. Lichnerowich [2] has called a Riemannian space satisfying the relation $R^h_{ijk,ab} - \lambda_{ab} R^h_{ijk} = 0$, where λ_{ab} is a non-zero tensor a recurrent space of second order. An $n\ (= 2m)$ dimensional Kaehlerian space K^n is a Riemannian space which admits a structure tensor field F_i^h satisfying the relations (see Yano [8]), given as

$$F_j^h F_h^i = -\delta_j^i\ , \tag{1.1}$$

$$F_{ij} = -F_{ji} \text{ and } F_{ij} = F_i^a g_{aj}\ , \tag{1.2}$$

$$F^h_{i,j} = 0. \tag{1.3}$$

Here, the comma (,) followed by an index denotes the operator of covariant differentiation with respect to the metric tensor g_{ij} of the Riemannian space.

The Riemannian curvature tensor, denoted by R^h_{ijk} is given as

$$R^h_{ijk} = \delta_i \begin{Bmatrix} h \\ j \quad k \end{Bmatrix} - \delta_j \begin{Bmatrix} h \\ i \quad k \end{Bmatrix} + \begin{Bmatrix} h \\ i \quad l \end{Bmatrix} \begin{Bmatrix} l \\ j \quad k \end{Bmatrix} - \begin{Bmatrix} h \\ j \quad l \end{Bmatrix} \begin{Bmatrix} l \\ i \quad k \end{Bmatrix}$$

whereas, the Ricci-tensor and the scalar curvature are respectively given by
$R_{ij} = R^h_{hij}$ and $R = R_{ij} g^{ij}$.

It is well known that these tensors satisfy the identities (see Tachibana [7])

$$F_i^a R_a^j = R_i^a F_a^j \tag{1.4}$$

and

$$F_i^a R_{aj} = -R_{ia} F_j^a. \tag{1.5}$$

In view of (1.1), the relation (1.4) gives

$$F_i^a R_a^b F_b^j = -R_i^j. \tag{1.6}$$

Also, multiplying (1.5) by g^{ij}, we obtain

$$F_i^a R_a^i = -R_a^j F_j^a. \tag{1.7}$$

If we define a tensor S_{ij} by

$$S_{ij} = -F_i^a R_{aj}. \tag{1.8}$$

we have

$$S_{ij} = -S_{ji} \tag{1.9}$$

The holomorphically projective curvature tensor P_{ijk}^h and the Bochner curvature tensor $K_{ijk}^h$are respectively given by (see Sinha [6])

$$P_{ijk}^h = R_{ijk}^h + \frac{1}{(n+2)}\left(R_{ik}\delta_j^h - R_{jk}\delta_i^h + S_{ik}F_j^h - S_{jk}F_i^h + 2S_{ij}F_k^h\right), \tag{1.10}$$

$$K_{ijk}^h = R_{ijk}^h + \frac{1}{(n+4)}\left(R_{ik}\delta_j^h - R_{jk}\delta_i^h + g_{ik}R_j^h - g_{jk}R_i^h + S_{ik}F_j^h - S_{jk}F_i^h + F_{ik}S_j^h - F_{jk}S_i^h + 2S_{ij}F_k^h + 2F_{jk}S_k^h\right) - \frac{R}{(n+2)(n+4)}\left(g_{ik}\delta_j^h - g_{jk}\delta_i^h + F_{ik}F_j^h - F_{jk}F_i^h + 2F_{ij}F_k^h\right). \tag{1.11}$$

Also, Behari and Ahuja [1] have defined conharmonic* curvature tensor $^*T_{ijk}^h$ as

$$^*T_{ijk}^h \overset{\text{def}}{=} R_{ijk}^h + \frac{1}{n-2}\left(g_{ik}R_j^h - g_{jk}R_i^h\right), \tag{1.12}$$

In our work, we use the followings:

Definition (1.1): A Kaehlerian space K^n satisfying (see Singh, [5])

$$R_{ijk,ab}^h = \lambda_{ab}R_{ijk}^h \tag{1.13}$$

For a non-zero recurrence tensor λ_{ab}, will be called a Kaehlerian recurrent space of second order or briefly a Kaehlerian bi-recurrent space.

Definition (1.2): A Kaehlerian space K^n will be called a Kaehlerian Ricci-recurrent space of second order or briefly a Kaehlerian bi-Ricci-recurrent space, if its Ricci tensor satisfies the relation (see Singh, [5])

$$R_{ij,ab} = \lambda_{ab}R_{ij}\ , \tag{1.14}$$

$where\ R_{ij} \neq 0,$ for a non-zero tensor λ_{ab} from (1.14), we deduce

$$R_{,ab} = \lambda_{ab}R. \tag{1.15}$$

Remark (1.1): A Kaehlerian bi-recurrent space is Kaehlerian bi – Ricci – recurrent but the converse is not necessarily true.

Definition (1.3): A Kaehlerian space K^n satisfying (see Singh, [5])

$$P_{ijk,ab}^h = \lambda_{ab}P_{ijk}^h\ , \tag{1.16}$$

where, λ_{ab} is a non-zero tensor, will be called a Kaehlerian projective recurrent space of second order or briefly a Kaehlerian projective bi-recurrent space.

Definition (1.4): A Kaehlerian space K^n satisfying (see Singh, [5])

$$K^h_{ijk,ab} = \lambda_{ab} K^h_{ijk} \qquad (1.17)$$

for a non-zero tensor λ_{ab}, is called a Kaehlerian Bochner bi-recurrent space.

2. Kaehlerian Conharmonic* Bi - Recurrent Spaces

Definition (2.1): A Kaehlerian space K^n satisfying

$$^{*}T^h_{ijk,ab} = \lambda_{ab}\ {}^{*}T^h_{ijk}, \qquad (2.1)$$

for some non-zero recurrence tensor field λ_{ab}, will be called a **Kaehlerian conharmonic* recurrent space of second order** or briefly a **Kaehlerian conharmonic* bi-recurrent space.**
We investigate that:

Theorem (2.1): Every Kaehlerian bi-recurrent space is Kaehlerian conharmonic* bi- recurrent.
Proof: A Kaehlerian bi-recurrent space is characterized by (1.13), which yields (1.14). By differentiating (1.12) covariently with respect to x^a and x^b, successively and using equation (1.14), we get after some simplification

$$^{*}T^h_{ijk,ab} = \lambda_{ab}\ {}^{*}T^h_{ijk}.$$

Which shows that the space is Kaehlerian conharmonic* bi-recurrent.

Theorem (2.2): If in a Kaehlerian space K^n, any two of the following properties are satisfied:

(a) The space is Kaehlerian bi-Ricci-recurrent,

(b) The space is Kaehlerian conharmonic* bi-recurrent,

(c) The space is Kaehlerian Projective bi-recurrent,

Then, it must satisfy the third.

Proof: Equation (1.12), in view of (1.10) becomes

$$^{*}T^h_{ijk} = P^h_{ijk} + \frac{1}{n-2}\left(g_{ik}R^h_j - g_{jk}R^h_i\right) - \frac{1}{(n+2)}\left(R_{ik}\delta^h_j - R_{jk}\delta^h_i + S_{ik}F^h_j - S_{jk}F^h_i + 2S_{ij}F^h_k\right). \qquad (2.2)$$

Differentiating (2.2) covariently with respect to x^a and x^b successively and using (1.3), we have

$$^{*}T^h_{ijk,ab} = P^h_{ijk,ab} + \frac{1}{n-2}\left(g_{ik}R^h_{j,ab} - g_{jk}R^h_{i,ab}\right) - \frac{1}{(n+2)}\left(R_{ik,ab}\delta^h_j - R_{jk,ab}\delta^h_i + S_{ik,ab}F^h_j - S_{jk,ab}F^h_i + 2S_{ij,ab}F^h_k\right). \qquad (2.3)$$

Now, multiplying (2.2) by λ_{ab} and subtracting the resulting equation from (2.3), we have

$$^{*}T^h_{ijk,ab} - \lambda_{ab}\ {}^{*}T^h_{ijk} = P^h_{ijk,ab} - \lambda_{ab}P^h_{ijk} + \frac{1}{(n-2)}\left[g_{ik}\left(R^h_{j,ab} - \lambda_{ab}R^h_j\right) - g_{jk}\left(R^h_{i,ab} - \lambda_{ab}R^h_i\right)\right] - \frac{1}{(n+2)}\left[\delta^h_j\left(R_{ik,ab} - \lambda_{ab}R_{ik}\right) - \delta^h_i\left(R_{jk,ab} - \lambda_{ab}R_{jk}\right) + F^h_j\left(S_{ik,ab} - \lambda_{ab}S_{ik}\right) - F^h_i\left(S_{jk,ab} - \lambda_{ab}S_{jk}\right) + 2F^h_k\left(S_{ij,ab} - \lambda_{ab}S_{ij}\right)\right]. \qquad (2.4)$$

Making use of equations (1.8), (1.14), (1.16), (2.1) and (2.4), the above theorem can be proved.

Theorem (2.3): If in a Kaehlerian space K^n, any two of the following properties are satisfied:

(a) The space is Kaehlerian bi-Ricci-recurrent,

(b) The space is Kaehlerian conharmonic* bi-recurrent,

(c) The space is Kaehlerian Bochner bi-recurrent. Then it must satisfy the third.

Proof: With the help of equations (1.11) and (1.12), we obtain

$$K_{ijk}^h = {}^*T_{ijk}^h - \frac{6}{(n-2)(n+4)}\left(g_{ik}R_j^h - g_{jk}R_i^h\right) + \frac{1}{(n+4)}\left(R_{ik}\delta_j^h - R_{jk}\delta_i^h + S_{ik}F_j^h - S_{jk}F_i^h + F_{ik}S_j^h - F_{jk}S_i^h + 2S_{ij}F_k^h + 2F_{ij}S_k^h\right) - \frac{R}{(n+2)(n+4)}\left(g_{ik}\delta_j^h - g_{jk}\delta_i^h + F_{ik}F_j^h - F_{jk}F_i^h + 2F_{ij}F_k^h\right). \quad (2.5)$$

Differentiating (2.5) covariently with respect to x^a and x^b successively and using (1.3), we obtain

$$K_{ijk,ab}^h = {}^*T_{ijk,ab}^h - \frac{6}{(n-2)(n+4)}\left(g_{ik}R_{j,ab}^h - g_{jk}R_{i,ab}^h\right) + \frac{1}{(n+4)}\left(R_{ik,ab}\delta_j^h - R_{jk,ab}\delta_i^h + S_{ik,ab}F_j^h - S_{jk,ab}F_i^h + F_{ik}S_{j,ab}^h - F_{jk}S_{i,ab}^h + 2S_{ij,ab}F_k^h + 2F_{ij,ab}S_k^h\right) - \frac{R_{,ab}}{(n+2)(n+4)}\left(g_{ik}\delta_j^h - g_{jk}\delta_i^h + F_{ik}F_j^h - F_{jk}F_i^h + 2F_{ij}F_k^h\right). \quad (2.6)$$

Multiplying (2.5) by λ_{ab} and subtracting the resulting equation from (2.6), we get

$$K_{ijk,ab}^h - \lambda_{ab}K_{ijk}^h = {}^*T_{ijk,ab}^h - \lambda_{ab}\ {}^*T_{ijk}^h - \frac{6}{(n-2)(n+4)}\left[g_{ik}\left(R_{j,ab}^h - \lambda_{ab}R_j^h\right) - g_{jk}\left(R_{i,ab}^h - \lambda_{ab}R_i^h\right)\right] + \frac{1}{(n+4)}\Big(\delta_j^h\left(R_{ik,ab} - \lambda_{ab}R_{ik}\right) - \delta_i^h\left(R_{jk,ab} - \lambda_{ab}R_{jk}\right) + F_j^h\left(S_{ik,ab} - \lambda_{ab}S_{ik}\right) - F_i^h\left(S_{jk,ab} - \lambda_{ab}S_{jk}\right) + F_{ik}\left(S_{j,ab}^h - \lambda_{ab}S_j^h\right) - F_{jk}\left(S_{i,ab}^h - \lambda_{ab}S_i^h\right) + 2F_k^h\left(S_{ij,ab} - \lambda_{ab}S_{ij}\right) + 2S_k^h\left(F_{ij,ab} - \lambda_{ab}F_{ij}\right)\Big) - \frac{\left(R_{,ab} - \lambda_{ab}R\right)}{(n+2)(n+4)}\left(g_{ik}\delta_j^h - g_{jk}\delta_i^h + F_{ik}F_j^h - F_{jk}F_i^h + 2F_{ij}F_k^h\right). \quad (2.7)$$

Making use of equations (1.8), (1.14), (1.15), (1.17) and (2.7), we can obtain the proof of the above theorem.

Theorem (2.4): A Kaehlerian conharmonic* bi-recurrent space will Kaehlerian bi-recurrent provided that it is Kaehlerian bi-Ricci-recurrent i.e.

$${}^*T_{ijk,ab}^h - \lambda_{ab}\ {}^*T_{ijk}^h = R_{ijk,ab}^h - \lambda_{ab}R_{ijk}^h.$$

Proof: Differentiating (1.12) covariantly with respect to x^a and x^bsuccessively, we get

$${}^*T_{ijk,ab}^h = R_{ijk,ab}^h + \frac{1}{(n-2)}\left[g_{ik}R_{j,ab}^h - g_{jk}R_{i,ab}^h\right]. \quad (2.8)$$

Multiplying (1.12) by λ_{ab} and subtracting the resulting equation from (2.8), we get

$${}^*T_{ijk,ab}^h - \lambda_{ab}\ {}^*T_{ijk}^h = R_{ijk,ab}^h - \lambda_{ab}R_{ijk}^h + \frac{1}{(n-2)}\left[g_{ik}\left(R_{j,ab}^h - \lambda_{ab}R_j^h\right) - g_{jk}\left(R_{i,ab}^h - \lambda_{ab}R_i^h\right)\right] \quad (2.9)$$

Let the space be Kaehlerian bi-Ricci-recurrent then, (2.9), in view of (1.8) and (1.14), yields

$${}^*T_{ijk,ab}^h - \lambda_{ab}\ {}^*T_{ijk}^h = R_{ijk,ab}^h - \lambda_{ab}R_{ijk}^h.$$

This relation shows that the Kaehlerian conharmonic* bi-recurrent space is Kaehlerian bi-recurrent.

Theorem (2.5): If in a Kaehlerian space K^n any two of the following properties are satisfied:

(a) The space is Kaehlerian bi-recurrent,

(b) The space is Kaehlerian bi-Ricci-recurrent,

(c) The space is Kaehlerian conharmonic* bi-recurrent,

Then it must satisfy the third property.

Proof: Kaehlerian bi-recurrent, Kaehlerian bi-Ricci-recurrent and Kaehlerian conharmonic* bi-recurrent space are respectively characterized by equations (1.13), (1.14) and (2.1).The statement of the theorem follows in view of equations (1.13), (1.14),(2.1) and (2.9).

Theorem (2.6): The necessary and sufficient condition that a K^n be Kaehlerian bi-Ricci-recurrent, is that

$$ {}^*T^h_{ijk,ab} - \lambda_{ab}\ {}^*T^h_{ijk} = R^h_{ijk,ab} - \lambda_{ab} R^h_{ijk}. \tag{2.10}$$

Proof: Let K^n be Kaehlerian bi-Ricci-recurrent, then the relation (1.14) is satisfied and so the equation (2.9) reduces to (2.10).

Conversely, if in a K^n, (2.10) is satisfied , then the equation (2.9) yields

$$ g_{ik}\left(R^h_{j,ab} - \lambda_{ab} R^h_j\right) - g_{jk}\left(R^h_{i,ab} - \lambda_{ab} R^h_i\right) = 0, \tag{2.11}$$

after doing some simplification in the above equation, it gives us

$$ R^h_{ijk,ab} - \lambda_{ab} R^h_{ijk} = 0 \text{ that is } R^h_{ijk,ab} = \lambda_{ab} R^h_{ijk}.$$

This shows that the space K^n is Kaehlerian bi-Ricci-recurrent,

This completes the Proof.

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IMAGE OF WRIGHT-TYPE HYPERGEOMETRIC AND MITTAG-LEFFLER FUNCTIONS UNDER A GENERALIZED W-E-K FRACTIONAL OPERATOR

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Abstract
In the present paper, the approach of the authors is based on the use of a generalized fractional calculus operator namely Wright-Erdelyi-Kober operator (W-E-K) to obtain the images of generalized special functions. Some special cases have also been discussed.

2010 Mathematics Subject Classification: 33E12, 26A33, 30C45.
Keywords: Fractional integral operators, generalized Wright-Erdelyi-Kober operator (W-E-K) and Wright type hypergeometric function.

1. Introduction

The hypergeometric functions play a very important role in solving numerous problems of mathematical physics, engineering and mathematical sciences [see, 6, 9, 10, 14, 15, 16, 17, 24, 26].
The Gauss hypergeometric function is defined [22] as

$$ {}_2F_1\ (a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} z^k, |z| < 1, c \neq 0, -1, -2, \ldots\ . \tag{1}$$

Here in Eqn. (1), the Pochhammer symbol is defined by

$$(\gamma)_n = \gamma(\gamma+1)(\gamma+2)(\gamma+3)\ldots(\gamma+n-1) \forall n = 1,2,3,\ldots \text{ and } (\gamma)_0 = 1. \tag{2}$$

Several generalizations of hypergeometric functions (1) - (2) have been made and also motivated us to make further investigations in this topic. Virchenko *et al.* [27] defined the generalized hypergeometric function ${}_2R_1^{\tau}(z)$ in a different manner, given by

$$ {}_2R_1^{\omega,\mu}(z) = \frac{\Gamma(c)\mu}{\Gamma(c-b)\,\Gamma(b)} \int_0^1 t^{\mu b-1}(1-t)^{c-b-1}(1-zt^{\omega})^{c-b-1}\,dt,$$

where, $\Re(c) > R(b) > 0.$ (3)

This is the analogue of Euler's formula for Gauss's hypergeometric functions [4].
Prajapati *et al.* [21], Shukla and Prajapati [25] and Srivastava and Tomovski [26] used the fractional calculus approach in the study of an integral operator and also generalization of the Mittag-Leffler functions [18, 25, 28]. The subject of fractional calculus [1, 2, 6, 9, 10, 14, 15, 16, 17] deals with the investigations of integrals and derivatives of any arbitrary real or complex order, by which, we unify and extend the notions of integer-order derivative and *n*-fold integrals. Kumar [11] has obtained various generalized formulae on application of multiple fractional integral operators analogous to generalized Erdelyi – Kober operators [23]. Again,

Kumar, Pathan and Kumari [13] have evaluated identities for generalized Erdelyi – Kober operators [23] and further obtained generalized results with multiple Mellin – transformations. Kumar and Kumari [12], have investigated some measures analogous to Carlson's Dirichlet Measures on using generalized Erdelyi – Kober operators [23].

The theory of fractional operators defined in Eqns. (11) to (16) has gained importance and popularity during the last four decades or so, mainly due to its vast potential of demonstrated applications in various seemingly diversified fields of science and engineering, such as fluid flow, rheology, diffusion, relaxation, oscillation, anomalous diffusion, reaction-diffusion, turbulence, diffusive transport, electric networks, polymer physics, chemical physics, electrochemistry of corrosion, relaxation processes in complex systems, propagation of seismic waves, dynamical processes in self-similar and porous structures. Recently some interesting results on fractional boundary value problems and fractional partial differential equations were also discussed by Nyamoradi *et al.* [19] and Baleanu *et al.* [1, 2].

The Mittag–Leffler function [18, 28] is a generalization of hypergeometric function which appears as solution of well-known fractional differential and integral equations representing some physical and physiological phenomena like diffusion, transport theory, probability, elasticity and control theory. The purpose of this paper is to increase the accessibility of different dimensions of fractional calculus and generalization of hypergeometric functions to the real world problems of engineering and science (see [6], [9], [15], [16], [17], [24], [26]).

2. Mathematical Preliminaries

Fox's *H*-Function: Fox has defined H-function in terms of a general Mellin-Barnes type integral. He also investigated the most general Fourier kernel associated with the *H*- function and obtained the asymptotic expansions of the kernel for large values of the argument. Fox has also derived theorems about the H - function as asymmetric Fourier kernel and established certain operational properties for this function.

The H - function is defined by Fox [5] as follows

$$H(z) = H_{p,q}^{m,n}\left[\left(z\left|\begin{matrix}(a_1,\alpha_1),\dots,(a_p,\alpha_p)\\(b_1,\beta_1),\dots,(b_q,\beta_q)\end{matrix}\right.\right)\right] = \frac{1}{2\pi i}\int_L \varphi(s) z^s ds \tag{4}$$

where, $\varphi(s) = \dfrac{\prod\limits_{j=1}^{m}\Gamma(b_j-\beta_j s)\prod\limits_{j=1}^{n}\Gamma(1-a_j+\alpha_j s)}{\prod\limits_{j=m+1}^{q}\Gamma(1-b_j+\beta_j s)\prod\limits_{j=n+1}^{p}\Gamma(a_j-\alpha_j s)}$, the point $z = 0$ is tacitly excluded,

$$\Omega = \sum_{j=1}^{n}\alpha_j - \sum_{j=n+1}^{p}\alpha_j + \sum_{j=1}^{m}\beta_j - \sum_{j=m+1}^{q}\beta_j > 0,\ |arg\ (z)| < \frac{1}{2}\Omega\pi. \tag{5}$$

Kalla et al. [7, 8] have used H – function (4) – (5) to derive various results in fractional calculus theory.

Wright-type Hypergeometric Function: The generalized form of the hypergeometric function has been investigated by Dotsenko [3] and Malovichko [14] and the series form of Eqn. (3) due to Dotsenko [3] is given by

$${}_2R_1^{\omega,\mu}(z) = {}_2R_1\,(a,b,c,\omega,\mu,z) = \frac{\Gamma(c)}{\Gamma(a)\,\Gamma(b)}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma\left(b+\frac{\omega}{\mu}n\right)}{\Gamma\left(c+\frac{\omega}{\mu}n\right)}\frac{z^n}{n!} \tag{6}$$

In 2001 Virchenko et al [27] have defined the Wright type Hypergeometric function by taking $\frac{\omega}{\mu} = \tau > 0$ in above equation (6) as

$$ {}_2R_1^{\tau}(z) = {}_2R_1\,(a,b,c,\tau,z)) = \frac{\Gamma(c)}{\Gamma(b)} \sum_{k=0}^{\infty} \frac{(a)_k \Gamma(b+\tau k)}{\Gamma(c+\tau k)} \frac{z^k}{k!}\ ;\ \tau > 0, |z| < 1. \qquad (7)$$

If $\boldsymbol{\tau}$ =1, then (7) reduces to Gauss's hypergeometric function (1) – (2).

Mittag-Leffler Functions:

The single parameter Mittag-Leffler function is defined by Mittag –Leffler ([15], [18]), as follows:

$$E_{\alpha}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(1+\alpha n)}, \text{ for } \alpha \in \mathbb{C}, \Re(\alpha) > 0. \qquad (8)$$

Its generalization with two complex parameters was introduced by Wiman [28] as follows:

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\beta+\alpha n)}, \text{ for } \alpha,\beta \in \mathbb{C}, \Re(\alpha,\beta) > 0. \qquad (9)$$

In 1971, Prabhakar [20] introduced the generalized triple parameter ML function $E_{\alpha,\beta}^{\gamma}(z)$ as follows:

$$E_{\alpha,\beta}^{\gamma}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n z^n}{\Gamma(\beta+\alpha n)}, \text{ for } \alpha,\beta,\gamma \in \mathbb{C}, \Re(\alpha,\beta,\gamma) > 0. \qquad (10)$$

3. Fractional Operators

Riemann-Liouville Fractional Operator: The Riemann-Liouville fractional integrals of arbitrary order μ for a function $f(t)$, is a natural consequence of the well-known formula (Cauchy-Dirichlet formula) that reduces the calculation of the μ- fold primitive of a function $f\,(x)$ to a single integral of convolution type (see [11, 12, 13, 24])

$$I_a^{\mu} f(x) = \frac{1}{(\mu-1)!} \int_a^x (x-y)^{\mu-1} f(y)dy\,, (x > a). \qquad (11)$$

The above integral is meaningful for any number μ provided its real part is greater than zero.

Weyl Fractional Integral Operator:

The Weyl fractional integral of f(x) of order α, is defined as (see [11, 12, 13, 24])

$W_{\infty}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^{\infty} (t-x)^{\alpha-1} f(t)\, dt\,,\ -\infty < x < \infty,$ where $\alpha \in \mathbb{C}, \Re(\alpha) > 0$, is also denoted by $I_x^{\alpha} f(x)$. (12)

Kober Fractional Integral Operator:

The Kober operator is the generalization of Riemann-Liouville and Weyl operators which was given by Saxena in (1967). These operators have been used by many authors in deriving the solution of single, dual and triple integral equations involving different special functions as their kernels. The operator is defined by (see [11, 12, 13])

$$E_{0,x}^{\alpha,\eta} f(x) = \frac{(x)^{-\eta-\alpha}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} t^{\eta} f(t)\, dt\ ,\quad \Re(\alpha) > 0. \qquad (13)$$

Erdelyi-Kober Fractional Integral Operator:

Generalization of Kober operator was introduced by Kalla and Saxena (1969) given as follows

$$I(\alpha,\eta{:}\,m)f(x) = \frac{m(x)^{-\eta-m\alpha+m+1}}{\Gamma(\alpha)} \times \left\{\int_0^x (x^m - t^m)^{\alpha-1} t^{\eta} f(t)dt\right\}, \Re(\alpha) > 0. \qquad (14)$$

Saigo Fractional Operator: Useful and interesting generalization of both the Riemann-Liouville and Erdlyi- Kober fractional integration operators is introduced by Saigo [23], in terms of Gauss's hypergeometric function as given below

Let α, β and $\eta \in \mathbb{C}$ (set of complex numbers) and let $x \in \mathbb{R}_+$, $\Re(\alpha) > 0$ and the fractional derivative of the first kind of a function

$$I_{0,x}^{\alpha,\beta,\eta} f(x) = \frac{x^{-\alpha-\beta}}{\Gamma\alpha}\int_0^x (x-t)^{\alpha-1}\, {}_2F_1\left(\alpha+\beta,-\eta\,;1-\frac{t}{x}\right) f(t)\,dt,\ \Re(\alpha)>0$$
$$= \frac{d^n}{dx^n} I_{0,x}^{\alpha+n,\beta+n,\eta-n} f(x)\,,0 < \Re(\alpha)+\eta \le 1 \ (n \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}). \tag{15}$$

Wright-Erdelyi-Kober Operators (W-E-K): (W-E-K) operators of fractional integration introduced by Kalla, Galue and Srivastava [7, 8]. For integer $m \ge 1$ and real parameters $\beta_k > 0,\ \lambda_k > 0, \delta_k \ge 0\ and\ \gamma_k$; k= 1. …. m. the multiplicity W-E-K fractional integral is defined by

$$\overline{I}\, f(z) = I_{\beta_k,\lambda_k,m}^{\gamma_k,\delta_k} f(z) = \int_0^\infty H_{m,m}^{m,0}\left[\left(\sigma \left| \begin{matrix} \left(\gamma_k+\delta_k+1-\frac{1}{\beta_k}, \frac{1}{\beta_k}\right)_{1,m} \\ \left(\gamma_k+1-\frac{1}{\lambda_k}, \frac{1}{\lambda_k}\right)_{1,m} \end{matrix}\right.\right)\right] f(z\sigma)d\sigma,$$

$if\ \sum_{k=1}^m \delta_k > 0.$

In the case, when $\forall\delta_k = 0$ and $\forall\lambda_k = \beta_k, k = 1,2,\dots.m$, then it is an identity operator : $\overline{I}\, f(z) = f(z)$.

The W-E-K fractional integral operator of the power function is given in [7, 8] as follows:

$I_{\beta_k,\lambda_k,m}^{\gamma_k,\delta_k}(z^p) = c_p z^p,$ where, $c_p = \prod_{k=1}^m \frac{\Gamma(\gamma_k+1+p/\lambda_k)}{\Gamma(\gamma_k+\delta_k+1+\frac{p}{\beta_k})}$;

including for $p = 0$, if for all $\gamma_k \ge -1$. (16)

4. Main Results

Theorem 1: For integer $m \ge 1$ and real parameters $\beta > 0,\ \lambda > 0, \delta \ge 0\ and\ \gamma$, the multiple W-E-K fractional integral operator for Wright-type hypergeometric function is as follows:

$$I_{\beta,\lambda,1}^{\gamma,\delta}\left\{ {}_2R_1(a,b,c,\tau,z)\right\} = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\ {}_3\Psi_3\left(\begin{matrix} (a,1),(b,\tau),\left(\gamma+1,\frac{1}{\lambda}\right) \\ (c,\tau),\left(\gamma+\delta+1,\frac{1}{\beta}\right),(1,1)\end{matrix} | z\right). \tag{17}$$

Proof: For $\beta > 0, \lambda > 0, \delta \ge 0$, use Eqn. (16) in left hand side of Eqn. (17), the image of a Wright-type hypergeometric function under a multiple W-E-K operator is

$$I_{\beta,\lambda,1}^{\gamma,\delta}\left\{ {}_2R_1(a,b,c,\tau,z)\right\} = \frac{\Gamma(c)}{\Gamma(b)} \sum_{k=0}^\infty \frac{(a)_k\Gamma(b+\tau k)}{\Gamma(c+\tau k)\,k!} I_{\beta,\lambda,1}^{\gamma,\delta}\{z^k\}$$

which implies that

$$I_{\beta,\lambda,1}^{\gamma,\delta}\left\{ {}_2R_1(a,b,c,\tau,z)\right\} = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\ {}_3\Psi_3\left[\left(\begin{matrix} (a,1)(b,\tau)\left(\gamma+1,\frac{1}{\lambda}\right) \\ (c,\tau)\left(\gamma+\delta+1,\frac{1}{\beta}\right)(1,1)\end{matrix}\right| z\right)\right]. \tag{18}$$

The Eqn. (18) proves the theorem 1.

Theorem 2: For integer $m \ge 1$ and real parameters $\beta > 0,\ \lambda > 0, \delta \ge 0\ and\ \gamma$, the multiple W-E-K fractional integral operator for Mittag-Leffler function is as follows:

$$I_{\beta,\lambda,1}^{\gamma,\delta}\{E^{\gamma}{}_{\alpha,\beta}(z)\} = {}_2\Psi_4\left[\left(\begin{matrix} (\gamma,1)\left(\gamma+1,\frac{1}{\lambda}\right) \\ (\gamma,0)(\beta,\alpha)\left(\gamma+\delta+1,\frac{1}{\beta}\right)(1,1)\end{matrix}\right| z\right)\right]. \tag{19}$$

Proof: For $\beta > 0, \lambda > 0, \delta \ge 0$, use Eqn. (16) in left hand side of Eqn. (19), the image of a Mittag-Leffler function under a multiple W-E-K operator is

$$I_{\beta,\lambda,1}^{\gamma,\delta}\{E^{\gamma}{}_{\alpha,\beta}(z)\} = I_{\beta,\lambda,1}^{\gamma,\delta}\left\{\sum_{n=0}^\infty \frac{(\gamma)_n}{\Gamma(n\alpha+\beta)}\frac{z^n}{n!}\right\}$$

which implies that

$$I_{\beta,\lambda,1}^{\gamma,\delta}\{E^{\gamma}{}_{\alpha,\beta}(z)\} = {}_2\Psi_4\left[\left(\begin{matrix} (\gamma,1)\left(\gamma+1,\frac{1}{\lambda}\right) \\ (\gamma,0)(\beta,\alpha)\left(\gamma+\delta+1,\frac{1}{\beta}\right)(1,1)\end{matrix}\right| z\right)\right]. \tag{20}$$

The Eqn. (20) proves the theorem 2.

5. Special Cases

1. If in equation (17), we take $\boldsymbol{\tau}=1$, $\delta = 0,$ and $\lambda = \beta,$ then we get well known result of Gauss's hypergeometric function given in Eqn. (1).

2. If in equation (18), we take $\delta = 0,$ and $\lambda = \beta,$ then we get well known results of Mittag-Leffler functions given in Eqns. (8) – (10).

Conclusion

The results proved in this paper give some contributions to the theory of the fractional calculus, especially Wright-type hypergeometric function and Mittag-Leffler function. The results proved in this paper appear to be new and likely to have useful applications to a wide range of problems of mathematics, statistics and physical sciences.

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APPLICATION OF BALANCING TRANSFORMATION TO IMAGE SECURITY

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Abstract

In this article a new kind of nonlinear transformation namely the higher-dimensional balancing transformation is introduced. The scrambling action of this transformation targeting on the phase space of the digital images is also discussed .Indeed, the application of higher dimensional balancing transformation in the storage and transportation of image information is highly helpful for the information security purpose.

2010 Mathematics Subject Classification: 11B37, 11B39, 11T71.
Keywords: Digital image; Scrambling transformations; Balancing numbers; Periodicity; Balancing matrix; Balancing transformation.

1. Introduction

Image scrambling is a useful approach to secure the image data security by crawling the image into an indiscernible appearance. A number of image scrambling methods have been developed by many researchers [3-6]. The Arnold cat map which is a chaotic map from the torus into itself is widely used for image encryption and was first demonstrated by Arnold in 1967 which is defined as a transformation $\Gamma : T^2 \to T^2$such that: $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} (mod\ N)$,

where $x, y \in \{0, 1, 2, \ldots, N-1\}$ and N is the size of a digital image [1]. Arnold transformation can be effectively used in the image scrambling because of its periodicity. In [4], Dong-xu et al. established a new digital image scrambling method based on Fibonacci numbers and studied about the matrix modulo of order n of Arnold transformation and Fibonacci transformation. In [3], Bing has studied some more properties of the period of Arnold transformation by means of introducing a new integer sequence. Zou et al. [9] introduced a subfamily of the generalized Fibonacci sequence family, called as distinguished generalized Fibonacci sequence. They also considered transformations of the members of the subfamily namely Fibonacci transformation and Lucas transformation, which were used for image scrambling. Mishra et al.[6] proposed a new spatial domain image scrambling method which is based on Fibonacci and Lucas series. This method has wide application in various spatial domain image processing techniques of data hiding and secret communications. They have also claimed that the transforms with higher periodicity may reduce the level of security as those can be decrypted by other maps even if the exact map is not known.

In this article, we introduce a new type of chaotic map based on balancing numbers which we later call as balancing Q_B-matrix transformation. Further, we claim that this transformation is used for better image security than the other transformations like Arnold, Fibonacci and Lucas transformations.

2. Balancing Q_B-matrix transformation

Balancing and their balancers are solutions of a simple Diophantine equation posed by Behera and Panda [2]. According to them, the pairs (n, r) where n is a balancing number and r is the corresponding balancer, are solutions of the Diophantine equation $1 + 2 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+r)$.
For any natural number n, all the balancing numbers generate through the recursive relation $B_{n+1} = 6B_n - B_{n-1}$ with initial values $B_0 = 0$ and $B_1 = 1$, where B_n denotes the n-th balancing number. The study of balancing sequence is quite interesting because they closely resembling some properties of natural numbers and trigonometric functions [7]. There is another way to represent balancing numbers through matrices. Ray [8] has introduced the balancing matrix $Q_B$whose entries are the first three balancing numbers 0, 1 and 6, that is $Q_B = \begin{pmatrix} 6 & -1 \\ 1 & 0 \end{pmatrix}$. He has also shown that the sequence of balancing matrices satisfies the same recurrence relation as that of balancing numbers, that is $Q_B^n = 6Q_{B-1}^n - Q_{B-2}^n$, where $Q_B^n = \begin{pmatrix} B_{n+1} & -B_n \\ B_n & -B_{n-1} \end{pmatrix}$.

Without loss of generality, we present the balancing matrix Q_B in a different way by interchanging the main diagonal elements as $Q_{B_1} = \begin{pmatrix} 0 & -1 \\ 1 & 6 \end{pmatrix}$. The matrix $Q_{B_1}^n = \begin{pmatrix} -B_{n-1} & -B_n \\ B_n & B_{n+1} \end{pmatrix}$ is so formed that its determinant is invariant without loss of generality to the Cassini formula $B_n^2 - B_{n-1}B_{n+1} = 1$ [7]. The following extensions of balancing matrices are so formed that the determinants are invariant without loss of Cassini formula

$Q_{B_2} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 6 & 0 \end{pmatrix}$; $Q_{B_3} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 6 & 0 & 0 \end{pmatrix}$; $Q_{B_4} = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 1 & 6 & 0 & 0 & 0 \end{pmatrix}$; and so on. In general for $p = 0, 1, 2, \ldots$, the $(p+1) \times (p+1)$ matrix represents the Q_{B_p} matrix whose determinant indeed 1.

In this study, authors develop a new transformation which they call as balancing transformation by replacing 2D Arnold transformation matrix in the Arnold transform by 2D balancing transform matrixQ_{B_1}. The balancing transformation is defined in the following way.

Definition 2.1. *For a given positive integer $N \geq 2$,*

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 6 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} (mod\ N),$$

where $x, y \in \{0, 1, 2, \ldots, N-1\}$ and N is the size of a digital image.

Similarly, the balancing $Q_B$transformation is defined as follows.
Definition 2.2. *For a given positive integer $N \geq 2$,*

$$\begin{pmatrix} x_1' \\ x_2' \\ \vdots \\ x_p' \end{pmatrix} \equiv Q_{B_p} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{pmatrix} (mod\ N), where\ x, y\ \in\ \{0, 1, 2, \ldots, N\ -\ 1\}.$$

2.1 Periodicity of balancing transformation

Let m_N denotes the period of the balancing $Q_B$transformation which is nothing but the smallest positive integer n such that the image p can comeback after n times transforms. The following tables show the periodicity of the balancing-Q transformation. Table-1 and Table-2 show the periods of 3-dimensional and 4-dimensional balancing Q_B-transformations for some positive integer $N\ \geq 2$.

Table 1: Periods of the 3-dimensional Arnold transformations for some *N*

N	2	3	4	5	6	7	8	9	11	12	25	50
m_N	7	13	7	31	91	21	14	39	133	91	155	1085

Table 2: Periods of the 3-dimensional balancing Q_B-transformations for some *N*

N	2	3	4	5	6	7	8	9	11	12	14	16	18	30
m_N	3	3	6	31	3	48	12	9	55	6	48	24	9	93

Table3: Periods of the 4-dimensional Arnold transformations for some *N*

N	2	3	4	5	6	7	8	9	10	11	12	25	50
m_N	7	9	7	31	63	57	14	27	217	133	63	155	1085

Table 4: Periods of the 4-dimensional balancing Q_B-transformations for some *N*

N	2	3	4	5	6	7	8	9	11	12	14	16	18	30
m_N	4	8	8	124	8	400	16	24	183	8	400	32	24	248

3. The image scrambling based on the phase spaces

It is well known that a phase space of a dynamical system is a space in which all possible states of a system are represented, with each possible state of the system corresponding to one unique point in the phase space. In a phase space, every degree of freedom or parameter of the system is represented as an axis of a multi dimensional space. Here a digital image can be regarded as a matrix. The position of an element of the matrix is the coordinates of the image pixel and the element of the matrix is the gray level of the pixel. Effective crawling is able to enhance the security of encryption algorithm.
The following transformation is called the balancing transformation based on the phase space:

$$P' = AP\ (mod\ T),$$

where

$$P' = \begin{pmatrix} p'_{11} & p'_{12} & \cdots & p'_{1n} \\ p'_{21} & p'_{22} & \cdots & p'_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ p'_{m1} & p'_{m2} & \cdots & p'_{mn} \end{pmatrix}, A = \begin{pmatrix} 0 & -1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 6 & \cdots & m \end{pmatrix};$$

is the transformation matrix in the m-dimensional balancing transformation,

$$P = \begin{pmatrix} P_{11} & P_{11} & \cdots & P_{1n} \\ P_{21} & P_{22} & \cdots & P_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ P_{m1} & P_{m2} & \cdots & P_{mn} \end{pmatrix},$$

T is the highest level of the gray level of all image elements in P and $p_{ij} \in \{0, 1, 2, \dots, T-1\}$.

The step wise encoding and decoding procedure of the proposed method is outlined as follows,

Algorithm 1 Image Encoding
1. Consider an image.
2. Balancing transformation is applied on the input image to obtain the temporary image.
3. If temporary image is the original image, then display the temporary image and its period.
4. Choose an encoding image, whose iteration lies in between original image and its period.

Algorithm 2 Image Decoding
1. Input encrypted image (Temporary Image = Encrypted Image).
2. Multiply temporary image with inverse balancing matrix.
3. Take modulus of output image of step-2 with T and take it as temporary image.
4. Continue step-2 and step-3 until final image number reached.
5. Show the final image.

The following example shows the whole process.

Example 3.1. *Consider the original images Fig. 1(a) and Fig. 2(a) of size* 204×204 *and implemented these figures in Matlab environment. Fig. 1(b), 1(c) and Fig. 2(b), 2(c) represent the resulted encrypted images after applying balancing transformation with number of iterations 203, 422 and 255, 460, respectively. To decryption of the original version of encrypted image can be obtained by multiplying the inverse balancing matrix with the encrypted matrix.*


Fig. 1(a)


Fig. 1(b)

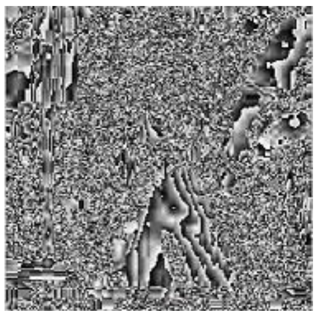
Fig. 1(c)


Fig. 2(a)


Fig. 2(b)

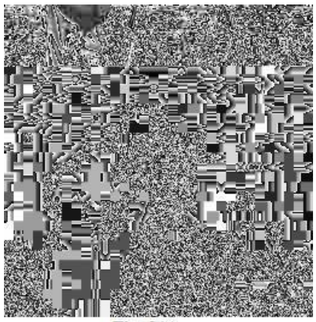
Fig. 2(c)

Conclusion

It is experimentally proved that the transformation with higher periodicity may reduce the level of security as those can be decrypted by other maps even if the exact map is not known [6]. Based on this argument, since balancing transform has lower periodicity, the level of security is high in the image scrambling as compared to Arnold transformation, Fibonacci transformation and Lucas transformation.

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SOME TRANSFORMATION FORMULAE OF RAMANUJAN'S THIRD ORDER AND SIXTH ORDER MOCK THETA FUNCTIONS

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Abstract

In this paper, we define the (p,q) analogues of Ramanujan's mock theta functions of third and sixth order and further generalize the corresponding mock theta functions by adding variables. It is observed that these generalized functions belong to the family of $F_{p,q}$-functions. We have established some transformation formulae of (p,q) −analogues of generalized mock theta functions of third and sixth order by relating them to the (p,q) series. This provides a new insight in the study of third and sixth order mock theta functions.

2010 AMS Subject Classification: 33D45, 33D99.

Keywords: Mock theta functions, Ramanujan theta functions, Twin basic number, (p,q)-series.

1. Introduction

The mock theta functions were Ramanujan's last gift to the world of Mathematics. In his last letter to G.H. Hardy [9, pp. 354-355], Ramanujan mentioned that he had discovered seventeen functions, having peculiar properties which were not exhibited by the usual theta functions. Hence he named these functions 'mock theta functions' and divided them into three classes; four functions of order three, ten functions in two groups , each containing five functions of order five, and three functions of order seven.

The discovery of these functions came to light only in 1935 when Watson mentioned them in his celebrated presidential address delivered at the meeting of London mathematical society. Watson[12] added three more functions to the list of four functions of order 3 given by Ramanujan. Later, some more mock theta functions were discovered by Gordon and McIntosh[7] .

Andrews and Hickerson [3] in 1991 in a long paper pointed out the existence of seven more q-series and some identities given in the 'Lost' notebook and called them mock theta functions of order six, on the basis of certain combinatorial considerations.

The (p,q) number, introduced by Chakrabarty and Jagannathan [14] is a generalization of Heine q- number which occurs in the theory of quantum algebras. Surprisingly, Wachs and White [17] also independently introduced the (p,q) analogue of q number during the process of generalization of Sterling numbers [18], around the same time as [14]. Some applications of the (p,q) hypergeometric series in the context of representations of two-parameter quantum groups have been considered by Nishizawa [15] and Sahai and Srivastava [16].

The main purpose of the present paper is to generalize Ramanujan's mock theta functions to their corresponding (p,q) analogues and to relate them to (p,q) series. In section 2, we define $F_{p,q}$ functions which are extensions of F_q function and also introduce (p,q) analogues of mock theta functions of third and sixth order respectively.

In section 3, we establish that (p,q) analogues of generalized third and sixth order mock theta functions are $F_{p,q}$-functions. In section 4, we derive transformation formulae of (p,q) analogues of generalized mock theta functions of third and sixth order. We also establish some formulae between two generalized mock theta functions of the same group.

Mathematical Preliminaries

We first present the definitions and notations employed in this paper.

The (p,q) −number is defined by [14]

$$[n]_{p,q} = \frac{P^n - q^n}{p-q} \tag{1.1}$$

The q –shifted factorial [6] is given by

$$(a;q)_n = \begin{cases} 1, & n = 0 \\ (1-a)(1-aq)\dots(1-aq^{n-1}) & \\ & n = 1,2,\dots \end{cases} \tag{1.2}$$

with

$$(a_1, a_2, \dots, a_k; q)_n = (a_1;q)_n (a_2;q)_n \dots (a_k;q)_n\ . \tag{1.3}$$

The basic hypergeometric series is defined by [6]

$$ {}_r\Phi_s\ (a_1, a_2 \dots, a_r; b_1, b_2, \dots, b_s;\ q, z)$$
$$= \sum_{n=0}^{\infty} \frac{(a_1,a_2,\dots,a_r;q)_n \left((-1)^n q^{\frac{n(n-1)}{2}}\right)^{1+s-r} z^n}{(b_1,b_2,\dots,b_s;q)_n (q;q)_n}\ . \tag{1.4}$$

Twin basic analogues of (1.2) and (1.3) are defined as follows [13]

$$\big((a,b);(p,q)\big)_n = \begin{cases} 1, & n = 0 \\ (a-b)(ap-bq)(ap^2-bq^2)\dots(ap^{n-1}-bq^{n-1}) & \\ & n = 1,2,\dots \end{cases} \tag{1.5}$$

$$\Big((a_{1p},a_{1q}),(a_{2p},a_{2q}),\dots,(a_{mp},a_{mq});(p,q)\Big)_n$$

$$= \Big((a_{1p},a_{1q});(p,q)\Big)_n \Big((a_{2p},a_{2q});(p,q)\Big)_n \dots \Big((a_{mp},a_{mq});(p,q)\Big)_n\ . \tag{1.6}$$

Particularly, putting $a = 1$ and $p = 1$ in equation (1.5), it becomes equivalent to the equation (1.2), in view of the relation

$$\big((1,b);(1,q)\big)_n = (b;q)_n \tag{1.7}$$

Then (p,q) analogue of (1.4) or the twin basic hypergeometric series has been defined as [13]

$$ {}_r\Phi_s\ \Big((a_{1p},a_{1q}),\dots,(a_{rp},a_{rq});\ (b_{1p,}b_{1q}),\dots,(b_{sp},b_{sq});(p,q),z\Big)$$

$$= \sum_{n=0}^{\infty} \frac{\big((a_{1p},a_{1q}),\dots,(a_{rp},a_{rq});(p,q)\big)_n}{\big((b_{1p},b_{1q}),\dots,(b_{sp},b_{sq});(p,q)\big)_n \big((p,q);(p,q)\big)_n} \times \big((-1)^n (q/p)^{n(n-1)/2}\big)^{1+s-r} z^n\ . \tag{1.8}$$

where $|q\ /\ p| < 1$, assuming that $0 < q < p \leq 1$. The numbers p and q can also take other values if there is no problem with the convergence of the particular series involved in a result.

Definition of third order mock theta functions and sixth order mock theta functions

The sixth order mock theta functions of Ramanujan [3] are

$\Phi(q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2} (q;q^2)_n}{(-q)_{2n}}$,

$$\Psi(q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{(n+1)^2} (q;q^2)_n}{(-q)_{2n+1}},$$

$\rho(q) = \sum_{n=0}^{\infty} \frac{q^{\frac{n(n+1)}{2}} (-q)_n}{(q;q^2)_{n+1}}$,

$$\sigma(q) = \sum_{n=0}^{\infty} \frac{q^{\frac{(n+1)(n+2)}{2}} (-q)_n}{(q;q^2)_{n+1}},$$

$\lambda(q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^n (q;q^2)_n}{(-q)_n}$,

$\mu(q) = \sum_{n=0}^{\infty} \frac{(-1)^n (q;q^2)_n}{(-q)_n}$,

and

$\gamma(q) = \sum_{n=0}^{\infty} \frac{q^{n^2} (q)_n}{(q^3;q^3)_n}$.

The third order mock theta functions of Ramanujan [9,12] are

$f(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q)_n^2}$,

$$\phi(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q^2;q^2)_n},$$

$\psi(q) = \sum_{n=1}^{\infty} \frac{q^{n^2}}{(q;q^2)_n}$,

$$\chi(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1-q+q^2)\ldots(1-q^n+q^{2n})},$$

$$\omega(q) = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q;q^2)_{n+1}^2},$$

$\nu(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(-q;q^2)_{n+1}}$,

and

$\rho(q) = \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(1+q+q^2)\ldots(1+q^{2n+1}+q^{4n+2})}$.

In next section we shall define their (p,q) analogues.

2. Definition of $F_{p,q}$ functions

The functions which satisfy the functional equation

$$D_{p,q,z}\,F(z,\alpha,\beta) = F(z,\alpha+1,\beta+1), \tag{2.1}$$

where

$$zD_{p,q,z}\,F(z,\alpha,\beta) = F(z,\alpha,\beta) - F(zpq,\alpha,\beta)$$

are called $F_{p,q}$ functions.

Definition of (p,q) analogue of third order mock theta function and sixth order mock theta function:

We define (p,q) analogues of third order mock theta functions as

$$f(p,q) = \sum_{n=0}^{\infty} \frac{p^{n^2}q^{n^2}}{\left((1,-q);(1,q)\right)_n^2}, \tag{2.2}$$

$$\phi(p,q) = \sum_{n=0}^{\infty} \frac{p^{n^2}q^{n^2}}{\left((1,-q^2);(1,q^2)\right)_n}, \tag{2.3}$$

$$\psi(p,q) = \sum_{n=1}^{\infty} \frac{p^{n^2}q^{n^2}}{\left((1,q);(1,q^2)\right)_n}, \tag{2.4}$$

$$\omega(p,q) = \sum_{n=0}^{\infty} \frac{p^{2n(n+1)}q^{2n(n+1)}}{\left((1,q);(1,q^2)\right)_{n+1}^2}, \tag{2.5}$$

$$\nu(p,q) = \sum_{n=0}^{\infty} \frac{p^{n(n+1)}q^{n(n+1)}}{\left((1,-q);(1,q^2)\right)_{n+1}}. \tag{2.6}$$

The (p,q) analogue of functions $\chi(q)$ and $\rho(q)$ have not been considered as they take the form of Ramanujan functions $\chi(q)$ and $\rho(q)$ respectively on taking $pq = r$, $/r/ < 1$. Thus the $\chi(p,q)$ and $\rho(p,q)$ are essentially the same as $\chi(q)$ and $\rho(q)$.

We further generalize third order (p,q) mock theta functions by adding independent variables and parameters as follows:

$$f\big(t,\alpha,\beta,\delta,\varepsilon,z,x;(p,q)\big) = \frac{1}{(t)_\infty}\sum_{n=0}^{\infty} \frac{(t)_n p^{n^2-4n+n\delta} q^{n^2-4n+n\beta} \alpha^n \varepsilon^n z^{2n} x^{2n}}{((1,-z);(1,q))_n \left(\left(1,-\frac{\alpha z}{q}\right);(1,q)\right)_n}, \tag{2.7}$$

$$\phi\big(t,\alpha,\beta,\delta,z,x;(p,q)\big) = \frac{1}{(t)_\infty}\sum_{n=0}^{\infty} \frac{(t)_n p^{n^2-3n+n\delta} q^{n^2-3n+n\beta} z^{2n} x^{2n}}{\left(\left(1,-\frac{\alpha z^2}{q}\right);(1,q^2)\right)_n}, \tag{2.8}$$

$$\psi\big(t,\alpha,\beta,\delta,z,x;(p,q)\big) = \frac{1}{(t)_\infty}\sum_{n=0}^{\infty} \frac{(t)_n p^{n^2-n+n\delta} q^{n^2-n+n\beta} z^{2n+1} x^{2n+1}}{\left(\left(1,\frac{\alpha z^2}{q^2}\right);(1,q^2)\right)_{n+1}}, \tag{2.9}$$

$$\nu\big(t,\alpha,\beta,\delta,z,x;(p,q)\big) = \frac{1}{(t)_\infty}\sum_{n=0}^{\infty} \frac{(t)_n\, q^{n^2-2n+n\beta} z^{2n} p^{n^2-2n+n\delta} x^{2n}}{\left(\left(1,-\frac{\alpha^2 z^2}{q^3}\right);(1,q^2)\right)_{n+1}}, \tag{2.10}$$

$$\omega(t,\alpha,\beta,\delta,\varepsilon,z,x;(p,q) = \frac{1}{(t)_\infty}\sum_{n=0}^{\infty} \frac{(t)_n p^{2n^2-5n-4+n\delta} x^{4(n+1)} q^{2n^2-5n+n\beta-4} \alpha^{2n} z^{4(n+1)} \varepsilon^{2n}}{\left(\left(1,\frac{z^2}{q}\right);(1,q^2)\right)_{n+1} \left(\left(1,\frac{\alpha^2 z^2}{q^3}\right);(1,q^2)\right)_{n+1}}. \tag{2.11}$$

Setting $t = 0, \alpha = q, \beta = 1, \delta = 1, z = q, x = p, \varepsilon = p$ these generalized twin basic functions reduce to corresponding third order (p,q) mock theta functions.

Similarly (p,q) −analogues of seven sixth order mock theta function can be defined as follows:

$$\Phi(p,q)=\sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2}p^{n^2}((1,q);(1,q^2))_n}{((1,-q);(1,q))_{2n}}, \tag{2.12}$$

$$\Psi(p,q)=\sum_{n=0}^{\infty}\frac{(-1)^n p^{(n+1)^2}q^{(n+1)^2}((1,q);(1,q^2))_n}{((1,-qp^{2n});(1,q))_{2n+1}}, \tag{2.13}$$

$$\rho(p,q)=\sum_{n=0}^{\infty}\frac{p^{\frac{n(n+1)}{2}}q^{\frac{n(n+1)}{2}}((1,-q);(1,q))_n}{((1,q);(1,q^2))_{n+1}}, \tag{2.14}$$

$$\sigma(p,q)=\sum_{n=0}^{\infty}\frac{p^{\frac{(n+1)(n+2)}{2}}q^{\frac{(n+1)(n+2)}{2}}((1,-q);(1,q))_n}{((1,q);(1,q^2))_{n+1}}, \tag{2.15}$$

$$\lambda(p,q)=\sum_{n=0}^{\infty}\frac{(-1)^n p^n q^n((1,q);(1,q^2))_n}{((1,-q);(1,q))_n}, \tag{2.16}$$

$$\mu(p,q)=\sum_{n=0}^{\infty}\frac{(-1)^n((1,q);(1,q^2))_n}{((1,-q);(1,q))_n}, \tag{2.17}$$

$$\gamma(p,q)=\sum_{n=0}^{\infty}\frac{p^{n^2}q^{n^2}((1,q);(1,q))_n}{((1,q^3);(1,q^3))_n}. \tag{2.18}$$

Next we define further generalization of seven sixth order (p,q) −mock theta functions as follows :

$$\Phi(t,\alpha,\beta,z,x;(p,q))=\frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n(-1)^n q^{n(n-3)+n\beta}p^{n(n-3)+n\alpha}z^{2n}x^{2n}\left(\left(1,\frac{z^2}{q}\right);(1,q^2)\right)_n}{\left(\left(1,\frac{-z^2}{q}\right);(1,q)\right)_{2n}}, \tag{2.19}$$

$$\Psi(t,\alpha,\beta,z,x;(p,q))=\frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n(t)_n\, p^{n(n-1)+n\alpha}q^{n(n-1)+n\beta}z^{2n+1}x^{2n+1}\left(\left(1,\frac{z^2}{q}\right);(1,q^2)\right)_n}{\left(\left(1,\frac{-z^2p^{2n}}{q}\right);(1,q)\right)_{2n+1}}, \tag{2.20}$$

$$\rho(t,\alpha,\beta,z,x;(p,q))=\frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n p^{\frac{n(n-3)}{2}+n\alpha}q^{\frac{n(n-3)}{2}+n\beta}z^n x^n((1,-z);(1,q))_n}{\left(\left(1,\frac{z^2}{q}\right);(1,q^2)\right)_{n+1}}, \tag{2.21}$$

$$\sigma((t,\alpha,\beta,z,x;(p,q))=\frac{1}{2(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n\, p^{\frac{n(n-1)}{2}+n\alpha}z^{n+1}\left(\left(1,\frac{-z}{q}\right);(1,q)\right)_{n+1}q^{\frac{n(n-1)}{2}+n\beta}x^{n+1}}{\left(\left(1,\frac{z^2}{q}\right);(1,q^2)\right)_{n+1}}, \tag{2.22}$$

$$\lambda(t,\alpha,\beta,z,x;(p,q))=\frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n(-1)^n p^{n\alpha}q^{n\beta}\left(\left(1,\ \frac{q^3}{z^2}\right);(1,q^2)\right)_n}{\left(\left(1,\frac{-q^2}{z}\right);(1,q)\right)_n}, \tag{2.23}$$

$$\mu(t,\alpha,\beta,z,x;(p,q))=\frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n(-1)^n\, p^{n(\alpha-1)}q^{n(\beta-1)}\left(\left(1,\frac{q^3}{z^2}\right);(1,q^2)\right)_n}{\left(\left(1,\frac{-q^2}{z}\right);(1,q)\right)_n}, \tag{2.24}$$

$$\gamma(t,\alpha,\beta,z,x;(p,q)) = \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n\, p^{n(n-3)+n\alpha}q^{n(n-3)+n\beta}z^{2n}x^{2n}}{((1,\nu^2 z);(1,q))_n\big((1,\nu^4 z);(1,q)\big)_n}, \tag{2.25}$$

where $\nu = e^{\frac{\pi i}{3}}$.

Putting $t = 0, \alpha = 1, \beta = 1, z = q$ and $x = p$ these functions take the form of (p,q) −analogues of sixth order mock theta functions.

3. Generalized (p,q) mock theta functions are $F_{p,q}$ –functions:

In this section, we shall show that our generalized (p,q) −mock theta functions are indeed $F_{p,q}$ –functions.

Here we give the detailed proof for the function $\Phi((t,\alpha,\beta,z,x;(p,q))$ only; the proofs for other functions are similar and hence omitted.

Theorem 1 : The generalized third and sixth order mock theta functions defined by (2.7) to (2.11) and (2.19) to (2.25) are $F_{p,q}$ functions.

Proof: We consider the generalized mock theta function $\Phi((t,\alpha,\beta, z, x;(p, q))$.

Applying the difference operator $D_{p,q,t}$ *to* $\Phi((t,\alpha,\beta,z, x;(p, q))$ we have

$$tD_{p,q,t}\,\Phi((t,\alpha,\beta,z, x;(p, q)) = \Phi((t,\alpha,\beta,z, x;(p, q)) - \Phi((tpq,\alpha,\beta,z, x;(p, q))$$

$$= \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n(t)_n\, p^{n(n-3)+n\alpha}q^{n(n-3)+n\beta}x^{2n}z^{2n}\left(\left(1,\frac{z^2}{q}\right);(1,q^2)\right)_n}{\left(\left(1,\frac{-z^2}{q}\right);(1,q)\right)_{2n}}$$

$$-\frac{1}{(tpq)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n(tpq)_n\, p^{n(n-3)+n\alpha}q^{n(n-3)+n\beta}z^{2n}x^{2n}\left(\left(1,\frac{z^2}{q}\right);(1,q^2)\right)_n}{\left(\left(1,\frac{-z^2}{q}\right);(1,q)\right)_{2n}}$$

$$= \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n(t)_n\, p^{n(n-3)+n\alpha}q^{n(n-3)+n\beta}x^{2n}z^{2n}\left(\left(1,\frac{z^2}{q}\right);(1,q^2)\right)_n}{\left(\left(1,\frac{-z^2}{q}\right);(1,q)\right)_{2n}}$$

$$-\frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n(t)_n\, p^{n(n-3)+n\alpha}z^{2n}x^{2n}q^{n(n-3)+n\beta}\left(\left(1,\frac{z^2}{q}\right);(1,q^2)\right)_n(1-tp^nq^n)}{\left(\left(1,\frac{-z^2}{q}\right);(1,q)\right)_{2n}}$$

$$= \frac{t}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n(t)_n\, p^{n(n-3)+n(\alpha+1)}q^{n(n-3)+n(\beta+1)}x^{2n}z^{2n}\left(\left(1,\frac{z^2}{q}\right);(1,q^2)\right)_n}{\left(\left(1,\frac{-z^2}{q}\right);(1,q)\right)_{2n}}$$

$= t\Phi((t, \alpha+1, \beta+1, z, x;(p, q))$.

So $D_{p,q,t}\,\Phi((t,\ \alpha,\ \beta,\ z,\ x\ ;\ (p,\ q)) = \Phi((t,\ \alpha+1,\ \beta+1,\ z,\ x;\ (p,\ q)).$

Hence $\Phi((t,\alpha,\beta,z,x;(p,q))$ is a $F_{p,q}$ -function.

Similarly all other functions in theorem 1 can be shown to be $F_{p,q}$ functions.

4. Transformation Formulae of Generalized (p,q) Mock Theta Functions:

In this section we shall establish some transformation formulae to show that a generalized (p,q) mock theta functions can be expressed in terms of another (p,q) mock theta functions.

Theorem : 2

$(i)\,\Phi\big(t,\alpha,\beta,z,x;(p,q)\big)$

$$= \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n(t)_n p^{n(n-3)+n\alpha}x^{2n}z^{2n}\ q^{n(n-3)+n\beta}\left(\left(1,\frac{z^2}{q}\right);(1,q^2)\right)_n}{\left(\left(1,\frac{-z^2p^{2n}}{q}\right);(1,q)\right)_{2n+1}}$$

$$+\frac{z}{qx}\ \Psi\big(t,\alpha,\beta,z,x;(p,q)\big),$$

$$(ii)\,\sigma\big(t,\alpha,\beta,z,x;(p,q)\big) = \frac{zx}{2}\left(1+\frac{z}{q}\right)D_{p,q,t}\ \rho\big(t,\alpha,\beta,z,x;(p,q)\big),$$

$$(iii)\,D_{p,q,t}\,\Phi\big(t,\alpha^2,\beta,\delta,z,x;(p,q)\big) = \left(1+\frac{\alpha^2z^2}{q^3}\right)\nu\big(t,\alpha,\beta,\delta,z,x;(p,q)\big),$$

$$(iv)\,\psi\left(t,\frac{-\alpha^2}{q},\beta,\delta,z,x;(p,q)\right) = xz\ D_{p,q,t}\ \nu\big(t,\alpha,\beta,\delta,z,x;(p,q)\big).$$

Proof of (i)

$\Phi\big(t,\alpha,\beta,z,x;(p,q)\big)$

$$= \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n(t)_n\ p^{n(n-3)+n\alpha}\ q^{n(n-3)+n\beta}x^{2n}z^{2n}\left(\left(1,\frac{z^2}{q}\right);(1,q^2)\right)_n(1+z^2p^{2n}q^{2n-1})}{\left(\left(1,\frac{-z^2p^{2n}}{q}\right);(1,q)\right)_{2n+1}}$$

$$= \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n(t)_n\ p^{n(n-3)+n\alpha}\ q^{n(n-3)+n\beta}x^{2n}z^{2n}\left(\left(1,\frac{z^2}{q}\right);(1,q^2)\right)_n}{\left(\left(1,\frac{-z^2p^{2n}}{q}\right);(1,q)\right)_{2n+1}}$$

$$+\frac{z}{qx}\frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n(t)_n\ p^{n(n-1)+n\alpha}\ q^{n(n-1)+n\beta}z^{2n+1}x^{2n+1}\left(\left(1,\frac{z^2}{q}\right);(1,q^2)\right)_n}{\left(\left(1,\frac{-z^2p^{2n}}{q}\right);(1,q)\right)_{2n+1}}$$

$$= \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n (t)_n\, p^{n(n-3)+n\alpha}\, q^{n(n-3)+n\beta} x^{2n} z^{2n}\left(\left(1,\frac{z^2}{q}\right);(1,q^2)\right)_n}{\left(\left(1,\frac{-z^2p^{2n}}{q}\right);(1,q)\right)_{2n+1}} + \frac{z}{qx}\ \Psi\big(t,\alpha,\beta,x,z;(p,q)\big),$$

which proves Theorem 2(i).

Proof of (ii)

$$\sigma\big(t,\alpha,\beta,z,x;(p,q)\big) = \frac{1}{2(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n\, p^{\frac{n(n-1)+n\alpha}{2}}\, z^{n+1}\, q^{\frac{n(n-1)+n\beta}{2}}\, x^{n+1}\left(\left(1,\frac{-z}{q}\right);(1,q)\right)_{n+1}}{\left(\left(1,\frac{z^2}{q}\right);(1,q^2)\right)_{n+1}}$$

$$= \frac{1}{2(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n\, p^{\frac{n^2-n+2n\alpha}{2}}\, z^{n+1}\, q^{\frac{n^2-n+2n\beta}{2}} x^{n+1}\left(\left(1,\frac{-z}{q}\right);(1,q)\right)_{n+1}}{\left(\left(1,\frac{z^2}{q}\right);(1,q^2)\right)_{n+1}}$$

$$= \frac{zx}{2(t)_\infty}\left(1+\frac{z}{q}\right)\sum_{n=0}^{\infty}\frac{p^{\frac{n(n-1)+n\alpha}{2}}\, q^{\frac{n(n-1)+n\beta}{2}}\, z^n\,\big((1,-z);(1,q)\big)_n x^n}{\left(\left(1,\frac{z^2}{q}\right);(1,q^2)\right)_{n+1}}$$

$$= \frac{zx}{2}\left(1+\frac{z}{q}\right)\, D_{p,q,t}\,\rho\big(t,\alpha,\beta,z,x;(p,q)\big),$$

which proves theorem 2 (ii).

Proof of (iii)

Replacing α by α^2 in $\Phi(\big(t,\alpha,\beta,\delta,z,x;(p,q)\big)$, we have

$$D_{p,q,t}\,\Phi(\big(t,\alpha^2,\beta,\delta,z,x;(p,q)\big) = \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n\, p^{n^2-2n+n\alpha}\quad q^{n^2-2n+n\beta)} z^{2n} x^{2n}}{\left(\left(1,\frac{-\alpha^2 z^2}{q}\right);(1,q^2)\right)_n}$$

$$= \left(1+\frac{\alpha^2 z^2}{q^3}\right)\nu\,(\big(t,\alpha,\beta,\delta,z,x;(p,q)\big),$$

which proves theorem 2 (iii).

Proof of (iv)

$$\psi\big(t,\alpha,\beta,\delta,z,x;(p,q)\big) = \frac{1}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(t)_n\, p^{n^2-n+n\delta}\, q^{n^2-n+n\beta} z^{2n+1} x^{2n+1}}{\left(\left(1,\frac{\alpha z^2}{q^2}\right);(1,q^2)\right)_{n+1}},$$

Replacing α by $\frac{-\alpha}{q}$ and then writing α^2 for α in $\psi\big(t,\alpha,\beta,\delta,z,x;(p,q)\big)$,

$$\psi\left(t, \frac{-\alpha^2}{q}, \beta, \delta, z, x; (p,q)\right) = \frac{zx}{(t)_\infty} \sum_{n=0}^{\infty} \frac{(t)_n \, p^{n^2-n+n\delta} \, q^{n^2-n+n\beta} z^{2n} x^{2n}}{\left(\left(1, \frac{-\alpha^2 z^2}{q^3}\right); (1, q^2)\right)_{n+1}}$$

$= xz \, D_{p,q,t} \, \nu\left(\left(t, \alpha, \beta, \delta, z, x; (p,q)\right)\right),$

which proves theorem 2(iv).

Conclusion

In this paper we have defined and further generalized (p,q) analogues of Ramanujan's third and sixth order mock-theta functions and have shown that these functions are $F_{p,q}$ functions. Subsequently we have established some transformation formulae between generalized functions of same order. These results are quite general in nature and reduce to earlier established relations on specializing the parameters.

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On a new theorem involving a product of some special functions

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Abstract

In this paper, we first establish an interesting theorem by making use of the well-known Orr theorem. Applications of this theorem to obtain certain integral and expansion formula have also been recorded in the elegant manner. Certain special cases of the main theorem are also given.

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1. Introduction

The function ${}_2F_1\,(a,\ b;\ c;\ z)$ is termed or named or known as Gauss function or simply hypergeometric function is represented as

$$ {}_2F_1\,(a, b; c; z) \;=\sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n\, n\,!} z^n, \tag{1}$$

where, c is not zero or negative integer. The series in (4.2.3) converges for $|\,z\,| < 1$.

On the boundary $|\,z\,| = 1$ of the region of convergence, a sufficient condition for absolute convergence of the series is Re $(c\text{-}a\text{-}b) > 0$.

In our investigations uses that

If

$$(1-z)^{a+b+c}\ {}_2F_1(\,2a,\ 2b;\ 2c\ ;\ z)=\sum_{n=0}^{\infty} a_n\, z^n \qquad , \tag{2}$$

then

$$ {}_2F_1\left(a, b\ ; c\ +1/2;z\right){}_2F_1\left(c-a, c-b\ ; c\ +1/2;z\right)$$
$$=\sum_{n=0}^{\infty} \frac{(c)_n a_n}{\left(c+1/2\right)_n} z^n \tag{3}$$

Srivastava ([12], Eqn. (1), p. 1] introduced the general class of polynomials

$$S_n^m[x]\ =\ \sum_{\ell=0}^{[n/m]} \frac{(\,-n)_{m\ell}}{\ell\,!} A_{n,\ell}\ , \qquad \ell=0,1,2,... \tag{4}$$

where m is an arbitrary positive integer and the coefficients $A_{n,\ell}(n,\ell \geq 0)$ are arbitrary constants, real or complex. On suitably specialization of the coefficients $A_{n,\ell}$, $S_n^m[X]$ yields a number of known polynomials as its special cases. These include, among others, the Hermite polynomials, the Jacobi polynomials, the Lagurre polynomials, the Bessel polynomials, the Gould-Hopper polynomials, the Brafman polynomials and several others ([13], *p*.158-161).

The $\bar{H}$ -function introduced by Inayat-Hussain [5, 6] is a generalization of the well known Fox *H*-function [4].

It is defined and represented in the following manner:

$$\overline{H}_{P,Q}^{M,N}\left[z \left| \begin{matrix} (a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P} \\ (b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q} \end{matrix} \right. \right]$$

$$= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \Phi(\xi)\, z^{\xi}\, d\xi\,, \tag{5}$$

where,

$$\Phi(\xi) = \frac{\prod_{j=1}^{M} \Gamma(b_j-\beta_j\xi)\prod_{j=1}^{N}\{\Gamma(1-a_j+\alpha_j\xi)\}^{A_j}}{\prod_{j=M+1}^{Q}\{\Gamma(1-b_j+\beta_j\xi)\}^{B_j}\prod_{j=N+1}^{P}\Gamma(a_j-\alpha_j\xi)}, \tag{6}$$

which contains fractional powers of some of the Gamma functions. Here and throughout the paper a_j $(j=1,\ldots,P)$ and $b_j(j = 1,\ldots,Q)$ are complex parameters, $\alpha_j \geq 0 (j = 1,\ldots P), \beta_j \geq 0 (j = 1,\ldots Q)$ not all zero simultaneously) and the exponent A_j $(j = 1,\ldots,N)$ and B_j $(j=M+1,\ldots,Q)$ can take on non-integer value. The contour in (5) is imaginary axis $Re(\xi) = 0$. It is suitably indented in order to avoid the singularities of the Gamma functions and those singularities on appropriate side. Again, for A_j $(j = 1,\ldots,N)$ not an integer, the poles of Gamma functions of the numerator in (6) are converted to the branch points. However, as long as there is no coincidence of poles from any $\Gamma(b_j-\beta_j\xi)(j = 1,\ldots,M)$ *and* $\Gamma\ 1-a_j+\alpha_j\xi)(j = 1,\ldots,N)$ pair, the branch cuts have been chosen so that the path of integration can be distorted in the usual manner. The following sufficient conditions for the absolute convergence of the defining integral for $\overline{H}$ -function given by equation (5) have been given by Buschman and Srivastava [1].

$$T = \sum_{j=1}^{M}\beta_j + \sum_{j=1}^{N}|A_j\alpha_j| - \sum_{j=M+1}^{Q}|B_j\beta_j| - \sum_{j=N+1}^{P}\alpha_j > 0, \tag{7}$$

and

$$|\arg z| < \frac{1}{2}T\pi\,. \tag{8}$$

On suitably specialization of the parameters involved in $\overline{H}$-function a large number of special functions have obtained.

The series representation of $\overline{H}$ -function [7] is as follows:

$$\overline{H}_{P,Q}^{M,N}[z] = \overline{H}_{P,Q}^{M,N}\left[z \left| \begin{matrix} (a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P} \\ (b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+!,Q} \end{matrix} \right. \right]$$

$$= \sum_{g=1}^{M}\sum_{k=0}^{\infty}\frac{(-1)^k\,\overline{\varphi}(\xi_{g,k})\,z^{\xi_{g,k}}}{k!\ \beta_g}, \tag{9}$$

where

$$\overline{\varphi}(\xi_{g,k}) = \frac{\prod_{\substack{j=1\\ j\neq g}}^{M}\Gamma(b_j-\beta_j\xi_{g,k})\prod_{j=1}^{N}\{\Gamma\Gamma 1-a_j+\alpha_j\xi_{g,k})\}^{A_j}}{\prod_{j=M+1}^{Q}\{\Gamma\Gamma 1-b_j+\beta_j\xi_{g,k})\}^{B_j}\prod_{j=N+1}^{P}\Gamma(a_j-\alpha_j\xi_{g,k})}, \tag{10}$$

and $\xi_{g,k} = \dfrac{b_g + k}{\beta_g}$. (11)

The *M*-series introduced by Sharma ([8], p.188, eqn. (3)) is defined as follows

$$ {}_P\overset{\alpha}{M}_Q\, [y] = {}_P\overset{\alpha}{M}_Q \left(\begin{matrix} a_1,\ldots,a_P,\alpha\,; \\ b_1,\ldots,b_Q\,; \end{matrix}\; y \right) $$

$$ = \sum_{k=0}^{\infty} \frac{(a_1)_k \ldots (a_P)_k}{(b_1)_k \ldots (b_Q)_k} \frac{y^k}{\Gamma(\alpha k + a)}, \tag{12} $$

here $\alpha \in C, \mathrm{Re}(\alpha) > 0$ *and* $(a_j)_k, (b_j)_k$ are the Pochhammer symbols. The series in (12) is defined when none of the parameters $b_j's$, j = 1,…,*Q*, is a negative integer or zero. If any numerator parameter a_j is a negative integer or zero, then the series terminates to a polynomial in *y*. From the ratio that it is evident that the series in (12) is convergent for all *z* if $P \le Q$, it is convergent for if *P* = *Q* +1 and divergent, if *P* > *Q*+1. When *P* = *Q* +1 and | *z* | = 1, then series can converge in some cases. Let $\tau = \sum_{j=1}^{P} a_j - \sum_{j=1}^{Q} b_j$. It can be shown that *P* = *Q* +1 the series is absolutely convergent for | *z* | = 1, if Re(τ) < 0, conditionally convergent for *z* = –1,if 0 ≤ Re(τ) < 1, and divergent for | *z* | = 1 if 1 ≤ Re(τ).
The following result will also be required in this sequel

$$ \int_0^1 {}_2F_1(\sigma,\delta;\gamma+1/2;x)\,{}_2F_1(\gamma-\alpha,\gamma-\delta;\gamma+1/2;x) $$

$$ S_M^N[x^\rho]\; {}_P\overset{\alpha}{M}_Q \left[z,x^v \,\middle|\, \begin{matrix} (a_1),\ldots,(a_P) \\ (b_1),\ldots,(b_Q) \end{matrix} \right] $$

$$.\,\overline{H}_{p',q'}^{m',n'} \left[z_3 x^\omega \,\middle|\, \begin{matrix} (c'_j,\gamma'_j;C'_j)_{1,n'},\ (c'_j,\gamma'_j)_{n'+1,p'} \\ (d'_j,\delta'_j)_{1,m'},\ (d'_j,\delta'_j;D'_j)_{m'+1,q'} \end{matrix} \right] \overline{H}_{p,q}^{m,n} \left[z_2 x^u \,\middle|\, \begin{matrix} (c_j,\gamma_j;C_j)_{1,n},\ (c_j,\gamma_j)_{n+1,p} \\ (d_j,\delta_j)_{1,m},\ (d_j,\delta_j;D_j)_{m+1,q} \end{matrix} \right] dx $$

$$ = \sum_{s=0}^{[N/M]} \sum_{h=1}^{m'} \sum_{r,t,k=0}^{\infty} \frac{a_t(\gamma)_t\, A'_{N,s}(-N)_{Ms}}{(\gamma+1/2)_t\, s!\, t!} $$

$$.\, \frac{\prod_{\substack{j=1 \\ j\neq h}}^{m'} \Gamma(d'_j - \delta'_j \xi_{h,r}) \prod_{j=1}^{n'} \{\Gamma(1-c'_j+\gamma'_j\xi_{h,r})\}^{C'_j} (-1)^r z_3^{\xi_{h,r}} (a_1)_k \ldots (a_P)_k z_1^k}{\prod_{j=m'=1}^{q'} \{(\Gamma(1-d'_j+\delta'_j\xi_{h,r})\}^{D'_j} \prod_{j=n'+1}^{p'} \Gamma(c'_j-\gamma'\xi_{h,r})\, r!\, \delta_h (b_1)_k \ldots (b_Q)_k \Gamma(\alpha k+1)} $$

$$.\, \overline{H}_{p+1,q+1}^{m,\,n+1} \left[z_2 \,\middle|\, \begin{matrix} (-t-\rho s-vk-\omega\xi_{h,r},u,1),(c_j,\gamma_j;C_j)_{1,n},(c_j,\gamma_j)_{n+1,p} \\ (d_j,\delta_j)_{1,m},(d_j,\delta_j;D_j)_{m+1,q},(-1-t-\rho s-vk-\omega\xi_{h,r},u;1) \end{matrix} \right], \tag{13} $$

provided that $\rho > 0, u > 0, v > 0, -\frac{1}{2} < \gamma - \sigma - \delta) < \frac{1}{2}, \mathrm{Re}\left[1 + u\frac{d_j}{\delta_j} + \omega\frac{d'_{j'}}{\delta'_{j'}}\right] > 0,$

$j = 1,\ldots, m$; $|\arg z_2|<\frac{1}{2}\pi T, T > 0$, $|\arg z_3|<\frac{1}{2}\pi T', T' > 0, T, T'$ are given in (7), M is an arbitrary positive integer and the coefficients $A'_{N,s}\,(N,s \geq 0)$ are arbitrary, real or complex, $P \leq Q$, $|z_1|<1$ and $\xi_{h,r}=(d'h+r)/\delta'_h$.

2. The Main Theorem

Theorem 1. With T defined by (7), $\rho > 0$, $u > 0$, $v > 0$, $w > 0$, $-\frac{1}{2}<(\gamma-\sigma-\delta)<\frac{1}{2}$, $T > 0$,

(i) $\operatorname{Re}\left(1+u\,\frac{d_j}{\delta_j}+\omega\frac{d'_{j'}}{\delta'_{j'}}\right) > 0, j = 1,\ldots,m\,; j' = 1,\ldots,m'$,

(ii) $|\arg z_1| < \frac{1}{2}T\pi$,

(iii) $P \leq Q$, $|z_1|<1$,

(iv) M is an arbitrary positive integer and the coefficients $A'_{N,s}(N,s \geq 0)$ are arbitrary, real or complex,

and if

$$(1-x)^{\sigma+\delta-\gamma}\;{}_2F_1(2\sigma,2\delta;2\gamma;x) = \sum_{t=0}^{\infty} a_t\, x^t \tag{14}$$

then

$$\int_0^1 {}_2F_1(\sigma,\delta;\gamma+1/2\,;x)\;{}_2F_1(\gamma-\sigma,\gamma-\delta;\gamma+1/2\,;x)S_N^M[x^\rho]\;{}_P\overset{\alpha}{M}{}_Q\left[{}^{a_1,\ldots,a_P,\alpha}_{b_1,\ldots,b_Q};z\,x^v\right]$$

$$.\bar{H}^{m',n'}_{p',q'}\left[z_3x^\omega\left|\begin{matrix}(c'_j,\gamma'_j;C'_j)_{1,n'},\ (c'_j,\gamma'_j)_{n'+1,p'}\\(d'_j,\delta'_j)_{1,m'},\ (d'_j,\delta'_j;D'_j)_{m'+1,q'}\end{matrix}\right.\right]\bar{H}^{m,n}_{p,q}\left[z_2x^u\left|\begin{matrix}(c_j,\gamma_j;C_j)_{1,n},\ (c_j,\gamma_j)_{n+1,p}\\(d_j,\delta_j)_{1,m},\ (d_j,\delta_j;D_j)_{m+1,q}\end{matrix}\right.\right]dx$$

$$=\sum_{s=0}^{[N/M]}\sum_{h=1}^{m'}\sum_{r,t,k=0}^{\infty}\frac{a_t(\gamma)_t\,A'_{N,s}(-N)_{Ms}}{(\gamma+1/2)_t\;s!\quad t!}$$

$$.\frac{\prod\limits_{\substack{j=1\\j\neq h}}^{m'}\Gamma(d'_j-\delta'_j\xi_{h,r})\prod\limits_{j=1}^{n'}\{\Gamma(1-c'_j+\gamma'_j\xi_{h,r})\}^{C'_j}(-1)^r(z_2)^{\xi_{h,r}}(a_1)_k\ldots(a_P)_k(z_1)^k}{\prod\limits_{j=m'+1}^{q'}\{\Gamma(1-d'_j+\delta'_j\xi_{h,r})\}^{D'_j}\prod\limits_{j=n'+1}^{p'}\Gamma(c'_j-\gamma'\xi_{h,r})r!\;\delta_h(b_1)_k\ldots(b_Q)_k\Gamma(\alpha z_1+1)}$$

$$.\;\bar{H}^{m,n+1}_{p+1,q+1}\left[z_2\left|\begin{matrix}(-t-\rho s-vk-\omega\xi_{h,r},u;1),\ (c_j,\gamma_j;C_j)_{1,n},(c_j,\gamma_j)_{n+1,p}\\(d_j,\delta_j)_{1,m},(d_j,\delta_j,D_j)_{m+1,q},(-1-t-\rho s-\omega\xi_{h,r},u;1)\end{matrix}\right.\right]. \tag{15}$$

Proof. We have ([10], p.75)

$${}_2F_1(\sigma,\delta;\gamma+1/2;x)\,{}_2F_1(\gamma-\sigma,\gamma-\delta;\gamma+1/2;x)$$

$$=\sum_{t=0}^{\infty}\frac{(\gamma)_t}{(\gamma+1/2)_t}a_t\,x^t, \tag{16}$$

where a_t is given by (14).

Multiplying both sides of (16) by

$$S_N^M[x^\rho]\ {}_P\overset{\alpha}{M}{}_Q\left[{}^{a_1,\dots,a_P,\alpha}_{b_1,\dots,b_Q};z_1x^v\right]\overline{H}{}^{m',n'}_{p',q'}\left[z_3x^\omega\left|\begin{matrix}(c'_j,\gamma'_j;C'_j)_{1,n'},(c'_j,\gamma'_j)_{n'+1,p'}\\(d'_j,\delta'_j)_{1,m'},(d'_j,\delta'_j;D'_j)_{m'+1,q'}\end{matrix}\right.\right]$$

$$.\overline{H}{}^{m,n}_{p,q}\left[z_2x^u\left|\begin{matrix}(c_j,\gamma_j;C_j)_{1,n},\ (c_j,\gamma_j)_{n+1,p}\\(d_j,\delta_j)_{1,m},\ (d_j,\delta_j;D_j)_{m+1,q}\end{matrix}\right.\right],$$

expressing the functions by their respective series form with the help of (4), (12) and (10) respectively with making use of Mellin-Barnes contour integral definition of the $\overline{H}$ -function as given in (5), and then interchanging the order of integrations and summations which is permissible under the conditions stated with (15), and evaluating the remaining simple integral with the help of (13), with a little simplification and adjustment of parameters, we arrive at the required result.

3. Application

(A) Taking $\sigma = \gamma$ in the main theorem, the values of a_t in (14) comes out to be equal to $(\delta)_t$, the result in (15) reduces to the following theorem

Theorem 1 (A)

(a) With T defined by (7), $\rho > 0$, $u > 0$, $v > 0$, $T > 0$, $,\omega>0,-\frac{1}{2}<(\gamma-\sigma-\delta)<\frac{1}{2}$,

(i) $\text{Re}\left(1+u\dfrac{d_j}{\delta_j}+\omega\dfrac{d'_{j'}}{\delta'_{j'}}\right)>0, j = 1,\dots,m\ ; j' = 1,\dots,m'$,

(ii) $|\arg z_1| < \frac{1}{2}T\pi$,

(iii) $P \le Q$, $|z_1|<1$,

(iv) M is an arbitrary positive integer and the coefficients $A'_{N,s}(N,s \ge 0)$ are arbitrary, real or complex and $\xi_{h,r}=\dfrac{d'_h+r}{\delta'_h}$, then

$$\int_0^1 {}_2F_1(\sigma,\delta;\gamma+1/2;x)\ S_N^M[x^\rho]\ {}_P\overset{\alpha}{M}{}_Q[z_1x^v]$$

$$.\overline{H}{}^{m',n'}_{p',q'}\left[z_3x^\omega\left|\begin{matrix}(c'_j,\gamma'_j;C'_j)_{1,n'},\ (c'_j,\gamma'_j)_{n'+1,p'}\\(d'_j,\delta'_j)_{1,m'},\ (d'_j,\delta'_j;D'_j)_{m'+1,q'}\end{matrix}\right.\right]\overline{H}{}^{m,n}_{p,q}\left[z_2x^u\left|\begin{matrix}(c_j,\gamma_j;C_j)_{1,n},\ (c_j,\gamma_j)_{n+1,p}\\(d_j,\delta_j)_{1,m},\ (d_j,\delta_j;D_j)_{m+1,q}\end{matrix}\right.\right]dx$$

$$=\sum_{s=0}^{[N/M]}\sum_{h=1}^{m'}\sum_{r,t,k=0}^{\infty}\frac{A'_{N,s}(-N)_{Ms}\,\Phi(k)\,(\sigma)_t(\delta)_t}{s!\ t!\,(\sigma+1/2)_t}$$

$$\cdot\frac{\prod\limits_{\substack{j=1\\ j\neq h}}^{m'}\Gamma(d'_j-\delta'_j\xi_{h,r})\prod\limits_{j=1}^{n'}\{\Gamma(1-c'_j+\gamma'_j\xi_{h,r})\}^{C'_j}(-1)^r(z_3)^{\xi_{h,r}}(a_1)_k\ldots(a_P)_k(z_1)^k}{\prod\limits_{j=m'+1}^{q'}\{\Gamma(1-d'_j+\delta'_j\xi_{h,r})\}^{D'_j}\prod\limits_{j=n'+1}^{p'}\Gamma(c'_j-\gamma'\xi_{h,r})r!\ \delta_h(b_1)_k\ldots(b_Q)_k\Gamma(\alpha k_1+1)}$$

$$.\bar{H}^{m,n+1}_{p+1,q+1}\left[z_2\left|\begin{matrix}(-t-\rho s-\omega\xi_{h,r}-vk,u;1),(c_j,\gamma_j;C_j)_{1,n},(c_j,\gamma_j)_{n+1,p}\\(d_j,\delta_j)_{1,m},(d_j,\delta_j,D_j)_{m+1,q},(-1-t-\rho s-\omega\xi_{h,k}-vk,u;1)\end{matrix}\right.\right]. \tag{17}$$

(B) Theorem 1 (B)

(b) With T defined by (7), $\rho > 0,\ u > 0,\ v > 0,\quad T>0,\ \omega>0$ *and* $\xi_{h,r}=d'_h+r/\delta'_h$,

(i) $\quad \mathrm{Re}\left(1+u\dfrac{d_j}{\delta_j}+\omega\dfrac{d'_{j'}}{\delta'_{j'}}\right)>0,\ j = 1,\ldots,m\ ;\ j'=1,\ldots,m',$

(ii) $\quad |\arg\ z_1| < \dfrac{1}{2}T\pi,$

(iii) $\quad P \le Q,\ |z_1|<1$,

(iv) M is an arbitrary positive integer and the coefficients $A'_{N,s}(N,s\ \ge 0)$ are arbitrary, real or complex and

$$\int_0^1 {}_2F_1(-\mu;x)\ S_N^M[x^\rho]\ {}_P\overset{\alpha}{M}_Q\left[\begin{matrix}a_1,\ldots,a_P,\alpha;\\ b_1,\ldots,b_Q;\end{matrix}\ z_1x^v\right]$$

$$.\bar{H}^{m',n'}_{p',q'}\left[z_3x^\omega\left|\begin{matrix}(c'_j,\gamma'_j;C'_j)_{1,n'},\ (c'_j,\gamma'_j)_{n'+1,p'}\\(d'_j,\delta'_j)_{1,m'},\ (d'_j,\delta'_j;D'_j)_{m'+1,q'}\end{matrix}\right.\right]\bar{H}^{m,n}_{p,q}\left[z_2x^u\left|\begin{matrix}(c_j,\gamma_j;C_j)_{1,n},\ (c_j,\gamma_j)_{n+1,p}\\(d_j,\delta_j)_{1,m},\ (d_j,\delta_j;D_j)_{m+1,q}\end{matrix}\right.\right]dx$$

$$=\sum_{s=0}^{[N/M]}\sum_{h=1}^{\mu'}\sum_{r,t,k=0}^{\infty}\frac{A'_{N,s}(-N)_{Ms}(-\mu)_t}{s!\ t!}$$

$$\cdot\frac{\prod\limits_{\substack{j=1\\ j\neq h}}^{m'}\Gamma(d'_j-\delta'_j\xi_{h,r})\prod\limits_{j=1}^{n'}\{\Gamma(1-c'_j+\gamma'_j\xi_{h,r})\}^{C'_j}(-1)^r(z_2)^{\xi_{h,r}}(a_1)_k\ldots(a_P)_k(z_1)^k}{\prod\limits_{j=m'+1}^{q'}\{\Gamma(1-d'_j+\delta'_j\xi_{h,r})\}^{D'_j}\prod\limits_{j=n'+1}^{p'}\Gamma(c'_j-\gamma'\xi_{h,r})r!\ \delta_h(b_1)_k\ldots(b_Q)_k\Gamma(\alpha k_1+1)}$$

$$.\bar{H}^{m,n+1}_{p+1,q+1}\left[z_2\left|\begin{matrix}(-t-\rho s-\omega\xi_{h,r}-vk,u;1),(c_j,\gamma_j;C_j)_{1,n},(c_j,\gamma_j)_{n+1,p}\\(d_j,\delta_j)_{1,m},(d_j,\delta_j,D_j)_{m+1,q},(-1-t-\rho s-\omega\xi_{h,r}-vk,u;1)\end{matrix}\right.\right]. \tag{18}$$

4. Expansion Formula

On evaluating on the left hand side of (18) with the help of a known result [2], we have the following interesting expansion formula

Theorem 1(C)

$$\sum_{s=0}^{[N/M]}\sum_{h=1}^{m'}\sum_{t=0}^{\mu}\sum_{r,k=0}^{\infty}\frac{A'_{N,s}(-N)_{Ms}(-\mu)_t}{s!\ t!}$$

$$\cdot \frac{\prod_{\substack{j=1\\j\neq h}}^{m'} \Gamma(d'_j - \delta'_j \xi_{h,r}) \prod_{j=1}^{n'} \{\Gamma(1-c'_j+\gamma'_j\xi_{h,r})\}^{C'_j} (-1)^r (z_2)^{\xi_{h,r}} (a_1)_k \ldots (a_P)_k (z_1)^k}{\prod_{j=m'+1}^{q'} \{\Gamma(1-d'_j+\delta'_j\xi_{h,r})\}^{D'_j} \prod_{j=n'+1}^{p'} \Gamma(c'_j - \gamma'\xi_{h,r})\, r!\ \delta_h (b_1)_k \ldots (b_Q)_k \Gamma(\alpha k_1+1)}$$

$$\cdot \overline{H}^{m,n+1}_{p+1,q+1}\left[z_2 \left| \begin{matrix} (-t-\rho s-\omega\xi_{h,r}-vk,u;1),\ (c_j,\gamma_j;C_j)_{1,n},(c_j,\gamma_j)_{n+1,p} \\ (d_j,\delta_j)_{1,m},(d_j,\delta_j,D_j)_{m+1,q},(-1-t-\rho s-\omega\xi_{h,k}-vk,u;1) \end{matrix}\right.\right]$$

$$= \sum_{s=0}^{[N/M]} \sum_{h=1}^{m'} \sum_{r,t,k=0}^{\infty} \frac{A'_{N,s}(-N)_{Ms}\, \Gamma(\mu+\rho s+\omega\xi_{h,r}+vk+1)}{s!}$$

$$\cdot \frac{\prod_{\substack{j=1\\j\neq h}}^{m'} \Gamma(d'_j - \delta'_j \xi_{h,r}) \prod_{j=1}^{n'} \{\Gamma(1-c'_j+\gamma'_j\xi_{h,r})\}^{C'_j} (-1)^r (z_2)^{\xi_{h,r}} (a_1)_k \ldots (a_P)_k (z_1)^k}{\prod_{j=m'+1}^{q'} \{\Gamma(1-d'_j+\delta'_j\xi_{h,r})\}^{D'_j} \prod_{j=n'+1}^{p'} \Gamma(c'_j - \gamma'\xi_{h,r})\, r!\ \delta_h (b_1)_k \ldots (b_Q)_k \Gamma(\alpha k_1+1)}$$

$$\cdot \overline{H}^{m,n+1}_{p+1,q+1}\left[z_2 \left| \begin{matrix} (0,u;1),\ (c_j,\gamma_j;C_j)_{1,n},(c_j,\gamma_j)_{n+1,p}\ (\ldots\ldots..) \\ (d_j,\delta_j)_{1,m},(d_j,\delta_j,D_j)_{m+1,q},(-1-\mu-\rho s-\omega\xi_{h,r}-vk,u;1) \end{matrix}\right.\right], \tag{19}$$

provided that both the sides exists.

5. Special Cases

(i) Letting $m = 1$, $n = 2 = p = q$, and replacing z by $-z$ in (14), and using

$$g(\theta(\theta,\sigma',\tau,z) = \frac{R_{\rho'-1}\Gamma(\varphi+1)\Gamma((1+\tau)/2)}{(-1)^{\varphi}\, 2^{2-\varphi}\, \Gamma(\theta)\Gamma(\theta-\tau/2)\sqrt{\pi}}$$

$$\overline{H}^{1,3}_{3,3}\left[-z\left| \begin{matrix} (1-\theta,1;1),\left(1-\theta+\frac{\tau}{2},1;1\right),(1-\sigma',1;\varphi+1) \\ (0,1),\left(-\frac{\tau}{2},1;1\right),(-\sigma',1;\varphi+1) \end{matrix}\right.\right],$$

where

$$R_{\rho'-1} = \frac{2^{1-\rho'}\,\pi^{-\rho'/2}}{\Gamma(\sigma'/2)}, \tag{20}$$

the function in (20) is connected with a certain class of Feynman integrals, we get the following result

$$\int_0^1 {}_2F_1(\sigma,\delta;\gamma+1/2;x)\, {}_2F_1(\gamma-\sigma,\gamma-\delta;\gamma+1/2;x)\quad S_N^M[x^{\rho}]$$

$$\cdot\ {}_P\overset{\alpha}{M}_Q\left[\begin{matrix}(a_1,\ldots,a_P,\alpha;\\ b_1,\ldots,b_Q;\end{matrix}\ x^{v}\right] g(\theta,\sigma',\tau,\varphi;z_3 x^{u})$$

$$\cdot\ \overline{H}^{m',n'}_{p',q'}\left[z_3 x^{\omega}\left| \begin{matrix} (c'_j,\gamma'_j;C'_j)_{1,n'},(c'_j,\gamma'_j)_{n'+1,p'} \\ (d'_j,\delta'_j)_{1,m'},(d'_j,\delta'_j;D'_j)_{m'+1,q'} \end{matrix}\right.\right] dx$$

$$= \sum_{s=0}^{[N/M]} \sum_{h=1}^{m'} \sum_{r,k,t=0}^{\infty} \frac{A'_{N,s}(-N)_{Ms}}{s!\quad t!}$$

$$\cdot\frac{\prod_{\substack{j=1\\j\neq h}}^{m'}\Gamma(d_j'-\delta_j'\xi_{h,r})\prod_{j=1}^{n'}\{\Gamma(1-c_j'+\gamma_j'\xi_{h,r})\}^{C_j'}(-1)^r(z_3)^{\xi_{h,r}}(a_1)_k\dots(a_P)_k(z_1)^k}{\prod_{j=m'+1}^{q'}\{\Gamma(1-d_j'+\delta_j'\xi_{h,r})\}^{D_j'}\prod_{j=n'+1}^{p'}\Gamma(c_j'-\gamma'\xi_{h,r})r!\ \delta_h(b_1)_k\dots(b_Q)_k\Gamma(\alpha k_1+1)}$$

$$\cdot\frac{R_{\rho'-1}\Gamma(\varphi+1)\Gamma((1+\tau)/2)(\gamma)_t\ a_t}{(-1)^{\varphi}2^{2+\varphi}\ \Gamma(\theta)\Gamma(\theta-\tau/2)\sqrt{\pi}\,(\gamma+1/2)_t}$$

$$\cdot\overline{H}_{4,4}^{1,4}\left[-z_2\left|\begin{matrix}(-t-\rho s-\omega\xi_{h,r}-vk,u;1),\ (1-\theta,1;1),\left(1-\theta+\frac{\tau}{2},1;1\right),(1-\sigma',1;\varphi+1)\\(0,1),\left(-\frac{\tau}{2},1;1\right),(-\sigma',1;\varphi+1),(-1-t-\rho s-\omega\xi_{h,r}-vk,\ u;1)\end{matrix}\right.\right],\tag{21}$$

valid under the conditions surrounding (14).

(ii) Setting m = 1, n = p, q = q+1 and replacing z_2 by $-z_2$ in the equation (14), and making use of the following transformation

$${}_p\overline{\psi}_q\left[z_2\left|\begin{matrix}(c_j,\gamma_j;C_j)_{1,p}\\(d_j,\delta_j;D_j)_{1,q}\end{matrix}\right.\right]=\overline{H}_{p,q+1}^{1,p}\left[-z_2\left|\begin{matrix}(1-c_j,\gamma_j;C_j)_{1,p}\\(0,1),(1-d_j,\delta_j;D_j)_{1,q}\end{matrix}\right.\right],\tag{22}$$

we have

$$\int_0^1 {}_2F_1(\sigma,\delta;\gamma+1/2;x)\,{}_2F_1(\gamma-\sigma,\gamma-\delta;\gamma+1/2;x)\quad S_N^M[x^{\rho}]$$

$$\cdot{}_P\overset{\alpha}{M}{}_Q\left[z_1x^v\left|\begin{matrix}a_1,\dots,a_P,\alpha\alpha\\b_1,\dots,b_Q\,;\end{matrix}\right.\right]{}_p\overline{\psi}_q\left[z_2x^u\left|\begin{matrix}(c_j,\gamma_j;C_j)_{1,p}\\(d_j,\delta_j;D_j)_{1,q}\end{matrix}\right.\right]$$

$$\cdot\ \overline{H}_{p',q'}^{m',n'}\left[z_3x^{\omega}\left|\begin{matrix}(c_j',\gamma_j';C_j')_{1,n'},(c_j',\gamma_j')_{n'+1,p'}\\(d_j',\delta_j')_{1,m'},(d_j',\delta_j';D_j')_{m'+1,q'}\end{matrix}\right.\right]dx$$

$$=\sum_{s=0}^{[N/M]}\sum_{h=1}^{m'}\sum_{r,k,t=0}^{\infty}\frac{A_{N,s}'(-N)_{Ms}\,\psi(h,r)\Phi(k)(\gamma)_t}{s!\qquad t!\ (\gamma+1/2)_t}$$

$$=\sum_{s=0}^{[N/M]}\sum_{h=1}^{m'}\sum_{r,k,t=0}^{\infty}\frac{A_{N,s}'(-N)_{Ms}(\gamma)_t}{s!\qquad t!\,(\gamma+1/2)_t}$$

$$\cdot\frac{\prod_{\substack{j=1\\j\neq h}}^{m'}\Gamma(d_j'-\delta_j'\xi_{h,r})\prod_{j=1}^{n'}\{\Gamma(1-c_j'+\gamma_j'\xi_{h,r})\}^{C_j'}(-1)^r(z_3)^{\xi_{h,r}}(a_1)_k\dots(a_P)_k(z_1)^k}{\prod_{j=m'+1}^{q'}\{\Gamma(1-d_j'+\delta_j'\xi_{h,r})\}^{D_j'}\prod_{j=n'+1}^{p'}\Gamma(c_j'-\gamma'\xi_{h,r})r!\ \delta_h(b_1)_k\dots(b_Q)_k\Gamma(\alpha k_1+1)}$$

$$\cdot\overline{H}_{p+1,q+2}^{1,p+1}\left[-z_2\left|\begin{matrix}(-t-\rho s-\omega\xi_{h,r}-vk,u;1),\ (1-c_j,\gamma_j;C_j)_{1,p}\\(0,1),(1-d_j,\delta_j;D_j)_{1,q},(-1-t-\rho s-\omega\xi_{h,r}-vk,\ u;1)\end{matrix}\right.\right],\tag{23}$$

valid under the conditions as obtainable from (14).

(iii) Taking m = 1, n = p = q+1, $\gamma_j=1=\delta_1$ and replacing z_2 by $-z_2$ in the equation (14) and making use of the following relation

$$
{}_p\overline{F}_q\left[z_2\left|\begin{matrix}(c_j,1;C_j)_{1,p}\\(d_j,1;D_j)_{1,q}\end{matrix}\right.\right]
$$

$$
=\frac{\prod_{j=1}^{q}\{\Gamma(d_j)\}^{D_j}}{\prod_{j=1}^{p}\{\Gamma(c_j)\}^{C_j}}\ \overline{H}^{1,p}_{p,q+1}\left[-z_2\left|\begin{matrix}(1-c_j,1;C_j)_{1,p}\\(0,1),(1-d_j,1;D_j)_{1,q}\end{matrix}\right.\right], \tag{24}
$$

we have

$$
\int_0^1 {}_2F_1\left(\sigma,\delta;\gamma+1/2;x\right){}_2F_1\left(\gamma-\sigma,\gamma-\delta;\gamma+1/2;x\right)\ S_N^M[x^\rho]
$$

$$
\cdot\ {}_P\overset{\alpha}{M}{}_Q\left[z_1x^v\left|\begin{matrix}a_1,\ldots,a_P,\alpha\\b_1,\ldots,b_Q\end{matrix}\right.\right]{}_p\overline{F}_q\left[z_2x^u\left|\begin{matrix}(c_j,1;C_j)_{1,p}\\(d_j,1;D_j)_{1,q}\end{matrix}\right.\right]
$$

$$
.\ \overline{H}^{m',n'}_{p',q'}\left[z_3x^\omega\left|\begin{matrix}(c'_j,\gamma'_j;C'_j)_{1,n'},(c'_j,\gamma'_j)_{n'+1,p'}\\(d'_j,\delta'_j)_{1,m'},(d'_j,\delta'_j;D'_j)_{m'+1,q'}\end{matrix}\right.\right]dx
$$

$$
=\sum_{s=0}^{[N/M]}\sum_{h=1}^{m'}\sum_{r,k,t=0}^{\infty}\frac{A'_{N,s}(-N)_{Ms}(\gamma)_t\prod_{j=1}^{q}\{\Gamma(d_j)\}^{D_j}}{s\,!\,t\,!\left(\gamma+1/2\right)_t\prod_{j=1}^{p}\{\Gamma(c_j)\}^{C_j}}
$$

$$
\cdot\frac{\prod_{\substack{j=1\\j\neq h}}^{m'}\Gamma(d'_j-\delta'_j\xi_{h,r})\prod_{j=1}^{n'}\{\Gamma(1-c'_j+\gamma'_j\xi_{h,r})\}^{C'_j}(-1)^r(z_3)^{\xi_{h,r}}(a_1)_k\ldots(a_P)_k(z_1)^k}{\prod_{j=m'+1}^{q'}\{\Gamma(1-d'_j+\delta'_j\xi_{h,r})\}^{D'_j}\prod_{j=n'+1}^{p'}\Gamma(c'_j-\gamma'\xi_{h,r})r\,!\ \delta_h(b_1)_k\ldots(b_Q)_k\Gamma(\alpha k_1+1)}
$$

$$
.\ \overline{H}^{1,p+1}_{p+1,q+2}\left[-z_2\left|\begin{matrix}(-t-\rho s-\omega\xi_{h,r}-vk,u;1),\ (1-c_j,\gamma_j;C_j)_{1,p}\\(0,1),\ (1-d_j,\delta_j;D_j)_{1,q},(-1-t-\rho s-\omega\xi_{h,r}-vk,\ u;1)\end{matrix}\right.\right], \tag{25}
$$

valid under the conditions as obtainable from (14).

Known Results

(i) Setting $v\to 0$ in (15), we find a known result due to Chaurasia and Soni ([3], p.1715, eqn. (2.2)).

(ii) Taking $v\to 0$, $N\to 0$, $w\to 0$, with $C_j = 1 = D_j$ in (15), we arrive at the known result derived by Chaurasia ([2], p.185, eqn. (6.5.2)).

(iii) Putting $\gamma_j=\delta_j=1=C_j=D_j$ for all j, $N\to 0$, $v\to 0$, the results (17) through (19) reduce to the known results obtained by Srivastava ([11], p.236, eqn. (1.2)).

(iv) Letting $w\to 0, v\to 0, \gamma_j=\delta_j=1=C_j=D_j$, for all j, the results (17) through (25) reduce to the known results due to Sharma [9].

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A NEW PATH TO REACH THE SOLUTION OF FRACTIONAL MECHANICAL OSCILLATOR USING ADOMAIN DECOMPOSITION METHOD

By

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Abstract
The present paper describes the fractional mechanical oscillator of a simple system represented by fractional differential equation (FDE). The order of derivatives is $0 < \alpha \leq 1$. The system can be solved in terms of Mittag-Leffler function depending on the parameter α. In this paper, a new and developed approach Adomain Decomposition Method (ADM) has been worked out by using adomain polynomial as a proposed solution. In this regards, the concept of Jumarie derivative in terms of Mittag-Leffler function for FDE is utilized. With the help of MATHEMATICA software, the total displacement of the system has been studied for different values of α's $(0 < \alpha \leq 1)$.

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1. Introduction

In the last few decades, fractional based calculus spreads widely in various fields of science and engineering. The fractional calculus involves with fractional derivatives (of order $\alpha, 0 < \alpha \leq 1$) which has inbuilt integration (often called differeintegrals) and non-local in character. The order of the derivative classifies and quantifies the strong influence of past history if α is away from 1 and requires information about the past states. The closure to the value of 1 indicates that the influence of history is minimal. This type of memory effect can be represented by means of convolution between the function and a memorykernel (power of time).

Many authors have developed and defined the concepts of fractional integrals and derivatives in their own way. Of them, Riemann-Liouville, Weyl and Grunworld-Letnikov are worth mentioning. However, Caputo (1967) reformulated the more classic definition of the Riemann-Liouville fractional derivatives in order to solve fractional differential equation (FDE) with integer order initial conditions. The definitions of Riemann-Liouville and Caputo's are as follows [8, 16, 24, 30]:

Fractional Integration:

A repeated n-folded integration is defined as

$$D^{-n} f(t) = I^n f(t) = \frac{1}{(n-1)!}\int_0^t (t-u)^{n-1} f(u)du.$$

For any real number α>0, the repeated α- fold integration is defined as

$$D^{-\alpha} f(t) = I^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-u)^{\alpha-1} f(u)du.$$

The following are some definitions of differeintegrals in the sense of several authors.
Riemann-Liouville (RL) Integration: For $\alpha > 0$

(i) *Forward Integration:*

$$ {}_aI_x^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-u)^{\alpha-1} f(u)du. $$

(ii) *Backward Integration:*

$$ {}_xI_b^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (u-x)^{\alpha-1} f(u)du. $$

Fractional derivatives

Riemann- Liouville left hand definition (RL LHD):

Let us select an integer $m > \alpha, \alpha$: fractional number such that

i) Integrate the function (m-α)- folds in the sense of forward RL

ii) Differentiate the above result by m

Then, *RL LHD* is defined as

$$ {}_0D_t^{\alpha} f(t) = \frac{d^m}{dt^m}\left[\frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f(u)}{(x-u)^{\alpha+1-m}} du\right], \quad (m-1 \leq \alpha < m). $$

Caputo right hand definition (Caputo RHD):

Let us select an integer $m > \alpha, \alpha$: fractional number such that

(i) Differentiate the function m times,

(ii) Integrate the above result $(m-\alpha)-$fold by RL LHD integration method.

Thus, Caputo's RHD is defined as

$$ {}_0^CD_t^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{\frac{d^m}{dt^m} f(u)}{(t-u)^{\alpha+1-m}} du, \ (m-1 \leq \alpha < m). $$

It is to be noted that the RL LHD and Caputo RHD are equivalent.

But the Caputo derivative of a constant function is zero, whereas both left and right RL derivative of a constant (K) is non zero. In this regards, a modified left (right) RL derivative has been developed by Jumarie [14]. Thus the Jumarie derivative is defined as,

$$ {}_0D_t^{\alpha} f(t) = \frac{1}{\Gamma(-\alpha)} \int_0^t (t-\xi)^{-\alpha-1} f(\xi)d\xi \ , \alpha < 0, $$

$$ \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\xi)^{-\alpha}[f(\xi)-f(0)]d\xi \ , 0 < \alpha < 1, $$

$$ [f^{(\alpha-n)}(t)]^{(n)}, n \leq \alpha < n+1, n \geq 1. $$

Using this definition, we get, ${}_0D_t^{\alpha}\{K\} = 0 \ , 0 \leq \alpha < 1$.

In 1903 Mittag-Leffler [8, 16, 24, 30] introduce a function defined by infinite series

$$ E_{\alpha}(at^{\alpha}) = 1 + \frac{at^{\alpha}}{\Gamma(1+\alpha)} + \frac{a^2t^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{a^3t^{3\alpha}}{\Gamma(1+3\alpha)} + ... $$

is the one parameter Mittag-Leffler function. This function can be used as a trial solution of FDE like exponential function of whole order differential equation.

Using the definition of Jumarie derivative, several properties of Mittag-Leffler function has been deduced. Thus, one can deduce the following properties

1) $D^{\alpha}(E_{\alpha}(at^{\alpha})) = aE_{\alpha}(at^{\alpha})$, 2) $E_{\alpha}(at^{\alpha})E_{\alpha}(bt^{\alpha}) = E_{\alpha}((a+b)t^{\alpha})$.

Moreover, the analytical solution of linear fractional differential equation has been solved with the help of Jumarie derivative in terms of Mittag-Leffler function [8, 24, 30]. For example, the solution of

$(D^{\alpha} - a_1)(D^{\alpha} - a_2) \dots (D^{\alpha} - a_n)y(t) = 0$, with all $a_i's$ distinct is

$y = \Sigma A_i E_{\alpha}(a_i t^{\alpha})$, where A_i are arbitrary constants and $E_{\alpha}(a_i t^{\alpha})$ is a one-parameter family of Mittag-Leffler function. As a particular case, the solution of FDE

$D^{2\alpha}y - 2aD^{\alpha}y + a^2 y = 0$ is

$y = (At^{\alpha} + B)E_{\alpha}(at^{\alpha}), A = A_1/\Gamma(1+\alpha)$ where A and B are arbitrary constants.

Most of the applications and physical manifestations of fractional calculus have been found in 20th and 21st century [3, 5, 6, 7, 9-13, 15, 16, 18-21, 25-30]. The applications like Abel's fractional integral equation of Tatochrome, Modelling of speech signals, Modelling of cardiac tissue electrode interface, fractional differentiation of edge detection related to image processing, the solution of time dependent viscous diffusion fluid problems are worth mentioning. Some other fields which includes fractional calculus are polymer science, fractal phenomena, ultrasonic wave propagation in human cancellous bone.

The present paper developsthe solution of fractional oscillator equation [4] using Adomain decomposition method (ADM) [1, 2, 8] along with Jumarie formulation of FDE, which will be discussed later on.

The fractional differential equation corresponding to the mechanical system [24] (see figure 1) is given by

$$\frac{m}{\sigma^{2(1-\alpha)}}\frac{d^{2\alpha}x(t)}{dt^{2\alpha}} + \frac{\beta}{\sigma^{(1-\alpha)}}\frac{d^{\alpha}x(t)}{dt^{\alpha}} + kx(t) = F(t) \tag{1}$$

where, m: mass measured in kg

β: damped coefficient measured in *N.s/m*

k: string constant measured in *N/m*

σ: auxiliary parameter (fractional time component of the system)

Equation (1) can be rewritten as

$$\frac{d^{2\alpha}x(t)}{dt^{2\alpha}} + \gamma\frac{d^{\alpha}x(t)}{dt^{\alpha}} + \omega^2 x(t) = \frac{F(t)}{m} \tag{2}$$

$$x(0) = 0, \dot{x}(0) = 0$$

where,

$\gamma = \dfrac{\beta\sigma^{1-\alpha}}{m}$,

$\omega^2 = \omega_0^2\sigma^{2(1-\sigma)}$,

$\omega_0^2 = \dfrac{k}{m}$, a fundamental frequency when $\alpha = 1$.

ω: driving frequency.

The different form of forcing function *F(t)* is defined as

$$\frac{F(t)}{m} = \omega_0^2, t \geq 0$$

$$0, t < 0 \tag{2a}$$

and

$$\frac{F(t)}{m} = F_0 \sin(\omega t) \tag{2b}$$

is a sinusoidal driving force, F_0 is the driven amplitude.

$\frac{d^{2\alpha}}{dt^{2\alpha}}, \frac{d^{\alpha}}{dt^{\alpha}}$ $(0 < \alpha < 1)$ is a fractional order derivative in the sense of Riemann Liouville. The parameter σ is introduced in order to make the time dimensionality to be consistent. This means that the fractional time derivatives operator $\frac{d^{\alpha}}{dt^{\alpha}}$ has $s^{-\alpha}$ $(0 < \alpha \leq 1)$ but if the new parameter is introduced in the form $\frac{1}{\sigma^{1-\alpha}}\frac{d^{\alpha}}{dt^{\alpha}}$ $(0 < \alpha \leq 1)$, then $[\frac{1}{\sigma^{1-\alpha}}\frac{d^{\alpha}}{dt^{\alpha}}] = 1/s$ (3)

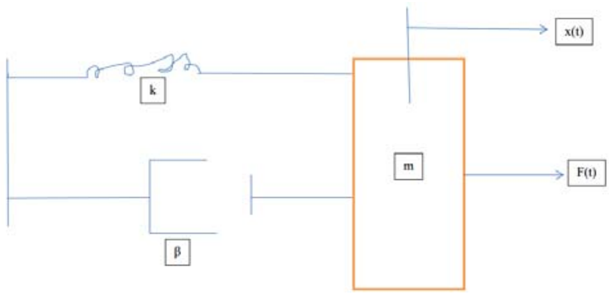

Figure 1. Damped Harmonic Oscillator

In fact (3) is dimensionally consistent if and only if $[\sigma] = s$. The present paper discusses the solution of (2) with forcing function as (2a).

In order to solve the system (2), a Mittag-Leffler function has been used, which depends on the order of the derivative α [7, 10, 24, 30]. However, a more developed and realistic approach viz.,Adomain Decomposition Method (ADM) is carried out to solve the system (2).The concept of Jumarie derivative in terms of Mittag-Leffler function is also utilized in the initial step. The major advantages of ADM is that, it does not involve any discretizations (and hence free from rounding off errors) and not huge amount of computer memory is required. Moreover, with Riemann-Liouville derivatives, the initial condition may not require fractional order state rather it would be an integer order state. However, ADM is close to physical reality and visualize the reaction system by decomposing the total gross reactions from zeroth mode to infinity mode. The sum of all these modes is the solution of (2).

The whole matter of the paper is organized as follows:

Section 1 is the introductory one. In section 2, a generalized dynamic system and the idea of solution of FDE using ADM (which involves adomain polynomial) is discussed. Section 3 gives the solution of the system (2) using the ADM. Finally in section 4, based on ADM a numerical simulation is done on fractional mechanical oscillator using MATHEMATICA software.

2. Generalized Dynamic system and its solution using Adomain Decomposition Method (ADM) [7, 8]

General physics law states that a system will react to external stimulus and will have opposition to changes and the process is describe by the extraordinary differential equation,

$$Fu = G, \tag{4}$$

where, F: general linear or non-linear differential equation which may be fractional in the sense of Riemann-Liouville. This operator can be decomposed as,

$$Fu = L_0 + Ru + Nu = G, \tag{5}$$

L_0: highest order derivative (may be positive integer or fractional order), which is invertible i.e.

$L_0 u = \dfrac{d^m u(x)}{dx^m} = D_x^m u(x)$, which is linear operator.

R: linear differential (remainder) operator of order less than that of L_0; this can also be fractional linear operator i.e.

$Ru \equiv a_1 D_x^{m-1} + a_2 D_x^{m-2} + a_3 D_x^{m-3} + \ldots + a_{m-1} D_x^1$, where a_0, a_1, a_2,... a_{m-1} are constants.

N: non- linear part which will be decomposed into infinite sum of adomain polynomial (this term may be linear or constant) i.e.

$Nu \equiv a_m u(x) + b_k [u(x)]^k + b_{k-1}[u(x)]^{k-1} + b_{k-2}[u(x)]^{k-2} + \ldots + b_0 u(x)$

G(x): sum of all external stimulus source/ sink.

The R and N are internal stimuli and when it get balanced with the external stimulus G then the process of the parameter remains static without any growth or decay. Otherwise, the process parameter will have a solution as infinite (or finite) decomposed modes; generated by the system itself to oppose stimulus generated internally by previous mode.

The idea of solution to (4) is a new approach called Adomain Decomposition Method (ADM). ADM method is an analytical method and has a certain advantage over standard numerical techniques. The decomposed parts of ADM method are related to system reactions of various modes from zeroth mode to infinity mode. The sum of all these modes is the solution of (4). In (4), if L_0 is the fractional differential operator in the sense of RL, then the initial conditions should be in fractional order say, $u^m(0)$, $u^{m-1}(0)$ for m to be fractional. These states are hard to visualize. But with this ADM, the RL formulation does not need these fractional initial states instead requires $u(0)$, $u^{(1)}(0)$, $u^{(2)}(0)$ etc.; the integer order states gives the solution and thus physically understandable.

Equation (5) can be rewritten as

$L_0 u = G - Ru - Nu.$

Applying invertible operator both sides, we get

$$u = \Phi + L_0^{-1} G - L_0^{-1}[R(u)] - L_0^{-1}[N(u)], \qquad (6)$$

Φ: solution of homogeneous equation $L_0 u=0$, so that $L_0\Phi=0$ and this comes from initial and boundary condition.

Then with ADM, we have,

$u(\lambda)=u_0+\lambda u_1+\lambda^2 u_2+\lambda^3 u_3+\ldots$

This is a Maclaurin's series with respect to λ with coefficient u_n $(n = 1,2,\ldots)$ and $u_n = u^{(n)}(0)/n!$ around $\lambda = 0$. Then, $N(u)$ is a Maclaurin's series with respect to λ and obtain

$$N(u) = \sum_{n=0}^{\infty} \lambda^n A_n \text{, where } A_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n} N\left(\sum_{k=0}^{\infty} \lambda^k u_k\right)\right].$$

For $N(u)$ linear, $A_0 = u_0$ and $A_n = u_n$.

For $N(u)$ to be non- linear,

$A_0=N(u_0)$, $A_1 = u_1 N'(u_0)$, $A_2 = u_2 N'(u_0) + \frac{1}{2} u_1^2 N''(u_0)$,

$A_3 = u_3 N'(u_0) + u_1 u_2 N''(u_0) + \dfrac{u_1^3}{3!} N'''(u_0)$ etc.

The parameter λ is just an identifier for collecting terms in suitable way, such that u_n depends on u_0, u_1, u_2,…,u_{n-1} and later on we will set λ=1. Parameterizing the equation (5), we get,

$$u = \Phi + L_0^{-1} G - \lambda L_0^{-1}[R(u)] - \lambda L_0^{-1}[N(u)]. \qquad (7)$$

Expanding the decomposition, (7) gives

$$u = \Phi + L_0^{-1}G - \lambda L_0^{-1}R\left(\sum_{n=0}^{\infty}\lambda^n u_n\right) - \lambda L_0^{-1}N\left(\sum_{n=0}^{\infty}\lambda^n A_n\right) \quad . \tag{8}$$

Comparing the like powers of λ in the expression for n = 0, we have,

$u_0 = \Phi + L_0^{-1}G$. Similarly, for n =1,2,…, we have

$$u_n = -L_0^{-1}R(u_{n-1}) - L_0^{-1}(A_{n-1}) \text{ for } n \geq 1. \tag{9}$$

3. Solution of fractional mechanical oscillator (2) using Adomain Decomposition Method

We rewrite the fractional mechanical oscillator (2) as

$$ {}_0D_t^{2\alpha}x + \gamma\, {}_0D_t^{\alpha}x + \lambda x = F_0, \tag{10}$$

$x(0) = 0, \dot{x}(0) = 0,$

where, $\gamma = \dfrac{\beta\sigma^{1-\alpha}}{m}, \lambda = \omega^2, F_0 = \omega_0^2.$

Equation (10) is again rewritten as $Fx=G$, where

$Fx=L_0x+Rx+Nx$ and $G=F_0$ (constant) and

$L_0 = {}_0D_t^{2\alpha}, R = \gamma\, {}_0D_t^{\alpha},$

$N(x)=\lambda x$ (constant).

The decomposed equation can be written as,

$L_0x = G-R(x)-N(x)$.

Applying invert operator on both sides, we get

$x = \phi + L_0^{-1}G - L_0^{-1}[R(x)] - L_0^{-1}[N(x)],$

where φ is the solution of the homogeneous equation $L_0(x) = 0$, so that $L_0(\varphi)=0$.

Consider the homogeneous equation

${}_0D_t^{2\alpha}x = 0,$

Let $x = AE_\alpha(mt^\alpha)$ ($\neq 0$) be a nontrivial trial solution of the homogeneous equation.

By using Jumarie derivative, we have $D^{2\alpha}x = Am^2E_\alpha(mt^\alpha)$.

Therefore, $x = A+Bt^\alpha$ and hence $x = 0$, by using initial conditions

$x(0) = 0, \dot{x}(0) = 0.$

By using the definition of Jumarie derivative, the ADM method thus generates the mode as

$$x_0 = \phi + L_0^{-1}G = {}_0D_t^{-2\alpha}F_0 = F_0\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}.$$

$A_0 = N(x_0) = \lambda x_0.$

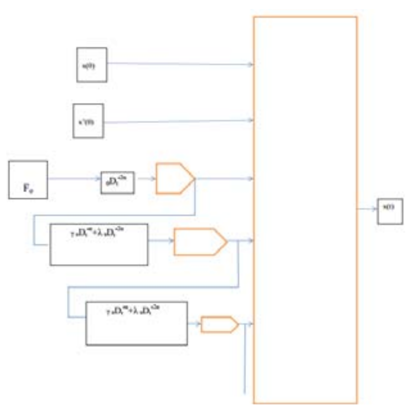

Figure 2. Block Diagram of fractional mechanical oscillator

Again, by using Jumarie derivative,

$$x_1 = -L_0^{-1}R(x_0) - L_0^{-1}A_0 = \left(-{}_0D_t^{-2\alpha}\gamma\, {}_0D_t^{\alpha} - {}_0D_t^{-2\alpha}\lambda\right)x_0$$

$$= \left(-\gamma\, {}_0D_t^{-\alpha} - \lambda\, {}_0D_t^{-2\alpha}\right)\left(F_0 \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\right) = -\gamma F_0 \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} - \lambda F_0 \frac{t^{4\alpha}}{\Gamma(4\alpha+1)},$$

$A_1 = x_1$,

$$x_2 = \left(-\gamma\, {}_0D_t^{-\alpha} - \lambda\, {}_0D_t^{-2\alpha}\right)x_1$$

$$= \left(-\gamma\, {}_0D_t^{-\alpha} - \lambda\, {}_0D_t^{-2\alpha}\right)\left(-\gamma F_0 \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} - \lambda F_0 \frac{t^{4\alpha}}{\Gamma(4\alpha+1)}\right)$$

$$= \gamma^2 F_0 \frac{t^{4\alpha}}{\Gamma(4\alpha+1)} + 2\gamma\lambda F_0 \frac{t^{5\alpha}}{\Gamma(5\alpha+1)} + \lambda^2 F_0 \frac{t^{6\alpha}}{\Gamma(6\alpha+1)},$$

$A_2 = x_2$

$$x_3 = \left(-\gamma\, {}_0D_t^{-\alpha} - \lambda\, {}_0D_t^{-2\alpha}\right)x_2$$

$$= \left(-\gamma\, {}_0D_t^{-\alpha} - \lambda\, {}_0D_t^{-2\alpha}\right)\left(\gamma^2 F_0 \frac{t^{4\alpha}}{\Gamma(4\alpha+1)} + 2\gamma\lambda F_0 \frac{t^{5\alpha}}{\Gamma(5\alpha+1)} + \lambda^2 F_0 \frac{t^{6\alpha}}{\Gamma(6\alpha+1)}\right)$$

$$= -\gamma^3 F_0 \frac{t^{5\alpha}}{\Gamma(5\alpha+1)} - 3\gamma^2\lambda F_0 \frac{t^{6\alpha}}{\Gamma(6\alpha+1)} - 3\gamma\lambda^2 F_0 \frac{t^{7\alpha}}{\Gamma(7\alpha+1)} - \lambda^3 \frac{t^{8\alpha}}{\Gamma(8\alpha+1)} \text{ etc.}$$

The following table indicates the modal force and displacement for the fractional mechanical oscillator with fractional order derivative α ($0 < \alpha \le 1$):

Table 1:

Mode	**Force**	**Displacement**
0	F_0	$F_0 \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$
1	$-F_0 \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$	$-\gamma F_0 \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} - \lambda F_0 \frac{t^{4\alpha}}{\Gamma(4\alpha+1)}$
2	$\gamma F_0 \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \lambda F_0 \frac{t^{4\alpha}}{\Gamma(4\alpha+1)}$	$\gamma^2 F_0 \frac{t^{4\alpha}}{\Gamma(4\alpha+1)} + 2\gamma\lambda F_0 \frac{t^{5\alpha}}{\Gamma(5\alpha+1)} + \lambda^2 F_0 \frac{t^{6\alpha}}{\Gamma(6\alpha+1)}$
3	$-\gamma^2 F_0 \frac{t^{4\alpha}}{\Gamma(4\alpha+1)} - 2\gamma\lambda F_0 \frac{t^{5\alpha}}{\Gamma(5\alpha+1)} - \lambda^2 F_0 \frac{t^{6\alpha}}{\Gamma(6\alpha+1)}$	$-\gamma^3 F_0 \frac{t^{5\alpha}}{\Gamma(5\alpha+1)} - 3\gamma^2\lambda F_0 \frac{t^{6\alpha}}{\Gamma(6\alpha+1)} - 3\gamma\lambda^2 F_0 \frac{t^{7\alpha}}{\Gamma(7\alpha+1)} - \lambda^3 \frac{t^{8\alpha}}{\Gamma(8\alpha+1)}$
...	...	

Therefore the solution of (10) is given by

$x = x_0 + x_1 + x_2 + \ldots$

$$= F_0 \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \gamma F_0 \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \left(\gamma^2 - \lambda\right) F_0 \frac{t^{4\alpha}}{\Gamma(4\alpha+1)} + \left(2\gamma\lambda - \gamma^3\right) F_0 \frac{t^{5\alpha}}{\Gamma(5\alpha+1)}$$

$$+ \left(\lambda^2 - 3\gamma^2\lambda\right) F_0 \frac{t^{6\alpha}}{\Gamma(6\alpha+1)} + (\gamma^4 - 3\gamma\lambda^2) F_0 \frac{t^{7\alpha}}{\Gamma(7\alpha+1)} + (4\gamma^3\lambda - \lambda^3) F_0 \frac{t^{8\alpha}}{\Gamma(8\alpha+1)} + \ldots.$$

$$F_0 = \frac{k}{m}, \gamma = \frac{\beta\sigma^{1-\alpha}}{m}, \lambda = \frac{k}{m}\sigma^{2(1-\alpha)}.$$

4. Numerical Simulation

Based on ADM, a numerical simulation is done by using MATHEMATICA software for fractional mechanical oscillator given by (10). The step length for t-axis has been taken to be 0.5. We take the parameter value for mechanical oscillator as m = 250 kg, F = 2450 N, x = 0.5 m, so that $k = F/x$ = 4900 N/m. It is to be noted that the parameter α, which represents the fractional order time derivative can be related to the auxiliary parameter σ, which characterizes the existence in the system of fractional structure. For $\beta = 0$, the system (10) gives the relation

$$\alpha = \frac{\sigma}{\sqrt{m/k}} = \sigma\omega_0, \quad 0 < \sigma \le \sqrt{m/k}.$$

Thus for $\beta = 0$; $\gamma = 0$, we have the sets of values of the parameters give as follows:

Table 2.

α	σ	F_0	λ
1	0.22	19.6	19.6
0.75	0.17	19.6	8.036
0.5	0.11	19.6	2.156
0.25	0.06	19.6	0.196

The corresponding graph for fractional mechanical oscillator with αth order element where, α=1, 0.75, 0.5, 0.25 is shown in Figure 3 with step length in t-axis as 0.5.

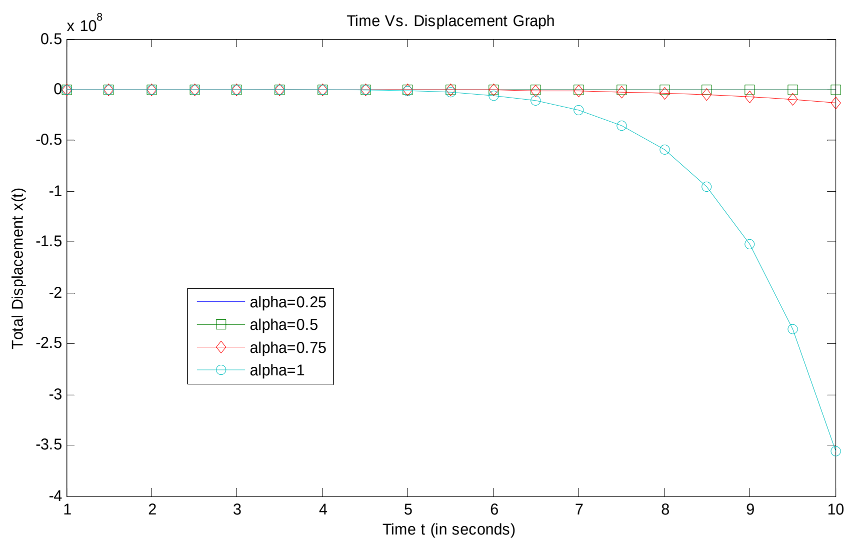

Figure 3. Time Vs. Total Displacement of mechanical oscillator system

Conclusion

In this study, a solution of fractional mechanical oscillator is represented by fractional differential equation (FDE) with αth order derivative in the sense of Riemann Liouville type is developed. It is generally given by driven harmonic oscillators and the equation is a second order differential equation of integer order. In order to tackle the solution to the system, a typical analytical method can be used by using Mittag-Leffler function as well as a series solution can be decomposed as Maclaurin's series at the origin with all its derivative components. But, in the present problem, a very new approach, Adomain Decomposition Method (ADM) is used to obtain the total displacement. It is illustrated in the block diagram (Figure 2) that the total displacement can be decomposed from zeroth mode to infinity mode, where the process parameter will have a solution generated by the system itself to oppose stimulus generated internally by the previous mode with initial conditions $x(0) = 0 = x'(0)$. However, it is demonstrated that using the capabilities of MATHEMATICA software, the total displacement of the oscillator by taking several values of parameters given in Table 2. It is clear from Figure 3, a sharp bent occurs for second order mechanical oscillator system i.e. at α =1. But in reality, it is slightly deviated when past states i.e. the fractional order element say α = 0.75, 0.5, 0.25 is taken into consideration.

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COMMON FIXED POINTS FOR FOUR NON-SELF-MAPPINGS

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Abstract
In this paper, we formulate a quasi-contraction type non-self mapping on Takahashi convex metric spaces and common fixed point theorems that applies to two pairs of mappings. The result generalizes the fixed point theorems of some previous authors.

2010Mathematics Subject Classification: 47H10, 54H25.
Key words: Common fixed point, metrically convex metric space, non-self mapping.

1. Introduction and Preliminaries

Gajić and Rakočević [1] proved a quasi-contraction common fixed point theorem for non-self mappings on Takahashi convex metric spaces for a pair of mappings. In their work they generalized the theorems by Jungck [2], Das and Naik [3], Ćirić et al. [4], Ćirić [5] and Imdad and Kum r [6]. In this study, we extend the theorem by Gajić and Rakočević [1] to apply for two pairs of mappings in metric spaces.

The following are the preliminaries required in this paper.
Given two non-self mappings $f, g: K \to X$ we say that $x \in K$ is *coincidence point* if $fx = gx$. We term the point $y \in X$ as a *point of coincidence* if $y = fx = gx$ where x is a coincidence point. We also say that f and g are *coincidentally commuting* if $fgx = gfx$ whenever x is a coincidence point.

If K is a subset of X, we denote the boundary of K as δK.
Here, we provide the definition of a Takahashi convex metric space which is useful for future discussion.
Definition 1.1. [7]. *Let X be a metric space and $I = [0, 1]$ be the closed unit interval. A mapping $W: X \times X \times I \to X$ is said to be a convex structure on X if for all $x, y \in X$; $\lambda \in I$,*

$$d(u, W(x, y, \lambda)) \le d(u, x) + d(u, y)$$

for every $u \in X$. The metric space (X, d), together with the convex structure is called the Takahashi convex metric space.

If (X, d) is a Takahashi convex metric space, then for every $x, y \in X$, we term

$$\text{seg}[x, y] := \{W(x, y, \lambda) : \lambda \in [0;\ 1]\}.$$

We will use the following property for a Takahashi convex structure in a metric space (X, d).
Lemma 1.2. [7] Let $x, y \in X$ and $z \in \text{seg}[x. y]$, then for all $u \in X$ we have

$$d(u, z) \le \max\{d(u, x), d(u, y)\}.$$

In Gajić and Rakočević [1], the following theorem was proved:

Theorem 1.3. *Let* (X, d) *be a complete Takahashi convex metric space with convex structure* W *which is continuous in the third variable. Let* C *be a non-empty closed subset of* X *and* δC *be the boundary of* C. *Let* $g, f: C \to X$ *and suppose* $\delta C \neq \emptyset$. *Let us assume that* f *and* g *satisfy the following conditions:*

(i) *For every* $x, y \in C, d(gx, gy) \leq M_\omega(x, y)$ *where*

$M_\omega(x, y) =$

$\max\{[\omega_1[d(fx, fy)], \omega_2[d(fx, gx)], \omega_3[d(fy, gy)], \omega_4[d(fx, gy)], \omega_5[d(gx, fy)]\}$,

$\omega_1: [0, +\infty) \to [0, +\infty), i = 1,2,3,4,5,$ *is a non-decreasing semicontinuous function from the right, such that* $\omega_i(r) < r$ *for* $r > 0$, *and* $lim_{r\to\infty}[r - w_i(r)] = +\infty$,

(ii) $\delta C \subseteq f(C)$,

(iii) $g(C) \cap C \subset f(C)$,

(iv) $fx \in \delta C \Rightarrow gx \in C$ *and*

(v) $f(C)$ *is closed in* X.

Then there exists a coincidence point v *in* C. *Moreover, if* f *and* g *are coincidentally commuting, then* v *remains a unique common fixed point of* f *and* g.

2. Results

This paper seeks to modify Theorem 1.3 to four non-self maps. We seek to prove the following theorem.

Theorem 2.1. *Let* (X, d) *be a complete Takahashi convex metric space with convex structure* W *which is continuous in the third variable. Let* K *be a non-empty closed subset of* X. *Let* δK *be the boundary of* K *with* $\delta K \neq \emptyset$. *Let mappings* $A, B, S, T : K \to X$. *Assume that* A, B, S *and* T *satisfy the following conditions:*

(i) *For every* $x, y \in K, d(Ax, By) \leq M_\omega(x, y)$, *where*

$M_\omega(x, y)$

$= max\{[\omega_1[d(Sx, Ty)], \omega_2[d(Ax, Sx)], \omega_3[d(By, Ty)], \omega_4[d(Ax, Ty)], \omega_5[d(Sx, By)]\}$,

$\omega_1: [0, +\infty) \to [0, +\infty), i = 1, 2, 3, 4, 5$ *is a non-decreasing semicontinuous function from the right, such that* $\omega_i(r) < \frac{1}{2} r$ *for* $r > 0$, *and* $lim_{r\to\infty}[r - 2w_i(r)] = +\infty$,

(ii) $\delta K \subseteq T(K), \delta K \subset S(K)$,

(iii) $Sx \in \delta K \Rightarrow Ax \in K,\ Tx \in \delta K \Rightarrow Bx \in K$,

(iv) $A(K) \cap K \subset T(K), B(K) \cap K \subset S(K)$ *and*

(v) $S(K), T(K)$ *are closed in* X.

Then there exists a coincidence point $z \in K$ *for* A, B, S *and* T. *Moreover, if each of the pairs* $\{A, S\}$ *and* $\{B, T\}$ *is coincidentally commuting, then* z *remains a unique common fixed point of* A, B, S *and* T.

Proof. Commencing with an arbitrary point $w \in \delta K$, we construct a sequence $\{x_n\}$ of points in K as follows:

From assumption (ii), there is a point $x_0 \in K$ such that $Sx_0 = w$. From (iii), $Ax_0 \in K$. According to (iv), we find $x_1 \in K$ such that $Tx_1 = Ax_0$. We locate Bx_1. We consider two scenarios.

(1) If $Bx_1 \in K$, then, using (iv), we can locate $x_2 \in K$ such that $Bx_1 = Sx_2$. We then find Ax_2. If it happens $Ax_2 \in K$, then, from (iv), we can find $x_3 \in K$ such that $Ax_2 = Tx_3$. If however $Ax_2 \notin K$, because W is continuous in the third variable, there is $\lambda_{22} \in (0,1)$ such that $W(Sx_2, Ax_2, \lambda_{22}) \in \text{seg}[Sx_2, Ax_2] \cap \delta K$. As $W(Sx_2, Ax_2, \lambda_{22}) \in \delta K$, by (ii), there is $x_3 \in K$ such that $Tx_3 = W(Sx_2, Ax_2, \lambda_{22}) \in \delta K$.

(2) (2) In the case where $Bx_1 \notin K$, because W is continuous in the third variable, there is $\lambda_{11} \in (0,1)$ such that $W(Tx_1, Bx_1, \lambda_{11}) \in \text{seg}[Tx_1, Bx_1] \cap K$. As $W(Tx_1, Bx_1, \lambda_{11}) \in \delta K$, by (ii), there is $x_2 \in K$ such that $Sx_2 = W(Tx_1, Bx_1, \lambda_{11}) \in \delta K$.

In general, we construct the rest of the sequence by proceeding inductively using the following procedure. If $Ax_{2n} \in K$, then, by (iv), we choose $x_{2n+1} \in K$ such that $Tx_{2n+1} = Ax_{2n}$. Similarly if $Bx_{2n+1} \in K$, then, by (iv), we choose $x_{2n+2} \in K$ such that $Sx_{2n+2} = Bx_{2n+1}$.

If however $Ax_{2n} \notin K$, it means, by (iii), there is $\lambda_{2n,2n} \in (0,1)$ and we can choose $x_{2n+1} \in K$ such that $Tx_{2n+1} = W\left(Sx_{2n}, Ax_{2n}, \lambda_{2n,2n}\right) \in K$.
Similarly if $Bx_{2n+1} \notin K$, it means there is $\lambda_{2n+1,2n+1} \in (0,1)$ and we can choose $x_{2n+2} \in K$ such that $Sx_{2n+2} = W\left(Tx_{2n+1}, Bx_{2n+1}, \lambda_{2n+1,2n+1}\right) \in \delta K$.
Now we first prove that

$$Ax_{2n} \neq Tx_{2n+1} \Rightarrow Bx_{2n-1} = Sx_n \tag{2.1}$$

Suppose we have $Bx_{2n-1} \neq Sx_{2n}$. Then we have $Sx_{2n} \in \delta K$, which by (iii) means $Ax_{2n} \in K$. By (iv), this implies that $Ax_{2n} = Tx_{2n+1}$, which is a contradiction. Using a similar argument we have

$$Bx_{2n+1} \neq Sx_{2n+2} \Rightarrow Ax_{2n} = Tx_{2n+1} \tag{2.2}$$

We now prove that the sequences $\{Sx_{2n}\}, \{Ax_{2n}\}, \{Bx_{2n+1}\}$ and $\{Tx_{2n+1}\}$ are bounded. For each $n \geq 1$, let

$$D_n = \left(\bigcup_{i=0}^{n-1}\{Ax_{2i}\}\right) \cup \left(\bigcup_{i=0}^{n-1}\{Bx_{2i+1}\}\right) \cup \left(\bigcup_{i=0}^{n-1}\{Sx_{2i}\}\right) \cup \left(\bigcup_{i=0}^{n-1}\{Tx_{2i+1}\}\right).$$

Let $\alpha_n = \text{diam}(D_n)$. We want to show that

$$\alpha_n \leq \max\{d\left(Sx_0, Ax_{2j}\right), d\left(Sx_0, Bx_{2j+1}\right\}, \ 0 \leq j \leq n-1 \tag{2.3}$$

Let us consider the case where $\alpha_n = 0, n \geq 1$.
If $\alpha_n = 0$, we have $Sx_0 = Ax_0 = Bx_1 = Tx_1$. We shall show that Sx_0 is a common fixed point of A and S. As the mappings A and S are coincidentally commuting at the coincidence point x_0, we have

$$Sx_0 = Ax_0 \Rightarrow SSx_0 = SAx_0 = ASx_0 \tag{2.4}$$

From (i), we have for some $t = 1, 4$ or 5,

$$\begin{aligned} d(SSx_0, Sx_0) = d(ASx_0, Bx_1) &\leq M_\omega(Sx_0, x_1) \\ &= \max\{\omega_1[d(SSx_0, Tx_1)], \omega_2[d(ASx_0, SSx_0)], \omega_3[d(Bx_1, Tx_1)], \\ &\qquad \omega_4[d(ASx_0, Tx_1)], \omega_5[d(SSx_0, Bx_1)]\} \\ &= \max\{\omega_1[d(SSx_0, Sx_0)], \omega_2[d(SSx_0, SSx_0)], \omega_3[d(Sx_0, Sx_0)], \\ &\qquad \omega_4[d(SSx_0, Sx_0)], \omega_5[d(SSx_0, Sx_0)]\} \\ &= \omega_t[d(SSx_0, Sx_0)] \\ &< \frac{1}{2} d(SSx_0, Sx_0) \text{ for } d(SSx_0, Sx_0) > 0 \\ &\Rightarrow \ d(SSx_0, Sx_0) = 0 \\ &\Rightarrow \ SSx_0 = Sx_0. \end{aligned}$$

Hence Sx_0 is a fixed point of S. From (2.4), we have $SSx_0 = ASx_{0,}$ which implies $d(ASx_0, Sx_0) = 0$, making Sx_0 a fixed point of A too.

Using a similar argument we have $Tx_1 = Sx_0$ being a common fixed point of T and B. Hence, $z = Sx_0$ is a common fixed point of all four mappings A, B, S and T.

To show the uniqueness of the fixed point, let z' be also a fixed point of A, B, S and T. Then for some $i = 1, 4$ or 5,

$$\begin{aligned} d(z,z') &= d(Az, Bz') \\ &\le \max\{\omega_1[d(Sz,Tz')], \omega_2[d(Az,Sz)], \omega_3[d(Bz',Tz')], \\ &\qquad \omega_4[d(Az,Tz')], \omega_5[d(Sz,Bz')],\} \\ &= \max\{\omega_i[d(z,z')]\} \\ &< \frac{1}{2} d(z,z') \text{ for } d(z,z') > 0 \\ &\Rightarrow d(z,z') = 0 \\ &\Rightarrow z = z'. \end{aligned}$$

Hence when $\alpha_n = 0, z = Sx_0$ is the unique common fixed point of A, B, S and T.

We now consider the cases when $\alpha_n > 0$.

Case 1: Consider the case where $\alpha_n = d(Sx_{2i}, Ax_{2j})$ for some $0 \le i, j \le n - 1$..
Subcase (1. i): If $i \ge 1$ and $Sx_{2i} = Bx_{2i-1}$ we have for some $s \in \{1, 2, \ldots, 5\}$

$$\begin{aligned} \alpha_n = d(Sx_{2i}, Ax_{2j}) &= d(Ax_{2j}, Bx_{2i-1}) \\ &\le M_\omega(x_{2j}, x_{2i-1}) \\ &\le \omega_s(\alpha_n) \\ &< \tfrac{1}{2}\alpha_n, \end{aligned}$$

which is a contradiction. Hence $i = 0$.

Subcase (1.ii): If however $i \ge 1$ and $Sx_{2i} \ne Bx_{2i-1}$, it implies $Sx_{2i} \in \mathrm{seg}[Ax_{2i-2}, Bx_{2i-1}]$ and hence by Lemma 1.2 we have

$$\alpha_n = d\big(Sx_{2i}, Ax_{2j}\big) \le \max\big\{d\big(Ax_{2j}, Bx_{2i-1}\big), d\big(Ax_{2i-2}, Ax_{2j}\big)\big\}.$$

Subcase (1.ii.1): If $d\big(Ax_{2j}, Bx_{2i-1}\big) \ge d\big(Ax_{2i-2}, Ax_{2j}\big)$, we have

$$\alpha_n = d\big(Sx_{2i}, Ax_{2j}\big) \le d\big(Ax_{2j}, Bx_{2i-1}\big),$$

which leads to the contradiction in Subcase (1.i).

Subcase (1.ii.2): Otherwise if $d\big(Ax_{2j}, Bx_{2i-1}\big) < d\big(Ax_{2i-2}, Ax_{2j}\big)$, then for

k: $2i - 2 < 2k + 1 < 2j$, and for some $s, t \in \{1,2,\ldots,5\}$, we have

$$\begin{aligned} \alpha_n = d\big(Sx_{2i}, Ax_{2j}\big) &\le d\big(Ax_{2i-2}, Ax_{2j}\big) \\ &\le d(Ax_{2i-2}, Bx_{2k+1}) + d(Ax_{2j}, Bx_{2k+1}) \\ &\le M_\omega(x_{2i-2}, x_{2k+1}) + M_\omega\big(x_{2j}, x_{2k+1}\big) \\ &\le \omega_s(\alpha_n) + \omega_t(\alpha_n) \end{aligned}$$

$$
\begin{aligned}
&< \frac{1}{2}\alpha_n + \frac{1}{2}\alpha_n \\
&= \alpha_n,
\end{aligned}
$$

which is a contradiction. Hence $i = 0$.

Case 2: The case where $\alpha_n = d(Ax_{2i}, Bx_{2j+1})$ leads to a contradiction by Subcase (1.i).

Case 3: The case where $\alpha_n = d(Ax_{2i}, Ax_{2j})$ leads to a contradiction by Subcase (1.ii.2).

Case 4: If $\alpha_n = d(Bx_{2i+1}, Bx_{2j+1})$ then for $k : 2i + 1 < 2k < 2j + 1$, and for some $s, t \in \{1,2, \dots ,5\}$, we have

$$
\begin{aligned}
\alpha_n &= d(Bx_{2i+1}, Bx_{2j+1}) \\
&\le d(Ax_{2k}, Bx_{2i+1}) + d(Ax_{2k}, Bx_{2j+1}) \\
&\le M_\omega(x_{2k}, x_{2i+1}) + M_\omega\big(x_{2k}, x_{2j+1}\big) \\
&\le \omega_{s(\alpha_n)} + \omega_t(\alpha_n) \\
&< \frac{1}{2}\alpha_n + \frac{1}{2}\alpha_n \\
&= \alpha_n,
\end{aligned}
$$

which is a contradiction.

Case 5: If $\alpha_n = d(Tx_{2i+1}, Bx_{2j+1})$ for some $0 \le i, j \le n - 1$, then:

Subcase (5.i): If $Tx_{2i+1} = Ax_{2i}$, we have $\alpha_n = d\big(Tx_{2i+1}, Bx_{2j+1}\big) = d\big(Ax_{2i}, Bx_{2j+1}\big)$, which is a contradiction by Subcase (1.i).

Subcase (5.ii): Otherwise if $Tx_{2i+1} \ne Ax_{2i}$ then $Tx_{2i+1} \in \text{seg}[Bx_{2i-1}, Ax_{2i}]$ and hence by Lemma 1.2,

$$\alpha_n = d\big(Tx_{2i+1}, Bx_{2j+1}\big) \le \max\{d\big(Bx_{2i-1}, Bx_{2j+1}\big), d(Ax_{2i}, Bx_{2j+1})\}.$$

This means we have either $d\big(Tx_{2i+1}, Bx_{2j+1}\big) \le d(Bx_{2i-1},\ Bx_{2j+1})$, which is a contradiction by Case 4 or $d\big(Tx_{2i+1}, Bx_{2j+1}\big) \le d(Ax_{2i},\ Bx_{2j+1})$, which is a contradiction by Subcase (1.i).

Case 6: If $\alpha_n = d(Tx_{2i+1}, Ax_{2j})$ for some $0 \le i, j \le n - 1$, then:

Subcase (6.i): If $Tx_{2i+1} = Ax_{2i}$ we have $\alpha_n = d(Tx_{2i+1}, Ax_{2j}) = d(Ax_{2i},\ Ax_{2j})$, which is not possible by Subcase (1.ii.2)

Subcase (6.ii): Otherwise if $Tx_{2i+1} \ne Ax_{2i}$ then $Tx_{2i+1} \in$ seg$[Bx_{2i-1}, Ax_{2i}]$ and hence $\alpha_n = d\big(Tx_{2i+1}, Ax_{2j}\big) \le \max\big\{d\big(Ax_{2j}, Bx_{2i-1}\big), d\big(Ax_{2i}, Ax_{2j}\big)\big\}$. This implies we have either $d\big(Tx_{2i+1}, Ax_{2j}\big) \le d(Ax_{2j}, Bx_{2i-1})$ which is a contradiction by Subcase (1.i) or else we have $d\big(Tx_{2i+1}, Ax_{2j}\big) \le d(Ax_{2i},\ Ax_{2j})$ which is a contradiction by Subcase (1.ii.2).

Case 7: If $\alpha_n = d(Tx_{2i+1}, Tx_{2j+1})$ for some $0 \le i, j \le n - 1$, then:

Subcase (7.i): If $Tx_{2j+1} = Ax_{2j}$, we have $\alpha_n = d(Tx_{2i+1},\ Tx_{2j+1}) = d(Tx_{2i+1},\ Ax_{2j})$ which a contradiction by Case 6.

Subcase (7.ii): Otherwise if $Tx_{2j+1} \neq Ax_{2j}$, then $Tx_{2j+1} \in \mathrm{seg}[Bx_{2j-1},\ Ax_{2j}]$ and hence $\alpha_n = d\big(Tx_{2i+1}, Tx_{2j+1}\big) \leq \max\{d\big(Tx_{2i+1}, Bx_{2j-1}\big), d(Tx_{2i+1}, Ax_{2j})\}$.

This implies we have either $d\big(Tx_{2i+1}, Tx_{2j+1}\big) \leq d(Tx_{2i+1}, Bx_{2j-1})$, which results in a contradiction by Case 5 or else we have $d\big(Tx_{2i+1}, Tx_{2j+1}\big) \leq d(Tx_{2i+1}, Ax_{2j})$, which is a contradiction by Case 6.

Case 8: If $\alpha_n = d(Sx_{2i}, Bx_{2j+1})$ for some $0 \leq i, j \leq n-1$, then:

Subcase (8.i): If $i \geq 1$ and $Sx_{2i} = Bx_{2i-1}$ then $\alpha_n = d(Sx_{2i}, Bx_{2j+1}) = d(Bx_{2i-1}, Bx_{2j+1})$, which is not possible as per Case 4. Hence $i = 0$.

Subcase (8.ii): If however $i \geq 1$ and $Sx_{2i} \neq Bx_{2i-1}$, it means that $Sx_{2i} \in \mathrm{seg}[Ax_{2i-2}, Bx_{2i-1}]$. This implies that
$\alpha_n = d\big(Sx_{2i}, Bx_{2j+1}\big) \leq \max\{d(Ax_{2i-1}Bx_{2j+1}), d\big(Bx_{2i-1}, Bx_{2j+1}\big)\}$. This leads to a contradiction by Subcase (1.i) when
$d\big(Sx_{2i}, Bx_{2j+1}\big) \leq\ d(Ax_{2i-2}, Bx_{2j+1})$ and a contradiction by Case 4 when it happens that
$d\big(Sx_{2i}, Bx_{2j+1}\big) \leq\ d(Bx_{2i-1}, Bx_{2j+1})$. Hence $i = 0$.

Case 9: If $\alpha_n = d(Sx_{2i}, Sx_{2j})$ for some $0 < i\ <\ j < n-1$, then:

Subcase (9.i): If $i \geq 1$ and $Sx_{2j} = Bx_{2j-1}$, then we have
$\alpha_n = d(Sx_{2i},\ Sx_{2j}) = d(Sx_{2i}, Bx_{2j-1})$, which leads to a contradiction according to Case 8. Hence $i = 0$.

Subcase (9.ii): If $i \geq 1$ and $Sx_{2j} \neq Bx_{2j-1}$, it implies that $Sx_{2j} \in \mathrm{seg}[Ax_{2j-2}, Bx_{2j-1}]$ and
$d\big(Sx_{2i}, Sx_{2j}\big) \leq \max\{d\big(Sx_{2i}, Ax_{2j-2}\big),\ d\big(Sx_{2i}, Bx_{2j-1}\big)\}$.
If it happens $d\big(Sx_{2i}, Sx_{2j}\big) \leq d\big(Sx_{2i}, Ax_{2j-2}\big)$, we get a contradiction by Case 1. However if it happens that $d\big(Sx_{2i}, Sx_{2j}\big) \leq d\big(Sx_{2i}, Bx_{2j-1}\big)$, then we get a contradiction by Case 8. Hence $i = 0$.

Case 10: If $\alpha_n = d(Sx_{2i},\ Tx_{2j+1})$, for some $0 \leq i, j \leq n-1$, we have
Subcase (10.i): If $i \geq 1$ and $Sx_{2i} = Bx_{2i-1}$ then we have
$\alpha_n = d\big(Sx_{2i},\ Tx_{2j+1}\big) = d(Tx_{2j+1}, Bx_{2i-1})$, which is not possible as per Case 8. Hence $i - 0$.
Subcase (10.ii): If however $i \geq 1$ and $Sx_{2i} \neq Bx_{2i-1}$ it implies that
$Sx_{2i} \in \mathrm{seg}[Ax_{2i-2}, Bx_{2i-1}]$ and
$d\big(Sx_{2i}, Tx_{2j+1}\big) \leq \max\{d\big(Tx_{2j+1}, Ax_{2i-2}\big), d(Tx_{2j+1}, Bx_{2i-1})\}$. This leads to contradictions by Case 6 and Case 5. Hence $i = 0$.
We have considered 10 possible cases for α_n and conclude that

$$\alpha_n \in \{d\big(Sx_0, Sx_{2j}\big), d\big(Sx_0, Ax_{2j}\big), d\big(Sx_0, Bx_{2j+1}\big), d(Sx_0, Tx_{2j+1})\},$$

for some $0 \leq j \leq\ n-1$. By the construction of the sequences, we have
$d\big(Sx_0, Sx_{2j}\big) \leq \max\{d\big(Sx_0, Ax_{2j-2}\big), d(Sx_0, Bx_{2j-1})\}$ and
$\big(Sx_0, Tx_{2j+1}\big) \leq \max\{d\big(Sx_0, Ax_{2j}\big), d(Sx_0, Bx_{2j-1})\}$. Thus we have now proved (2.3) that is,

$$\alpha_n \leq \max\{d\big(Sx_0, Ax_{2j}\big), d\big(Sx_0, Bx_{2j+1}\big)\}, 0 \leq j \leq 1.$$

Consider the case where $\max\{d(Sx_0, Ax_{2j})\} \leq \max\{d(Sx_0, Bx_{2j+1})\}$, $0 \leq j \leq n-1$. Then we have for some $0 \leq j \leq n-1$, and for some $u \in \{1,2,\dots,5\}$

$$\begin{aligned}\alpha_n &\leq d(Sx_0, Bx_{2j+1})\\ &\leq d(Sx_0, Ax_0) + d(Ax_0, Bx_{2j+1})\\ &\leq d(Sx_0, Ax_0) + \omega_u[\alpha_n]\\ &\leq d(Sx_0, Ax_0) + 2\omega_u[\alpha_n]\end{aligned}$$

$\Rightarrow \alpha_n - 2\omega_u[\alpha_n] \leq d(Sx_0, Ax_0)$.

Alternatively, if $\max\{d(Sx_0, Ax_{2j})\} > \max\{d(Sx_0, Bx_{2j+1})\}$, $0 \leq j \leq n-1$, then for some $0 \leq j \leq n-1$ and for some $v \in \{1,2,\dots,5\}$ and Subcase (1.ii.2) we have

$$\begin{aligned}\alpha_n &\leq d(Sx_0, Ax_{2j})\\ &\leq d(Sx_0, Ax_0) + d(Ax_0, Ax_{2j})\\ &\leq d(Sx_0, Ax_0) + 2\omega_v[\alpha_n]\end{aligned}$$

$\Rightarrow \alpha_n - 2\omega_v[\alpha_n] \leq d(Sx_0, Ax_0)$.

Thus in both cases we have for some $s \in \{1,2,3,4,5\}$

$$\alpha_n - 2\omega_s[\alpha_n] \leq d(Sx_0, Ax_0) \tag{2.5}$$

By assumption (i), there is $r_0 \in [0, +\infty)$ such that for each $s \in \{1,2,\dots,5\}$, we have $r - 2\omega_s[r] > d(Sx_0, Ax_0)$ for $r > r_0$. Thus, there is a subsequence $\{a_n\}$ of $\{\alpha_n\}$ and $s \in \{1,2,\dots,5\}$ such that for each n we have

$$a_n - 2\omega_s[a_n] \leq d(Sx_0, Ax_0).$$

Thus by (2.5), $a_n \leq r_0$, $n = 1,2,\dots$, and also

$$a := \lim_{n\to+\infty} a_n = diam(D) \leq r_0.$$

We have hence proved that $\{Sx_{2n}\}, \{Tx_{2n+1}\}$, $\{Ax_{2n}\}$ and $\{Bx_{2n+1}\}$ are bounded sequences.

To prove that $\{Sx_{2n}\}, \{Tx_{2n+1}\}$, $\{Ax_{2n}\}$ and $\{Bx_{2n+1}\}$ converge in K, we reflect on the set

$$E_n = \left(\bigcup_{i=n}^{+\infty}\{Ax_{2i}\}\right) \cup \left(\bigcup_{i=n}^{+\infty}\{Bx_{2i+1}\}\right) \cup \left(\bigcup_{i=n}^{+\infty}\{Sx_{2i}\}\right) \cup \left(\bigcup_{i=n}^{+\infty}\{Tx_{2i+1}\}\right),$$

$n = 2,3,\dots$.

By (2.3) we have

$$e_n := \mathrm{diam}(E_n) \leq \sup_{j\geq n}\{d(Sx_{2n}, Ax_{2j}), d(Sx_{2n}, Bx_{2j+1})\}, n = 2,3,\dots$$

If $Sx_{2n} = Bx_{2n-1}$, we have, as in Case 1 and Case 8, for each $j \geq n$, and for some $u \in \{1,2,\dots,5\}$

$$\begin{aligned}e_n &\leq \sup_{j\geq n}\{d(Ax_{2j}, Bx_{2n-1}), d(Bx_{2j+1}, Bx_{2n-1}), d(Ax_{2j}, Bx_{2n-1}),\}, n = 2,3,\dots\\ &\leq 2\omega_u[e_{n-1}].\end{aligned} \tag{2.6}$$

If however $Sx_{2n} \neq Bx_{2n-1}$, it implies $Sx_{2n} \in \mathrm{seg}[Ax_{2n-2}, Bx_{2n-1}]$. Hence, as in Case 1 and Case 8, for each $j \geq n$ and for some $u \in \{1,2,\dots,5\}$, we have,

$$e_n \leq \sup_{j\geq n}\{d(Ax_{2n-2}, Ax_{2j}), d(Bx_{2n-1}, Ax_{2j}), d(Ax_{2n-2}, Bx_{2j+1}), d(Bx_{2n-1}, Bx_2$$

$\leq 2\omega_v(e_{n-2}).$ (2.7)

By (2.6) and (2.7), there is a subsequence $\{\varepsilon_n\}$ of $\{e_n\}$ and some $s \in \{1,2,\dots,5\}$ such that for each n, we have

$\varepsilon_n \leq 2\omega_s[\varepsilon_{n-2}], n = 2,3,\dots$

$\leq \varepsilon_{n-2}.$ (2.8)

We note that $e_n \geq e_{n+1}$ for every n. Let $\lim_{n\to\infty} e_n = \lim_{n\to\infty} \varepsilon_n = e$. We claim that $e = 0$. If $e > 0$, then by (2.8) and assumption (i) we have

$$\lim_{n\to\infty} \varepsilon_n < \lim_{n\to\infty} \varepsilon_{n-2} \Rightarrow e < e,$$

which is a contradiction. Hence $e = 0$.

This means that the sequences $\{Sx_{2n}\}, \{Tx_{2n+1}\}$, $\{Ax_{2n}\}$ and $\{Bx_{2n+1}\}$ converge to a point z. Since $\{Sx_{2n}\}, \{Tx_{2n+1}\} \in K$ and $S(K), T(K)$ are closed in the complete metric space X, we conclude that

$$\lim_{n\to\infty} Sx_{2n} = \lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} Ax_{2n} = \lim_{n\to\infty} Bx_{2n+1} = z \in S(K) \cap T(K). \quad (2.9)$$

As $z \in S(K)$, there is a point $u \in K$ such that $Su = z$. We show that u is a coincidence point of A, B and S.

$$\begin{aligned} d(Au, Bx_{2n+1}) &\leq \max\{\omega_1[d(Su, Tx_{2n+1})]; \omega_2[d(Au, Su)], \omega_3[d(Bx_{2n+1}, Tx_{2n+1})], \\ &\qquad \omega_4[d(Au, Tx_{2n+1})], \omega_5[d(Su, Bx_{2n+1})]\} \\ &= \max\{\omega_1[d(z, Tx_{2n+1})]; \omega_2[d(Au, z)], \omega_3[d(Bx_{2n+1}, Tx_{2n+1})], \\ &\qquad \omega_4[d(Au, Tx_{2n+1})], \omega_5[d(z, Bx_{2n+1})]\}. \end{aligned}$$

Taking $n \to +\infty$ and applying (2.9) we get

$$\begin{aligned} d(Au, z) &\leq \max\{\omega_1[d(z, z)], \omega_2[d(Au, z)], \omega_3[d(z, z], \omega_4[d(Au, z)], \omega_5[d(z, z)]\} \\ &\leq \omega_i[d(Au, z)] \text{ for some } i \in \{2,4\} \\ &< d(Au, z) \text{ for } d(Au, z) > 0 \\ &\Rightarrow d(Au, z) = 0 \end{aligned}$$

$\Rightarrow Au = z.$

Using a similar procedure, when we expand $d(Ax_{2n}, Bu)$, we get $Bu = z$ making u a coincidence point of A, B and S. By the coincidental commutativity of S and A we have

$$SAu = ASu \Rightarrow Sz = Az.$$

From (2.9), $z \in T(K)$ means there is $v \in K$, such that $Tv = z$. We show that $Bv = z$.

$$\begin{aligned} d(z, Bv) &= d(Au, Bv) \\ &\leq \max\{\omega_1[d(Su, Tv)], \omega_2[d(Su, Au)], \omega_3[d(Tv, Bv)], \\ &\qquad \omega_4[d(Au, Tv)], \omega_5[d(Su, Bv)]\} \\ &= \max\{\omega_1[d(z, z)], \omega_2[d(z, z)], \omega_3[d(z, Bv)], \omega_4[d(z, z)], \omega_5[d(z, Bv)]\} \\ &\leq \omega_j[d(z, Bv)] \text{ for } j = 3 \text{ or } 5, \\ &< d(z, Bv) \text{ for } d(z, Bv) > 0 \\ &\Rightarrow Bv = z. \end{aligned}$$

Thus v is a coincidence point of B and T. By the coincidental commutativity property, we have $BTv = TBv \Rightarrow Bz = Tz$.

$$\begin{aligned} d(Az, Bz) &\leq \max\{\omega_1[d(Sz, Tz)], \omega_2[d(Sz, Az)], \omega_3[d(Tz, Bz)], \\ &\qquad \omega_4[d(Az, Tz)], \omega_5[d(Sz, Bz)]\} \\ &= \max\{\omega_1[d(Az, Bz)], \omega_2[d(Az, Az)], \omega_3[d(Bz, Bz)], \end{aligned}$$

$$\omega_4[d(Az,Bz)],\omega_5[d(Az,Bz)]\}$$
$$\le \omega_i[d(Az,Bz)] \text{ for } i \in \{1,4,5\}$$
$$< d(Az,Bz) \text{ for } d(Az,Bz) > 0$$
$$\Rightarrow Az = Bz.$$

Hence we have

$$Az = Bz = Sz = Tz. \tag{2.10}$$

Now we consider the following:

$$\begin{aligned}
d(z,Bz) &= d(Au,Bz)\\
&\le \max\{\omega_1[d(Su,Tz)],\omega_2[d(Au,Su)],\omega_3[d(Bz,Tz)],\\
&\qquad \omega_4[d(Au,Tz)],\omega_5[d(Su,Bz)]\}\\
&\le \max\{\omega_1[d(z,Bz)],\omega_2[d(z,z)],\omega_3[d(Bz,Bz)],\\
&\qquad \omega_4[d(z,Bz)],\omega_5[d(z,Bz)]\}\\
&\le \omega_j[d(z,Bz)] \text{ for } j \in \{1,4,5\}\\
&< d(z,Bz) \text{ for } d(z,Bz) > 0
\end{aligned}$$
$$\Rightarrow d(z,Bz) = 0$$
$$\Rightarrow Bz = z.$$

From (2.10) we conclude that

$$Az = Bz = Sz = Tz = z.$$

This means that z is a common fixed point of A, B, S and T.

We now show that z is unique. Suppose z' is also a common fixed point of A, B, S and T. We get

$$\begin{aligned}
d(z,z') &= d(Az,Bz')\\
&\le \max\{\omega_1[d(Sz,Tz')],\ \omega_2[d(Az,Sz)],\ \omega_3[d(Bz',Tz')],\\
&\qquad \omega_4[d(Az,Tz')],\omega_5[d(Sz,Bz')]\}\\
&\le \max\{\omega_1[d(z,z')],\ \omega_2[d(z,z)],\ \omega_3[d(z',z')],\omega_4[d(z,z')],\omega_5[d(z,z')]\}\\
&\le \omega_k[d(z,z')] \text{ for } k \in \{1,4,5\}\\
&< d(z,z') \text{ for } d(z,z') > 0\\
&\Rightarrow d(z,z') = 0\\
&\Rightarrow z = z'.
\end{aligned}$$

This proves that the common fixed point of A, B, S and T is unique.

If we define $\omega_t[r] = hr$ for $0 \le 2h < 1$, we get the following corollary:

Corollary 2.2. *Let (X,d) be a complete Takahashi convex metric space with convex structure W which is continuous in the third variable. Let K be a non-empty closed subset of X and δK be the boundary of K, with $\delta K \neq \emptyset$. Let the mappings $A, B, S, T: K \to X$. Suppose that A, B, S and T satisfy the following conditions:*

(i) *For every $x, y \in K$ we have $d(Ax,By) \le hM(x,y)$, where $0 \le 2h < 1$ and* $M(x,y) = max\{d(Sx,Ty), d(Ax,Sx), d(By,Ty), d(Ax,T y), d(Sx,By)\}$,
(ii) $\delta K \subseteq T(K), \delta K \subseteq S(K)$,
(iii) $Sx \in \delta K \Rightarrow Ax \in K, Tx \in \delta K \Rightarrow Bx \in K$ *and*
(iv) $S(K), T(K)$ *are closed in* X.

Then there exists a coincidence point z *for* $\ ,B,S$ *and* T *in* K. *Moreover, if each of the pairs* $\{A,S\}$ *and* $\{B,T\}$ *is coincidentally commuting, then* z *remains a unique common fixed point of* $\ ,B,S$ *and* T.

We deduce another corollary by letting $A = B$. When this is the situation, in the proof for Theorem 2.1, Case 4 is identical to Subcase (1.i). Moreover, Subcase (1.i) enables us to change the property in Theorem 2.1(i) from $\omega_1[r] < r/2$ for $r > 0$ to $\omega_1[r] < r$ for $r > 0$.

Corollary 2.3. *Let* (X,d) *be a complete Takahashi convex metric space with convex structure* W *which is continuous in the third variable. Let* K *be a non-empty closed subset of* X *and* δK *be the boundary of* K, *with* $\delta K \neq \emptyset$. *Let the mappings* $A,S,T: K \to X$. *Suppose that* $\ ,S$ *and* T *satisfy the following conditions:*

(i) *For every* $x,y \in K, d(Ax,Ay) \leq M_\omega(x,y)$ *where*
$M_\omega(x,y) =$
$max\{\omega_1[d(Sx,Ty)],\omega_2[d(Ax,Sx)],\omega_3[d(Ay,Ty)],\omega_4[d(Ax,Ty)],\omega_5[d(Sx,Ay)]\}$ *where* $\omega_1: [0,+\infty) \to [0,+\infty), i = 1,2,3,4,5,$ *is a non-decreasing semicontinuous function from the right, such that* $\omega_i(r) < r$ *for* $r > 0$, *and* $lim_{r\to\infty}[r - w_i(r)] = +\infty$,
(ii) $\delta K \subseteq T(K), \delta K \subset S(K)$,
(iii) $Sx \in \delta K \Rightarrow Ax \in K,\ Tx \in \delta K \Rightarrow Ax \in K$,
(iv) $\ (K) \cap K \subset T(K),\ \ (K) \cap K \subset S(K)$ *and*
(v) $S(K), T(K)$ *is closed in* X.

Then there exists a coincidence point $z \in K$ *for* A, S *and* T. *Moreover, if each of the pairs* $\{A,S\}$ *and* $\{A,T\}$ *is coincidentally commuting, then* z *remains a unique common fixed point of* $\ ,S$ *and* T.

Remark 1: If we set $S = T$ in Corollary 2.3, we get Theorem 1.3 by Gajić and Rakočević [1].

Remark 2: If we set $S = T = I$ in Corollary 2.3, we get the theorem as proved by Ćirić [5].

We form the following corollary by setting $\ = B = I$ in Theorem 2.1, that is, setting $\ = I$ in Corollary 2.3.

Corollary 2.4. *Let* (X,d) *be a complete Takahashi convex metric space with convex structure* W *which is continuous in the third variable. Let* K *be a non-empty closed subset of* X *and* δK *be the boundary of* K, *with* $\delta K \neq \emptyset$. *Let the mappings* $S,T: K \to X$. *Suppose that* S *and* T *satisfy the following conditions:*

(i) *For every* $x,y \in K, d(x,y) \leq M_\omega(x,y)$ *where*
$M_\omega(x,y) = \max\{\omega_1[d(Sx,Ty)],\omega_2[d(x,Sx)],\omega_3[d(y,Ty)],\omega_4[d(x,Ty)],\omega_5[d(Sx,y)]\}$ and $\omega_1: [0,+\infty) \to [0,+\infty), i = 1,2,3,4,5,$ *is a non-decreasing semicontinuous function from the right, such that* $\omega_i(r) < r$ *for* $r > 0$, *and* $lim_{r\to\infty}[r - w_i(r)] = +\infty$,
(ii) $Sx \in \delta K \Rightarrow x \in K,\ Tx \in \delta K \Rightarrow x \in K$,
(iii) $K \subset T(K), K \subset S(K)$ *and*
(iv) $S(K), T(K)$ *is closed in* X.

Then there exists a unique common fixed point of S *and* T.

We form yet another corollary from Corollary 2.4 by setting $S = T$.

Corollary 2.5. *Let* (X,d) *be a complete Takahashi convex metric space with convex structure* W *which is continuous in the third variable. Let* K *be a non-empty closed subset of* X *and* δK

be the boundary of K, with $\delta K \neq \emptyset$. Let the mapping $S: K \to X$. Suppose that S satisfies the following conditions:

(i) *For every $x, y \in K, d(x,y) \leq M_\omega(x,y)$ where*
$M_\omega(x,y) = \max\{\omega_1[d(Sx,Sy)], \omega_2[d(x,Sx)], \omega_3[d(y,Sy)], \omega_4[d(x,Sy)], \omega_5[d(Sx,y)]\}$ and $\omega_1: [0,+\infty) \to [0,+\infty), i = 1,2,3,4,5,$ *is a non-decreasing semicontinuous function from the right, such that $\omega_i(r) < r$ for $r > 0$, and $lim_{r\to\infty}[r - w_i(r)] = +\infty$,*

(ii) $Sx \in \delta K \Rightarrow x \in K$,

(iii) $K \subset S(K)$ *and*

(iv) $S(K)$ *is closed in X.*

Then there exists a unique fixed point of S.

If we let $B = I$ in Theorem 2.1, to get the following corollary:

Corollary 2.6.
Let (X,d) be a complete Takahashi convex metric space with convex structure W which is continuous in the third variable. Let K be a non-empty closed subset of X. Let δK be the boundary of K with $\delta K \neq \emptyset$. Let mappings $A, S, T : K \to X$. Assume that A, S and T satisfy the following conditions:

(vi) *For every $x, y \in K, d(Ax,y) \leq M_\omega(x,y)$, where*
$M_\omega(x,y)$
$= max\{[\omega_1[d(Sx,Ty)], \omega_2[d(Ax,Sx)], \omega_3[d(y,Ty)], \omega_4[d(Ax,Ty)], \omega_5[d(Sx,y)]\}$,
$\omega_1: [0,+\infty) \to [0,+\infty), i = 1,2,3,4,5$ *is a non-decreasing semicontinuous function from the right, such that $\omega_i(r) < \frac{1}{2} r$ for $r > 0$, and $lim_{r\to\infty}[r - 2w_i(r)] = +\infty$,*

(vii) $\delta K \subseteq T(K)$,

(viii) $Sx \in \delta K \Rightarrow Ax \in K,\ Tx \in \delta K \Rightarrow Bx \in K$,

(ix) $A(K) \cap K \subset T(K), K \subset S(K)$ *and*

(x) $S(K), T(K)$ *are closed in X.*

Then there exists a coincidence point $z \in K$ for A, S and T. Moreover, if the pair $\{A, S\}$ is coincidentally commuting, then z remains a unique common fixed point of A, S and T.

When we set S = T in Theorem 2.1, we get the following corollary:

Corollary 2.7. *Let (X,d) be a complete Takahashi convex metric space with convex structure W which is continuous in the third variable. Let K be a non-empty closed subset of X. Let δK be the boundary of K with $\delta K \neq \emptyset$. Let mappings $A, B, S : K \to X$. Assume that A, B and S satisfy the following conditions:*

(xi) *For every $x, y \in K, d(Ax,By) \leq M_\omega(x,y)$, where*
$M_\omega(x,y)$
$= max\{[\omega_1[d(Sx,Sy)], \omega_2[d(Ax,Sx)], \omega_3[d(By,Sy)], \omega_4[d(Ax,Sy)], \omega_5[d(Sx,By)]\}$,
$\omega_1: [0,+\infty) \to [0,+\infty), i = 1,2,3,4,5,$ *is a non-decreasing semicontinuous function from the right, such that $\omega_i(r) < \frac{1}{2} r$ for $r > 0$, and $lim_{r\to\infty}[r - 2w_i(r)] = +\infty$.*

(xii) $\delta K \subset S(K)$,

(xiii) $Sx \in \delta K \Rightarrow Ax, Bx \in K$,

(xiv) $A(K) \cap K \subset S(K), B(K) \cap K \subset S(K)$ *and*

(xv) $S(K)$, *is closed in X.*

Then there exists a coincidence point $z \in K$ for A, B and S. Moreover, if each of the pairs $\{S, A\}$ and $\{S, B\}$ is coincidentally commuting, then z remains a unique common fixed point of A, B and S.

We form another corollary by setting $x = y$ in Theorem 2.1.

Corollary 2.8. *Let (X, d) be a complete Takahashi convex metric space with convex structure W which is continuous in the third variable. Let K be a non-empty closed subset of X. Let δK be the boundary of K with $\delta K \neq \emptyset$. Let mappings $S, T, A, B : K \to X$. Assume that S, T, A and B satisfy the following conditions:*

(i) For every $x, y \in K, d(Ax, Bx) \leq M_\omega(x)$, where
$M_\omega(x)$
$= max\{[\omega_1[d(Sx, Tx)], \omega_2[d(Ax, Sx)], \omega_3[d(Bx, Tx)], \omega_4[d(Ax, Tx)], \omega_5[d(Sx, Bx)]\}$,
$\omega_1: [0, +\infty) \to [0, +\infty), i = 1, 2, 3, 4, 5$, *is a non-decreasing semicontinuous function from the right, such that $\omega_i(r) < \frac{1}{2}r$ for $r > 0$, and $lim_{r\to\infty}[r - 2w_i(r)] = +\infty$.*
(ii) $\delta K \subseteq T(K), \delta K \subset S(K)$,
(iii) $Sx \in \delta K \Rightarrow Ax \in K,\ Tx \in \delta K \Rightarrow Bx \in K$,
(iv) $A(K) \cap K \subset T(K), B(K) \cap K \subset S(K)$ and
(v) $S(K), T(K)$ is closed in X.
Then there exists a coincidence point $z \in K$ for A, B, S and T. Moreover, if each of the pairs $\{S, A\}$ and $\{T, B\}$ is coincidentally commuting, then z remains a unique common fixed point of A, B, S and T.

Here we give an example on the use of our result (Theorem 2.1).

Example 2.1: Let $X = [0, +\infty), K = [1, 3]$ and $d(x, y) = |x - y|$.
Let $\omega_i[r] = \frac{1}{3}r$ for $i \in \{1, 2, 3, 4, 5\}$. We note that $\omega_i[r] < \frac{1}{2}r$. Define $A, B, S, T: K \to X$ by

$$Sx = \begin{cases} 2x^4 - 1 & for\ x \in [1,2] \\ 7 & for\ x \in (2,3] \end{cases}, \qquad Ax = \begin{cases} x^2 & for\ x \in [1,2] \\ 1 & for\ x \in (2,3] \end{cases}$$

$$Sx = \begin{cases} 2x^6 - 1 & for\ x \in [1,2] \\ 7 & for\ x \in (2,3] \end{cases}, \qquad Ax = \begin{cases} x^3 & for\ x \in [1,2] \\ 2 & for\ x \in (2,3] \end{cases}.$$

We have $S(K) = [1, 31], T\ (K) = [0, 127]$ both of which are closed. We also have $\delta K = \{1,3\} \subseteq S(K), T(K)$.
We find out that $\{x \in K: Sx \in \delta K\} = \{1, 2^{1/4}\}$ and $A(\{1, 2^{1/4}\}) = \{1, \sqrt{2}\} \in K$. Similarly $\{x \in K: Tx \in \delta K\} = \{1, 2^{1/6}\}$ and $B(\{1, 2^{1/6}\}) = \{1, \sqrt{2}\} \in K$.
We note that $\{A, S\}$ and $\{B, T\}$ are both coincidentally commuting at $x = 1$, that is, $SA(1) = AS(1) = 1$ and $TB(1) = BT(1) = 1$. We also note that all four mappings are discontinuous at $x = 2$. Without loss of generality let $y \geq x$.

Consider $x, y \in (2, 3]$. Then we have

$$d(Ax, By) = d(2,2) = 0 \leq \frac{1}{3}d(Sx, Ty).$$

For $x, y \in [1,2]$, we have

$$
\begin{aligned}
d(Ax, By) &= |x^2 - y^3| \\
&= |x^2 - y^3| \times \frac{|x^2 + y^3|}{|x^2 + y^3|} \\
&= \frac{|x^4 - y^6|}{|x^2 + y^3|} \\
&\leq \frac{1}{2} \times \frac{1}{2} |2x^4 - 2x^6| \\
&= \frac{1}{4} d(Sx, Ty) \\
&\leq \frac{1}{3} d(Sx, Ty).
\end{aligned}
$$

Finally, for $x \in [1, 2]$, $y \in (2, 3]$, we get

$$
\begin{aligned}
d(Ax, By) &= |x^2 - 2| \\
&= |x^2 - 2| \frac{|x^2 + 2|}{|x^2 + 2|} \\
&= \frac{|x^4 - 4|}{|x^2 + 2|} \\
&< \frac{1}{3} \times \frac{1}{2} |2x^4 - 8| \\
&= \frac{1}{6} d(Sx, Ty) \\
&\leq \frac{1}{3} d(Sx, Ty).
\end{aligned}
$$

Thus in all cases, for every $x, y \in K$, we have

$$
d(Ax, By) \leq \max\{\, \omega_1[d(Sx, Ty)], \omega_2[d(Ax, Sx)], \omega_3[d(By, Ty)], \omega_4[d(Ax, Ty)], \omega_5[d(Sx, By)]\}.
$$

Thus all the conditions of Theorem 2.1 are satisfied and 1 is the unique common fixed point of A, B, S and T.

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ON A BI DIMENSIONAL BASIS INVOLVING SPECIAL FUNCTIONS FOR PARTIAL IN SPACE AND TIME FRACTIONAL WAVE MECHANICAL PROBLEMS AND APPROXIMATION

By

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Abstract
In this work, we construct a fractional time dependent wave mechanical problem consisting partial in space and time fractional derivatives and solved it on introducing a bi dimensional basis function involving Hermite and Mittag – Leffler functions. Then, we use it to approximate the solution of above two variable wave mechanical problem and then discuss its various cases in which it is computable.

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1. Introduction

The Hermite functions play an important role in the wave mechanical treatment of the harmonic oscillator. (see Mott and Sneddon [11, p. 50], Eravanis [1]). It also has great importance in the application to the quantum theory of radiation [19]. In this connection, the time independent Schrödinger equation corresponding to harmonic oscillator of point mass m with vibrational frequency ν is given by (Sneddon [17])

$$\frac{d^2\Psi_n}{dx^2} + \left(\frac{2T}{h\nu} - x^2\right)\Psi_n = 0, \tag{1}$$

$where,$ T is the total energy of the oscillator and h is the Plank's constant. Here, the wave functions Ψ_n have the property that

$$\Psi_n \to 0 \text{ as } |x| \to \infty; \tag{2}$$

and

$$\int_{-\infty}^{\infty} |\Psi_n|^2 dx = 2\pi\sqrt{\frac{m\nu}{h}} \tag{3}$$

In Eqn. (1), there is the Hermite function $\Psi_n = e^{-\frac{x^2}{2}} H_n(x)$ and $\frac{2T}{h\nu} = 2n + 1$, where, $H_n(x)$ being the Hermite polynomials [13] for all $n = 0, 1, 2, \ldots$, satisfying the differential equation

$$\frac{d^2 H_n(x)}{dx^2} - 2x\frac{d}{dx}H_n(x) + 2nH_n(x) = 0 \tag{4}$$

The $H_n(x)$ is defined by

$$H_n(x) = \begin{cases} (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{x^{-2}}, (the\ operational\ formula) \\ (2x)^n \sum_{r=0}^{\infty} \frac{\Gamma\left(-\frac{n}{2}+r\right)\Gamma\left(\frac{1}{2}-\frac{n}{2}+r\right)}{\Gamma\left(-\frac{n}{2}\right)\Gamma\left(\frac{1}{2}-\frac{n}{2}\right)\Gamma(1+r)} (-x^{-2})^r, (the\ series\ formula) \end{cases} \tag{5}$$

The function Ψ_n also satisfies the recursion formula

$$2x\Psi_n = 2n\Psi_{n-1} + \Psi_{n+1} \tag{6}$$

L. C. Evans [6] introduce a fundamental solution $\Theta(x,t) = \begin{cases} \frac{1}{\sqrt{4\pi t}} e^{-\frac{|x|^2}{4t}} (x \in \mathbb{R}, t > 0) \\ 0 \qquad (x \in \mathbb{R}, t < 0) \end{cases}$ for the heat equation $\frac{\partial}{\partial t}\Theta(x,t) - \frac{\partial^2}{\partial x^2}\Theta(x,t) = 0.$ (7)

Some two variable analogue of Hermite polynomials are studied through operational rules [3], [4], [14], in this study, the one variable Hermite polynomials [13] have the relation with two variable Hermite polynomials due to operational techniques.

Motivated by above work, by bi dimensional basis function, we construct a fractional time dependent wave mechanical problem consisting partial in space and time fractional derivatives and then study its properties and application in approximation theory to discuss its various cases in which it is computable.

Now, for constructing a fractional time dependent wave mechanical problem consisting partial in space and time fractional derivatives and to extend the work of Sneddon [17] and Evans [6], and on motivation of the theory due to [3], [4], [12] and [14], we introduce following new bi dimensional basis solution of that problem

$$\Psi_n^\alpha(x,t;\lambda_n) = \begin{cases} e^{-\frac{x^2}{2}} H_n(x) E_\alpha(\lambda_n t^\alpha), 0 < \alpha \leq 2, \lambda_n = -(2n+1), x \in (-\infty,\infty), t > 0, & (8,\mathrm{I}) \\ e^{-\frac{x^2}{2}} H_n(x), \qquad x \in (-\infty,\infty), t = 0, & (8,\mathrm{II}) \\ 0, \qquad x \in (-\infty,\infty), t < 0. & (8,\mathrm{III}) \end{cases}$$

Here, in Eqn. (8, I), the function $E_\alpha(\lambda_n t^\alpha)$ is the Mittag – Leffler function [10], satisfying the fractional differential equation (see, Gorenflo and Mainardi [7], Mainardi [8] and Mainardi and Gorenflo [9])

$${}_0^C\mathcal{D}_t^\alpha u(t) = \lambda_n u(t), u(t) = E_\alpha(\lambda_n t^\alpha), \tag{9}$$

where, ${}_0^C\mathcal{D}_t^\alpha$, is the Caputo derivative, defined by (see Diethelm [5, p. 49], Samko et al. [16])

$${}_0^C\mathcal{D}_t^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(k-\alpha)} \int_0^t (t-x)^{k-\alpha-1} D_x^k f(x) dx, k-1 < \alpha \leq k, \\ 0, \qquad \alpha > k, \end{cases}$$

And $\lim_{\alpha \to k} {}_0^C\mathcal{D}_t^\alpha f(t) = D_t^k f(t) = \frac{d^k}{dt^k} f(t), \forall k \in \mathbb{N} = \{1,2,3 \ldots \ldots\}$. (10)

1 Wave Mechanical Problem and its Properties

For $t > 0$, consider the function defined in Eqn. (8, I) and make the following operation to get

$$\left(\frac{\partial^2}{\partial x^2} - x^2\right) \Psi_n^\alpha(x,t;\lambda_n) = e^{-\frac{x^2}{2}} \left[\frac{\partial^2 H_n(x)}{\partial x^2} - 2x\frac{\partial}{\partial x} H_n(x) + 2nH_n(x)\right] E_\alpha(\lambda_n t^\alpha)$$
$$-(2n+1)\Psi_n^\alpha(x,t;\lambda_n) \tag{11}$$

Now, use the differential equations (4) and (9) in right hand side of the Eqn. (11) to get the required fractional time dependent wave mechanical problem consisting partial in space and time fractional derivatives in the form

$$\left(\frac{\partial^2}{\partial x^2} - x^2\right) \Psi_n^\alpha(x,t;\lambda_n) = {}_0^C\mathcal{D}_t^\alpha \Psi_n^\alpha(x,t;\lambda_n) \tag{12}$$

Clearly, at stationary position the Eqn. (12) is converted into the Eqn. (1) as setting the transformation $-\lambda_n = 2n+1 = \frac{2T}{h\nu}$ and the function $\Psi_n^\alpha(x,t;\lambda_n)$ has one of the conditions that $\Psi_n^\alpha(x,t;\lambda_n) \to 0 \; as \; |x| \to \infty$. (13)

Again, from Eqn. (12) we may write

$$\left(\frac{\partial^2}{\partial x^2}-x^2\right)\Psi_m^\alpha(x,t;\lambda_m)={}_0^C\mathcal{D}_t^\alpha\Psi_m^\alpha(x,t;\lambda_m) \quad (14)$$

Theorem 1: By the basis function $\Psi_n^\alpha(x,t;\lambda_n)$, defined in Eqn. (8) and satisfying the Eqn. (12) for a fix n, there exists an eigen value problem

${}_0^C\mathcal{D}_t^\alpha U(x,t;\lambda_n)=\lambda_n U(x,t;\lambda_n), where, \lambda_n=-(2n+1).$

Then $U=\Psi_n^\alpha(x,t;\lambda_n)$. (15)

Proof:

We multiply by $\Psi_n^\alpha(x,t;\lambda_n)$ in both sides of Eqn. (14) and that by $\Psi_m^\alpha(x,t;\lambda_m)$ in both sides of Eqn. (12) and again on subtracting them, and then integrating with respect to x both of the sides to get

$$\int_{-\infty}^{\infty}\left[\Psi_m^\alpha(x,t;\lambda_m)\frac{\partial^2}{\partial x^2}\Psi_n^\alpha(x,t;\lambda_n)-\Psi_n^\alpha(x,t;\lambda_n)\frac{\partial^2}{\partial x^2}\Psi_m^\alpha(x,t;\lambda_m)\right]dx$$

$$=\int_{-\infty}^{\infty}\left[\Psi_m^\alpha(x,t;\lambda_m){}_0^C\mathcal{D}_t^\alpha\Psi_n^\alpha(x,t;\lambda_n)-\Psi_n^\alpha(x,t;\lambda_n){}_0^C\mathcal{D}_t^\alpha\Psi_m^\alpha(x,t;\lambda_m)\right]dx \quad (16)$$

An integrating by parts in left hand side of Eqn. (16) shows that

$$\left[\Psi_m^\alpha(x,t;\lambda_m)\frac{\partial}{\partial x}\Psi_n^\alpha(x,t;\lambda_n)-\Psi_n^\alpha(x,t;\lambda_n)\frac{\partial}{\partial x}\Psi_m^\alpha(x,t;\lambda_m)\right]\Bigg|_{x=-\infty}^{x=\infty}-$$

$$\int_{-\infty}^{\infty}\left[\frac{\partial}{\partial x}\Psi_m^\alpha(x,t;\lambda_m)\frac{\partial}{\partial x}\Psi_n^\alpha(x,t;\lambda_n)-\frac{\partial}{\partial x}\Psi_n^\alpha(x,t;\lambda_n)\frac{\partial}{\partial x}\Psi_m^\alpha(x,t;\lambda_m)\right]dx$$

$$=\int_{-\infty}^{\infty}\left[\Psi_m^\alpha(x,t;\lambda_m){}_0^C\mathcal{D}_t^\alpha\Psi_n^\alpha(x,t;\lambda_n)-\Psi_n^\alpha(x,t;\lambda_n){}_0^C\mathcal{D}_t^\alpha\Psi_m^\alpha(x,t;\lambda_m)\right]dx \quad (17)$$

Now, use the condition (13) in first part of left hand side of Eqn. (17) to find the Sturm – Liouville Equation

$$\int_{-\infty}^{\infty}\left[\Psi_m^\alpha(x,t;\lambda_m){}_0^C\mathcal{D}_t^\alpha\Psi_n^\alpha(x,t;\lambda_n)-\Psi_n^\alpha(x,t;\lambda_n){}_0^C\mathcal{D}_t^\alpha\Psi_m^\alpha(x,t;\lambda_m)\right]dx=0 \quad (18)$$

The Eqn. (18) shows that in integrand there exists Wronskian determinant such that

$$\begin{vmatrix}\Psi_m^\alpha(x,t;\lambda_m) & \Psi_n^\alpha(x,t;\lambda_n)\\ {}_0^C\mathcal{D}_t^\alpha\Psi_m^\alpha(x,t;\lambda_m) & {}_0^C\mathcal{D}_t^\alpha\Psi_n^\alpha(x,t;\lambda_n)\end{vmatrix}=0$$

Therefore, due to Sturm – Liouville Equation (18) (see R. V. Churchill [2]), for a fix n, there exists an eigen value problem

${}_0^C\mathcal{D}_t^\alpha\Psi_n^\alpha(x,t;\lambda_n)=\lambda_n\Psi_n^\alpha(x,t;\lambda_n), where, \lambda_n=-(2n+1)$, which on comparing with the Eqn. (15) gives us $U=\Psi_n^\alpha(x,t;\lambda_n)$. Hence the theorem has proved.

Theorem 2: If $\Psi_n^\alpha(x,t;\lambda_n)$ satisfies the Eqn. (12), then, for $0<\alpha\le 2$ there exists a quadrature formula with weight function e^{-t} of the determinant due to molecular vibrations

$$B_{n,n-1}^\alpha=2^n n!\sqrt{\pi}\sum_{k=0}^{\infty}P_k^\alpha\left(\left(\frac{(2n-1)}{(2n+1)}\right)\right)\{-(2n+1)\}^k,$$

$$P_k^\alpha\left(\left(\frac{(2n-1)}{(2n+1)}\right)\right)=\sum_{l=0}^{k}\frac{\Gamma(\alpha k+1)\left(\frac{(2n-1)}{(2n+1)}\right)^l}{\Gamma(\alpha l+1)\Gamma(\alpha(k-l)+1)}.$$

Proof: Apply the Eqn. (15) in the Eqn. (18) to get

$$\int_{-\infty}^{\infty}\left[\Psi_m^\alpha(x,t;\lambda_m){}_0^C\mathcal{D}_t^\alpha\Psi_n^\alpha(x,t;\lambda_n)-\Psi_n^\alpha(x,t;\lambda_n){}_0^C\mathcal{D}_t^\alpha\Psi_m^\alpha(x,t;\lambda_m)\right]dx$$

$$=2(m-n)\int_{-\infty}^{\infty}[\Psi_m^\alpha(x,t;\lambda_m)\Psi_n^\alpha(x,t;\lambda_n)-\Psi_n^\alpha(x,t;\lambda_n)\Psi_m^\alpha(x,t;\lambda_m)]dx=0, when\ m\neq n. \quad (19)$$

Hence if we *let* $\varphi_{m,n}^\alpha(t)=\int_{-\infty}^{\infty}\Psi_m^\alpha(x,t;\lambda_m)\Psi_n^\alpha(x,t;\lambda_n)\,dx$, *in Eqn. (19), and then*

$\varphi_{m,n}^{\alpha}(t) = \int_{-\infty}^{\infty} \Psi_m^{\alpha}(x,t;\lambda_m)\Psi_n^{\alpha}(x,t;\lambda_n)dx = \int_{-\infty}^{\infty} \Psi_n^{\alpha}(x,t;\lambda_n)\Psi_m^{\alpha}(x,t;\lambda_m)dx, \, when \; m \neq n.$ (20)

This Eqn. (20) shows that the product $\Psi_m^{\alpha}(x,t;\lambda_m)\Psi_n^{\alpha}(x,t;\lambda_n)$ is commutative with respect to m and n.

In particular, the Eqn. (20) follows that

$$\varphi_{n-1,n+1}^{\alpha}(t) = 0 \Longrightarrow \varphi_{n-1,n+1}^{\alpha}(t) = \int_{-\infty}^{\infty} \Psi_{n-1}^{\alpha}(x,t;\lambda_{n-1})\Psi_{n+1}^{\alpha}(x,t;\lambda_{n+1})dx = 0. \tag{21}$$

Again, from Eqns. (8) and (20), we find that

$$\varphi_{m,n}^{\alpha}(t) = \{E_\alpha(\lambda_n t^\alpha)E_\alpha(\lambda_m t^\alpha)\}\int_{-\infty}^{\infty} e^{-x^2}H_n(x)H_m(x)dx \tag{22}$$

Eqn. (22) gives us

$$\varphi_{m,n}^{\alpha}(t) = \{E_\alpha(\lambda_n t^\alpha)E_\alpha(\lambda_m t^\alpha)\}2^n n!\sqrt{\pi}\delta_{mn} \tag{23}$$

Again, by Eqn. (6), we may write the relation

$$2x\Psi_n^{\alpha}(x,t;\lambda_n) = -\frac{E_\alpha(\lambda_n t^\alpha)}{E_\alpha(\lambda_{n-1}t^\alpha)}\,{}_0^C\mathcal{D}_t^\alpha \Psi_{n-1}^{\alpha}(x,t;\lambda_{n-1}) + \frac{E_\alpha(\lambda_n t^\alpha)}{E_\alpha(\lambda_{n-1}t^\alpha)}\Psi_{n-1}^{\alpha}(x,t;\lambda_{n-1}) + \frac{E_\alpha(\lambda_n t^\alpha)}{E_\alpha(\lambda_{n+1}t^\alpha)}\Psi_{n+1}^{\alpha}(x,t;\lambda_{n+1}) \tag{24}$$

Then, on applying the Eqns. (21), (23) and (24) and Theorem 1, for $t \in (0,\infty)$ we get the determinant (this type of determinant has used to study molecular vibrations ([1] and [19]))

$$A_{n,n-1}^{\alpha}(t) = \int_{-\infty}^{\infty} 2x\,\Psi_n^{\alpha}(x,t;\lambda_n)\Psi_{n-1}^{\alpha}(x,t;\lambda_{n-1})dx = E_\alpha(\lambda_{n-1}t^\alpha)E_\alpha(\lambda_n t^\alpha)2^n n!\sqrt{\pi} \tag{25}$$

Thus from the Eqn. (25), we find the quadrature formula with weight e^{-t} of the determinant $A_{n,n-1}^{\alpha}(t)$, given by

$$B_{n,n-1}^{\alpha} = \int_0^{\infty} e^{-t}A_{n,n-1}^{\alpha}(t)dt$$

$$= \int_0^{\infty}\int_{-\infty}^{\infty} 2x\,e^{-t}\,\Psi_n^{\alpha}(x,t;\lambda_n)\Psi_{n-1}^{\alpha}(x,t;\lambda_{n-1})dxdt = 2^n n!\sqrt{\pi}\sum_{k=0}^{\infty} P_k^{\alpha}\left(\left(\frac{(2n-1)}{(2n+1)}\right)\right)\{-(2n+1)\}^k \tag{26}$$

where, $$P_k^{\alpha}\left(\left(\frac{(2n-1)}{(2n+1)}\right)\right) = \sum_{l=0}^{k}\frac{\Gamma(\alpha k+1)\left(\frac{(2n-1)}{(2n+1)}\right)^l}{\Gamma(\alpha l+1)\Gamma(\alpha(k-l)+1)} = \sum_{l=0}^{k}\binom{\alpha k}{\alpha k - \alpha l}\left(\frac{(2n-1)}{(2n+1)}\right)^l. \tag{27}$$

Hence the theorem has proved.

Remark1. By the polynomial given in Eqn. (27), we may find the Riordon arrays [15] and this result may also used it in combinatorial theory.

Corollary1. There exists the quadrature formula of the determinant

$$lim_{\alpha\to 1} B_{n,n-1}^{\alpha} = 2^n n!\sqrt{\pi}\frac{1}{(1+4n)}, \tag{28}$$

and $$lim_{\alpha\to 2} B_{n,n-1}^{\alpha} = 2^{n-1}n!\,\pi F_4\left[\begin{matrix}1,\frac{1}{2};\\ \frac{1}{2},\frac{1}{2};\end{matrix} -(2n+1), -(2n-1)\right], \text{ provided that } n > 0. \tag{29}$$

In Eqns. (26) and (27) put the limit $\alpha \to 1$, we have

$$B_{n,n-1}^{1} = \int_0^{\infty}\int_{-\infty}^{\infty} 2x\,e^{-t}\,\Psi_n^{\alpha}(x,t;\lambda_n)\Psi_{n-1}^{\alpha}(x,t;\lambda_{n-1})dxdt$$

$$= 2^n n!\sqrt{\pi}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{\Gamma(k+l+1)\left(-(2n-1)\right)^l\left(-(2n+1)\right)^k}{\Gamma(l+1)\Gamma(k+1)} \tag{30}$$

Now use the Srivastava's formula (see Srivastava and Manocha [18, *p.* 52]) in Eqn. (30), finally, we find the relation (28).

Now, set the limit $\alpha \to 2$, in Eqns. (26) and (27), we have

$B_{n,n-1}^2 = \int_0^\infty \int_{-\infty}^\infty 2x\, e^{-t}\, \Psi_n^\alpha(x,t;\lambda_n)\Psi_{n-1}^\alpha(x,t;\lambda_{n-1})dxdt$

$= 2^{n-1}n!\,\pi \sum_{k=0}^\infty \sum_{l=0}^\infty \frac{(1)_{k+l}\left(\frac{1}{2}\right)_{k+l}}{\left(\frac{1}{2}\right)_l\left(\frac{1}{2}\right)_k} \frac{\left(-(2n+1)\right)^k}{k!} \frac{\left(-(2n-1)\right)^l}{l!}$

$= 2^{n-1}n!\,\pi F_4 \left[\begin{matrix} 1, \frac{1}{2}; \\ \frac{1}{2}, \frac{1}{2}; \end{matrix} -(2n+1), -(2n-1)\right],$

which is valid for $\sqrt{|2n-1|} + \sqrt{|2n+1|} < 1.$ (31)

The Eqn. (31) follows that

$|2n+1| < \left(1 - \sqrt{|2n-1|}\right)^2 = |1+2n| - 1 - 2\sqrt{|2n-1|}$ which implies that

$2\sqrt{|2n-1|} < -1, \Longrightarrow \sqrt{|2n-1|} < -\frac{1}{2}$, $or\ |2n-1| > \frac{1}{4}$, $or\ 2n-1 > \frac{1}{4}, \Rightarrow\ n > \frac{5}{8} > 0.$

Hence the result (29) has followed.

3. Application for evaluation of the Solution of given Problem

On making an appeal to the Eqn. (18), the wave mechanical problem (12) in the form $\left(\frac{\partial^2}{\partial x^2} - x^2\right)\Phi = {}_0^C\mathcal{D}_t^\alpha \Phi$, has the solution $\Phi = K\Psi_n^\alpha(x,t;\lambda_n)$,

where, $-\lambda_n = 2n+1.$ (32)

Hence due to Eqn. (32), we may write

$|\Phi|^2 = K^2\Psi_n^\alpha(x,t;\lambda_n)\Psi_n^\alpha(x,t;\lambda_n)$ and thus we have

$\int_0^\infty \int_{-\infty}^\infty e^{-t}\, |\Phi|^2 dxdt = 2^n n!\,\sqrt{\pi}K^2 \int_0^\infty e^{-t}\, \{E_\alpha(\lambda_n t^\alpha)E_\alpha(\lambda_n t^\alpha)\}dt$

$= 2^n n!\,\sqrt{\pi}K^2 \sum_{k,l=0}^\infty \frac{\Gamma(\alpha(k+l)+1)(\lambda_n)^{k+l}}{\Gamma(\alpha k+1)\Gamma(\alpha l+1)}$ (33)

Therefore, by Eqn. (33), we obtain

$K = \left\{\frac{\int_0^\infty \int_{-\infty}^\infty e^{-t}|\Phi|^2 dxdt}{2^n n!\sqrt{\pi}\sum_{k,l=0}^\infty \frac{\Gamma(\alpha(k+l)+1)(\lambda_n)^{k+l}}{\Gamma(\alpha k+1)\Gamma(\alpha l+1)}}\right\}^{1/2}$, $\lambda_n = -(2n+1)$ that use in Eqn. (32), we obtain

$\Phi = \left\{\frac{M}{2^n n!\sqrt{\pi}\sum_{k,l=0}^\infty \frac{\Gamma(\alpha(k+l)+1)(\lambda_n)^{k+l}}{\Gamma(\alpha k+1)\Gamma(\alpha l+1)}}\right\}^{1/2} \Psi_n^\alpha(x,t;\lambda_n),$

where, let $M = \int_0^\infty \int_{-\infty}^\infty e^{-t}\, |\Phi|^2 dxdt > 0.$ (34)

4. Different Shapes of the Solution of the Problem (32) in Various Conditions

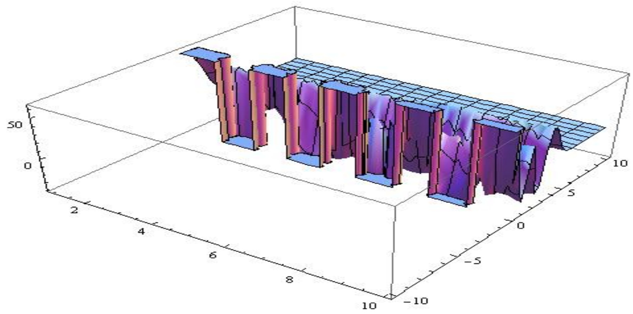
Set $M = \frac{1}{2}$ in Eqn. (34), and apply the formula (5), we find that

$$\Phi = \left\{\frac{1}{2^{n+1}\Gamma(n+1)\sqrt{\pi}\sum_{k,m=0}^\infty \frac{\Gamma(\alpha(k+m)+1)\left(-(2n+1)\right)^{k+m}}{\Gamma(\alpha k+1)\Gamma(\alpha m+1)}}\right\}^{1/2}$$

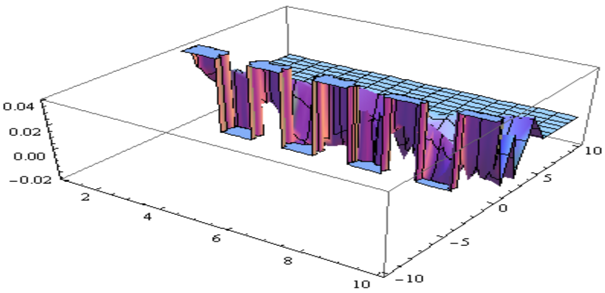
$$\times\, e^{-\frac{x^2}{2}}(2x)^n \sum_{r=0}^\infty \frac{\Gamma\left(-\frac{n}{2}+r\right)\Gamma\left(\frac{1}{2}-\frac{n}{2}+r\right)}{\Gamma\left(-\frac{n}{2}\right)\Gamma\left(\frac{1}{2}-\frac{n}{2}\right)\Gamma(1+r)}(-x^{-2})^r \sum_{s=0}^\infty \frac{(-(2n+1)t^\alpha)^s}{\Gamma(1+\alpha s)}, \forall x \in (-\infty,\infty), t > 0. \quad (35)$$

Now set different conditions of the parameters and variables in Eqn. (35) to find many shapes of the function Φ as:

Case1. When $\alpha = .5, t$ varies from 1 to 10, and the horizontal dimensions are $n \in \mathbb{N} = \{1,2,3,\ldots\}$ and $x \in (-\infty,\infty)$, and Φ fluctuates vertically up and down, then shape of the graph due to the function defined in Eqn. (8, I) is

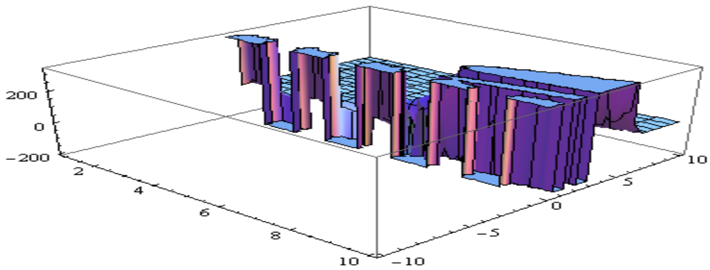


Case2. Again in above put**,** $\alpha = 1.5, t$ varies from 1 to 10, and the horizontal dimensions are $n \in \mathbb{N} = \{1,2,3, \ldots\}$ and $x \in (-\infty, \infty)$, and Φ fluctuates vertically up and down, then shape of the graph is

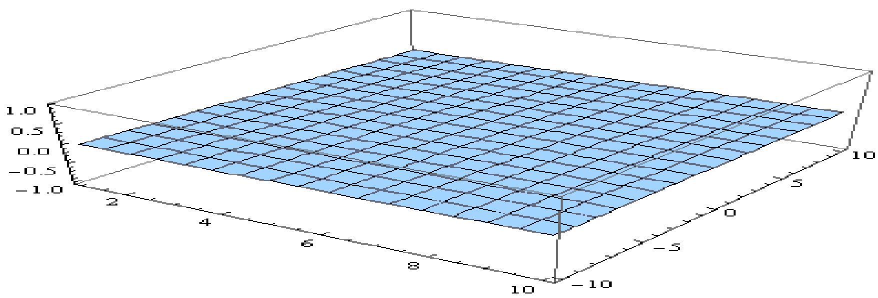


Remark2. It is remarkable that in above cases 1 and 2, if we take the horizontal dimensions $t \in \mathbb{N} = \{1,2,3, \ldots\}$ and $x \in (-\infty, \infty)$ in place of $n \in \mathbb{N} = \{1,2,3, \ldots\}$ and $x \in (-\infty, \infty)$, then the solution becomes indeterminate and not computable.

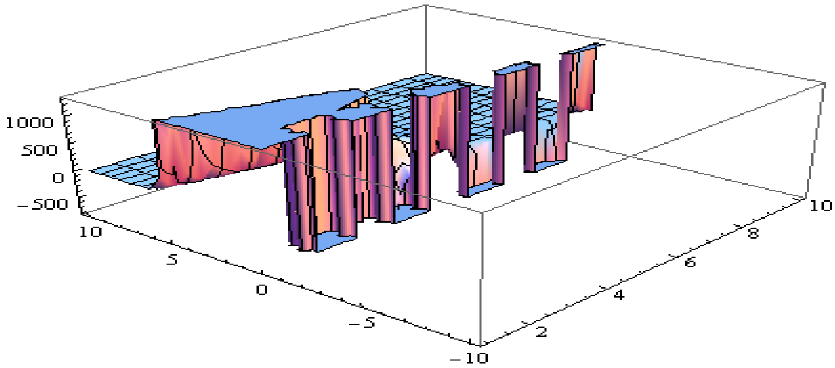
Case3. When $t = 0$ and the horizontal dimensions are $n \in \mathbb{N} = \{1,2,3, \ldots\}$ and $x \in (-\infty, \infty)$, and Φ fluctuates vertically up and down, then shape of the graph due to the function defined in Eqn. (8, II) is



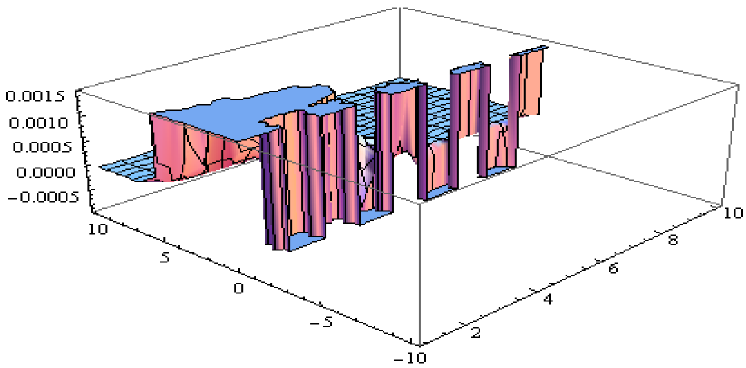
Case4. When $t < 0$ and the horizontal dimensions are $n \in \mathbb{N} = \{1,2,3, \ldots\}$ and $x \in (-\infty, \infty)$, then the function Φ is always zero, see the following graph (as we also taken in our definition (8,III)



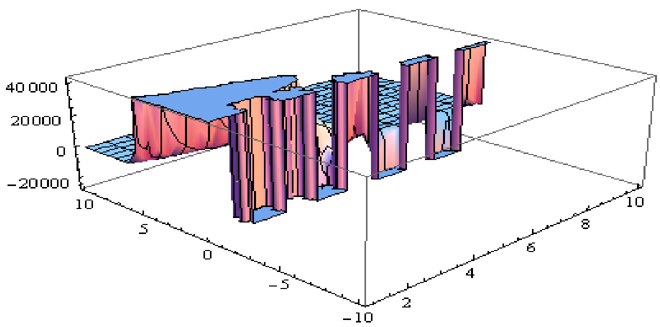
Case5. When $\alpha = .5, t$ is fixed as $t = 5$ and the horizontal dimensions are $n \in \mathbb{N} = \{1,2,3,\dots\}$ and $x \in (-\infty, \infty)$, and Φ fluctuates vertically up and down, then shape of the graph due to the function defined in Eqn. (8, I) is



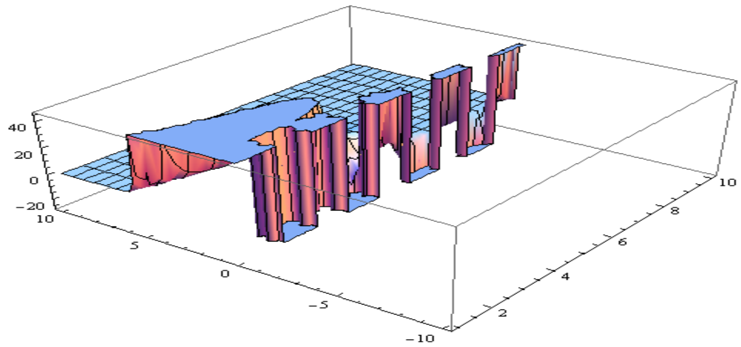
Case6. When $\alpha = 1.5, t$ is fixed as $t = 5$ and the horizontal dimensions are $n \in \mathbb{N} = \{1,2,3,\dots\}$ and $x \in (-\infty, \infty)$, and Φ fluctuates vertically up and down, then shape of the graph due to the function defined in Eqn. (8, I) is



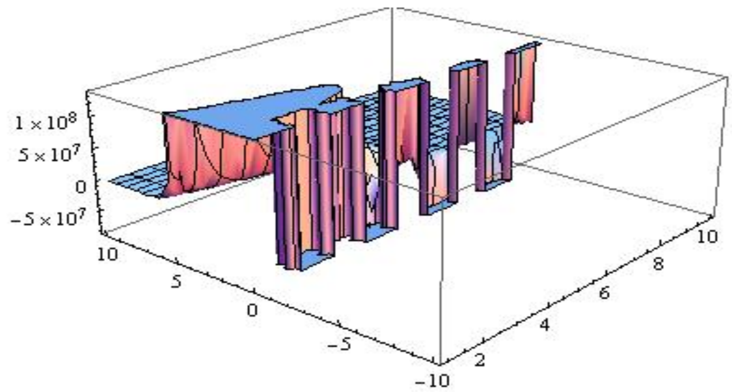
Case7. When $\alpha = .5, t$ is fixed as $t = 10$ and the horizontal dimensions are $n \in \mathbb{N} = \{1,2,3,\dots\}$ and $x \in (-\infty, \infty)$, and Φ fluctuates vertically up and down, then shape of the graph due to the function defined in Eqn. (8, I) is



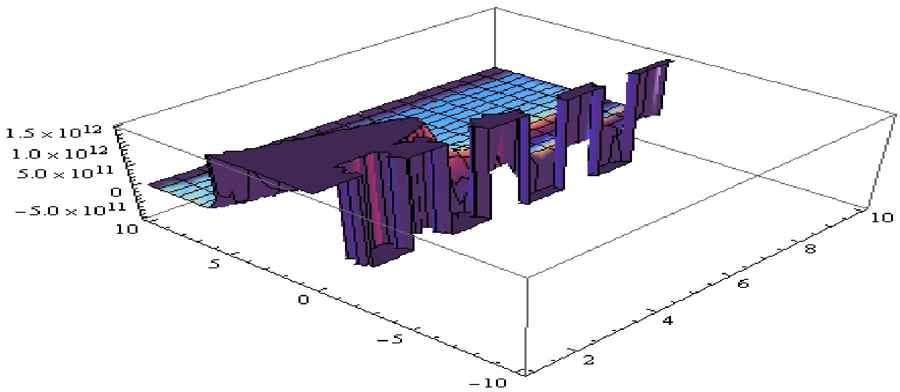
Case8. When $\alpha = 1.5, t$ is fixed as $t = 10$ and the horizontal dimensions are $n \in \mathbb{N} = \{1,2,3,\dots\}$ and $x \in (-\infty, \infty)$, and Φ fluctuates vertically up and down, then shape of the graph due to the function defined in Eqn. (8, I) is



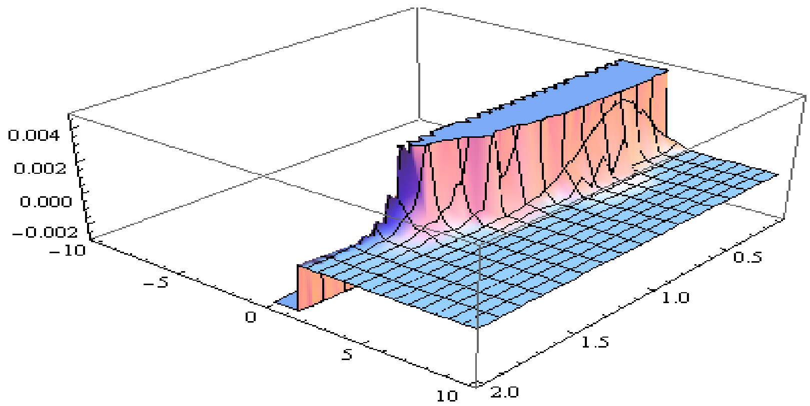
Case9. When $\alpha = .5, t$ is fixed as $t = 50$ and the horizontal dimensions are $n \in \mathbb{N} = \{1,2,3, \dots\}$ and $x \in (-\infty, \infty)$, and Φ fluctuates vertically up and down, then shape of the graph due to the function defined in Eqn. (8, I) is



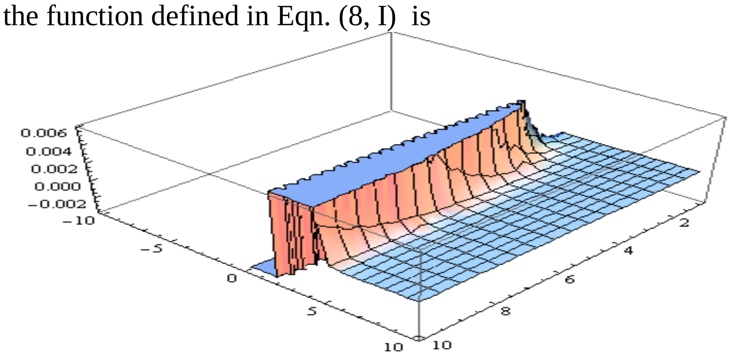
Case10. When $\alpha = 1.5, t$ is fixed as $t = 50$ and the horizontal dimensions are $n \in \mathbb{N} = \{1,2,3, \dots\}$ and $x \in (-\infty, \infty)$, and Φ fluctuates vertically up and down then shape of the graph due to the function defined in Eqn. (8, I) is



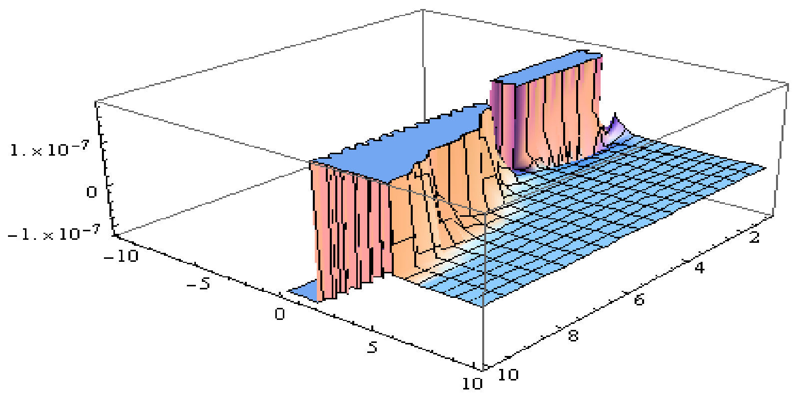
Case11. When t is fixed as $t = 10$, replace n by $\frac{1}{n}$, $n \in \mathbb{N}$ and the horizontal dimensions are $\alpha \in \{0.1, \dots, 2.0\}$ and $x \in (-\infty, \infty)$, and Φ fluctuates vertically up and down, then shape of the graph due to the function defined in Eqn. (8, I) is



Case12. When α is fixed as $\alpha = .5$, replace n by $\frac{1}{n}$, $n \in \mathbb{N}$ and the horizontal dimensions are $t \in \mathbb{N} = \{1,2,3,\dots\}$ and $x \in (-\infty, \infty)$, and Φ fluctuates vertically up and down, then shape of the graph due to the function defined in Eqn. (8, I) is



Case13. When α is fixed as $\alpha = 1.5$, replace n by $\frac{1}{n}$, $n \in \mathbb{N}$ and the horizontal dimensions are $t \in \mathbb{N} = \{1,2,3,\dots\}$ and $x \in (-\infty, \infty)$, and Φ fluctuates vertically up and down, then shape of the graph due to the function defined in Eqn. (8, I) is



Conclusion

In previous section, we claim 13 cases and by MATHEMATICA, we find that the solution (35) of the fractional wave problem (32) has many values and shapes of the wave in different dimensions and different values of α and t. Therefore, our introduced bi dimensional basis solution, involving Hermite and Mittag – Leffler functions, of fractional wave problem is a magic function which has various shapes in different conditions.

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AN INTRODUCTION TO FUNDAMENTALS OF SOFT SET THEORY AND ITS HYBRIDS WITH APPLICATIONS

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Abstract
This work is a comprehensive outline of fundamentals of soft set theory and some of its hybrids such as soft multiset, multi soft set, fuzzy soft set and intuitionistic fuzzy soft set. Most of the previous results and several new observations are given. Finally, a number of extant mathematical models constructed for various soft set theories to solve real-life problems are described and illustrated.

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Keywords: Soft set, soft multi set, multi soft set, fuzzy soft set, intuitionistic fuzzy soft set.

1. Introduction

A number of real-life problems in areas such as engineering, medical sciences, social sciences, economics, management science, etc., involves imprecise and uncertain data which cannot be effectively analyzed by extant mathematical methods such as probability theory, fuzzy set theory, rough set theory, vague set theory, game theory, and interval mathematics, largely due to inadequacy of the parameterization tools associated with these theories (see [32] for details).

Molodtsov [32] in 1999 developed the concept of soft set theory as a new mathematical tool with adequate parameterization for dealing with problems involving uncertainties. Following Molodtsov [32], Maji *et al*. [27] were the first to elaborately describe various operations of soft sets and their basic properties. Pei and Miao [35], Ali *et al*. [2], Ge and Yang [19], Sezgin and Atagun [44], etc., modified and improved the findings of Maji *et al.* [27]. Ali *et al*. [2] also introduced new operations on soft sets along with a new notion of the complement of a soft set. Other contributions on the topic include [20, 38, 44, 45, 46, 47, 49, 50, 51]. In the sequel, various hybrid structures, formed by combining soft sets with multisets, fuzzy sets, intuitionistic fuzzy sets, etc., have been developed.
Alkhazaleh *et al*. [4] initiated the concept of soft multiset and discussed its basic operations. Pinaki Majumdar [37] and, Babitha and Sunil [7] redefined the notion of soft multiset and its operations using count function. Herawan *et al.* [21, 22, 23] introduced the concept of multi soft set, AND-and, OR-operations on multi soft sets which is applied for finding reducts and core of attributes in a multi-valued information system.

In recent years, a number of researchers have contributed toward fuzzification of soft set theory, leading to a more generalized concept. Maji *et al.* [24] in 2001 proposed the concept of fuzzy soft set and discussed some properties and results regarding fuzzy soft union, intersection, complement, etc., which were revised and improved by Ahmad and Kharal [1]. The notion of intuitionistic fuzzy soft set, a combination of soft sets and intuitionistic fuzzy sets was introduced and some basic operations were presented in [8, 25]. Other related works include [3, 14, 25, 28, 30, 36, 40, 54].

Various applications of the soft set theory were made by several authors. Maji *et al*. [26] gave a practical application of soft set reduction using the rough set theory of Pawlak [33]. Later, Cagman and Enginoglu [9] introduced new products on soft sets and applied them in decision-making problem. Cagman *et al.* [11] extended the work in [9] to fuzzy soft set theory and also applied it to solve decision-making problems. Other works on the application of soft set theory and related concepts include [7, 13, 16, 22, 29, 37, 40, 41, 49, 53,54].

In an attempt to broaden the application of soft set theory especially in decision-making, some researchers such as Cagman and Enginoglu [10,12], Manash *et al*. [29], Chetia and Das[15], Rajarajeswari and Dhanalakshima [39] and Herawan *et al*. [23] introduced the concept of soft matrix, fuzzy soft matrix, intuitionistic fuzzy soft matrix, multi soft matrix and illustrated their applications.

In this work, a comprehensive study of the fundamentals of soft set theory, some of its hybrids such as soft multiset, fuzzy soft set, etc., and their applications are presented. The paper is organized as follows: In section 2, basics of the soft set and some of its hybrids are reviewed. In section 3, various operations on soft sets and their properties are systematized. In Section 4, matrix representations of the soft set, soft multiset, fuzzy soft set and intuitionistic fuzzy soft set are presented. Finally, section 5 presents some applications of soft set theory to decision-making problems.

2. Soft set and its hybrids

This section reviews the notions of soft set, soft multiset, multi soft set, fuzzy soft set, and intuitionistic fuzzy soft set.

2.1 Soft Set

Definition 2.1.1 [32]: Soft set

Let U be an initial universe set of objects and E a set of parameters or attributes with respect to U. Let $P(U)$ denote the power set of U and $A \subseteq E$. A pair (F, A) is called a soft set over U, where F is mapping given by

$$F : A \to P(U).$$

In other words, a soft set over U is a parameterized family of subsets of U. For $e \in A$, $F(e)$ may be considered as the set of e-elements or e-approximate elements of the soft set (F, A). Thus, (F, A), $A = \{e_1, e_2, e_3\} \subset E$ can be viewed as

$$(F, A) = \{F(e) \in P(U) : e \in E,\ F(e) = \varnothing \text{ if } e \notin A\}.$$

Example 2.1[27]

Assume that $U = \{h_1, h_2, h_3, h_4, h_5, h_6\}$ be a universe set consisting of a set of six houses under consideration, $E = \{e_1, e_2, e_3, e_4, e_5\}$ be a set of parameters with respect to U, where each parameter $e_i, i = 1, 2, \ldots, 5$ stands for ‘expensive’, ‘beautiful’, ‘cheap’, ‘modern’, ‘wooden’, respectively, and $A = \{e_1, e_2, e_3\} \subset E$. Suppose a soft set (F, A) describes the attractiveness of the houses. Let $F(e_1) = \{h_2, h_4\}, F(e_2) = \{h_1, h_3, h_5\}$ and $F(e_3) = \{h_3, h_4, h_5\}$. Then the soft set ($F$,$A$) is a parameterized family $\{F(e_i): i=1,2,3\}$ of subsets of U, written as $(F, A) = \{e_1=\{h_2,h_4\},\ e_2=\{h_1,h_3,h_5\},\ e_3=\{h_3,h_4,h_5\}\}$. The soft set ($F$, A) can also be represented as a set of ordered pairs viz.,

$$(F,A)=\{(e_1,F(e_1)),(e_2,F(e_2)),(e_3,F(e_3))\}$$

$$\text{or,}(F,A)=\{(e_1,\{h_2,h_4\}),(e_2,\{h_1,h_3,h_5\}),(e_3,\{h_3,h_4,h_5\})\}.$$

Other notations for (F,A) are F_A or (F_A, E).

Definition 2.1.2 [27]: Soft subset/soft equal

Let (F,A) and (G,B) be two soft sets over a common universe U.

(a) (F,A) is a soft subset of (G,B), denoted $(F,A)\tilde{\subseteq}(G,B)$, *if*

(i) $A\subseteq B$, *and*

(ii) $\forall\ e\in A, F(e)$ and $G(e)$ *are identical approximations.*

(b) (F,A) is soft equal to (G,B), denoted (F,A) = (G,B), if $(F,A)\tilde{\subseteq}(G,B)$ *and* $(G,B)\tilde{\subseteq}(F,A)$.

Pei and Miao [35] pointed out that in, general, $F(e)$ and $G(e)$ may not be identical and modified the definition of the soft subset in the following way:

Definition 2.1.3 [35]: Soft subset redefined

For two soft sets (F, A) and (G, B) over a universe U, $(F,A)\tilde{\subseteq}(G,B)$ if

(i) $A\subseteq B$, *and*

(ii) $\forall\ e\in A,\ F(e)\subseteq G(e)$.

Example 2.2

Let $U=\{u_1,u_2,u_3,u_4,u_5,u_6\}$ be a universe set and $E=\{e_1,e_2,e_3,e_4,e_5\}$ be a set of parameters. Let $A=\{e_1,e_3,e_5\}\subset E$ and $B=\{e_1,e_2,e_3,e_5\}\subset E$.

Suppose (F, A) and(G, B) are two soft sets over U where

$F(e_1)=\{u_2,u_4\}, F(e_2)=\{u_3,u_4,u_5\}, F(e_5)=\{u_1\}$ and

$G(e_1)=\{u_2,u_4\}, G(e_2)=\{u_1,u_3\}, G(e_3)=\{u_3,u_4,u_5\}, G(e_5)=\{u_1,u_4\}$. Then

$(F,A)\tilde{\subset}(G,B)$ since. But $(G,B)\tilde{\not\subset}(F,A)$. Hence $(F,A)\neq(G,B)$

Definition 2.1.4 [27]: Not Set

Let $E=\{e_1,e_2,e_3,\ldots,e_n\}$ be a set of parameters. The "Not set of E', denoted $\neg E$, is defined by $\neg E=\{\neg e_1,\neg e_2,\neg e_3,\ldots,\neg e_n\}$, where $\neg e_i$ means " not e_i", $\forall\ i=1,2,3,\ldots,n$

Definition 2.1.5 [27]: Null and Absolute Soft Sets

Let (F, A) be a soft set over a universe U.

(a) (F, A) is called a null soft set, denoted $\tilde{\emptyset}$, *if* $\forall\ e\in A,\ F(e)=\emptyset$ *(null set).*

This definition is confusing since the parameter set for $\tilde{\emptyset}$ *is not clear. Accordingly, Ali et. al.[2] provided the following definition:*

(F, A) is called a relative null softest (with respect to the parameter set A), denoted $\tilde{\emptyset}_A$, *if* $\forall\ a\in A,\ F(a)=\emptyset$.

(b) (F, A) is called an absolute soft set, denoted $\tilde{A}$, *if* $\forall\ e\in A,\ F(e)=U$.

In line with the modification provided above, Ali et. al.[2] redefined the whole softest as follows:
(F, A) is called a relative whole soft set, denoted U_A, *if* $\forall\ \alpha\ \in A, F(\alpha) = U$.

In fact, this is equivalent to the one given in [27] since Ã *is with respect to the parameter set A (see [19] for details).*

Definition 2.1.6 [27]: Complement Soft Set

The complement of a soft set (*F*, *A*), denoted $(F,A)^C$, is defined as, $(F,A)^C = \left(F^C, \neg A\right)$ where $F^C : \neg A \to P(U)$ is a mapping given by $F^C(\alpha) = U - F(\neg\alpha)\ \forall \alpha \in \neg A$.
Later, Ali et al. [2] introduced a new notion of complement called "relative complement", which is defined in the next definition.

Definition 2.1.7 [2]: Relative Complement

The relative complement of a soft set (*F*, *A*) denoted by $(F,A)^r$ is defined by $(F,A)^r = \left(F^r, A\right)$ where $F : A \to P(U)$ is a mapping given by $F^r(\alpha) = U - F(\alpha),\ \forall\ \alpha \in A$
.
It is to be noted that a number of definitions provided in [27] were required to be modified since several related results were found counter-intuitive (see [2, 9, 11, 19, 27]).

2.2 Soft Multiset

The notion of soft multiset (soft *m*set for short) was first introduced by Alkhazaleh *et al*. [4] and defined in the following way.

Definition 2.2.1 [4] : Soft *m*set

Let $\{U_i : i \in I\}$ be a collection of universes such that $\bigcap_{i\in I} U_i = \varnothing$ and let $\left\{E_{U_i} : i \in I\right\}$ be a collection of sets of parameters.
Let $U = \prod_{i\in I} P\left(U_i\right)$ where $P\left(U_i\right)$ is the power set of $U_i, E = \prod_{i\in I} E_{U_i}$ and $A \subseteq E$.
A pair (*F*, *A*) is called a soft multiset over *U* where *F* is a mapping given by $F : A \to U$.

Example 2.4 [4]

Suppose that U_1, U_2 and U_3 are three universes. Let a soft *m*set (*F*, *A*) describe the *attractiveness* of 'houses', 'cars' and 'hotels' which Mr. *X* says is considering for *accommodation* purchase, *transportation* purchase, and *venue* purchase to hold a wedding celebration, respectively.
Let $U_1 = \{h_1, h_2, h_3, h_4, h_5, h_6\}$, $U_2 = \{c_1, c_2, c_3, c_4, c_5\}$ and $U_3 = \{v_1, v_2, v_3, v_4\}$.
Let $E_{U_i} = \left\{E_{U_1}, E_{U_2}, E_{U_3}\right\}$ be a collection of sets of decision parameters related to the above universes, where

$$E_{U_1} = \Big\{e_{U_1},1 = \text{expensive},\ e_{U_1},2 = \text{cheap},\ e_{U_1},3 = \text{beautiful},$$

$$e_{U_1},4 = \text{wooden},\ e_{U_1},5 = \text{in green surroundings}\Big\},$$

$$E_{U_2} = \{ e_{U_2},1 = \text{expensive},\ e_{U_2},2 = \text{cheap},\ e_{U_2},3 = \text{model2000 and above},$$

$$e_{U_2},4 = \text{black},\ e_{U_2},5 = \text{Made in Japan},\ e_{U_2},6 = \text{ Made in Malaysia} \}$$

and

$$E_{U_3} = \{ e_{U_3},1 = \text{expensive},\ e_{U_3},2 = \text{cheap},\ e_{U_3},3 = \text{majestic},$$

$$e_{U_3},4 = \text{in Kuala Lumpue},\ e_{U_3},5 = \text{in Kajang} \}.$$

Let $U = \prod_{i=1}^{3} P(U_i)$, $E = \prod_{i=1}^{3} E_{U_i}$ and $A \subseteq E$ such that

$$A = \{ a_1 = (e_{U_1,1}, e_{U_2,1}, e_{U_3,1}),\ a_2 = (e_{U_1,1}, e_{U_2,2}, e_{U_3,1}),$$

$$a_3 = (e_{U_1,2}, e_{U_2,3}, e_{U_3,1}),\ a_4 = (e_{U_1,5}, e_{U_2,4}, e_{U_3,2}),$$

$$a_5 = (e_{U_1,4}, e_{U_2,3}, e_{U_3,3}),\ a_6 = (e_{U_1,2}, e_{U_2,3}, e_{U_3,2}),$$

$$a_7 = (e_{U_1,3}, e_{U_2,1}, e_{U_3,1}),\ a_6 = (e_{U_1,1}, e_{U_2,3}, e_{U_3,2})\ \}.$$

Suppose that

$$F(a_1) = (\{h_2, h_3, h_6\}, \{c_2\}, \{v_2, v_3\}),$$

$$F(a_2) = (\{h_2, h_3, h_6\}, \{c_1, c_3, c_4, c_5\}, \{v_2\}),$$

$$F(a_3) = (\{h_1, h_4, h_5\}, \{c_1, c_3\}, \phi),$$

$$F(a_4) = (\{h_1, h_4, h_6\}, \phi, \{v_1, v_4\}),$$

$$F(a_5) = (\{h_1, h_4\}, \{c_1, c_2\}, \{v_1\}).$$

$$F(a_6) = (\{h_1, h_4, h_5\}, \{c_1, c_2\}, \{v_1, v_2, v_2, v_4\}),$$

$$F(a_7) = (\{h_1, h_4\}, \varnothing, \{v_2\}), \text{and}$$

$$F(a_8) = (\{h_2, h_3, h_6\}, \{c_1, c_3\}, \{v_1, v_4\}).$$

Then we can view the soft set (*F*, *A*) as consisting of the following collection of approximations:

$$(F, A) = \{ (a_1, (\{h_2, h_3, h_6\}, \{c_2\}, \{v_2, v_3\})), (a_2, (\{h_2, h_3, h_6\}, \{c_1, c_2, c_4, c_5\}, \{v_2\})),$$

$$(a_3, (\{h_1, h_4, h_5\}, \{c_1, c_3\}, \phi)), (a_4 (\{h_1, h_4, h_6, \phi\}, \{v_1, v_4\})),$$

$$(a_5, (\{h_1, h_4\}, \{c_1, c_3\}, \{v_1\})),\ (a_6, (\{h_1, h_4, h_5\}, \{c_1, c_3\}, \{v_1, v_2, v_3, v_4\})),$$

$$(a_7, (\{h_1, h_4\}, \phi, \{v_3\})),\ (a_8, (\{h_2, h_3, h_6\}, \{c_1, c_2\}, \{v_1, v_4\})) \}.$$

Pinaki Majumdar [37] redefined the notion of soft *m*set using count function as follows:

Definition 2.2.2 [37] :Soft *m*set
Let *U* be a universe set, *E* a set of parameters and *J* the set of all non-negative integers. Let $\tilde{P}(U)$ be the collection of all multisets defined on *U* and $A \subseteq E$.
A triple $\langle F, A, C_F \rangle$ is a soft multiset characterized by its soft count function $C_F : A \to J^U$ which is defined as $C_F(e) = C_F^e \in J^U$, $C_F^e : U \to J$ is the parameterized count function and $F : A \to \tilde{P}(U)$ is defined such that corresponding to each e ϵ E, every element u∈U occurs exactly $C_F^e(U)$ times in *F*(*e*).

Example 2.5[37]
Let the universe set *U* and the parameter set *E* be given as follows:

$$U = \{u_1, u_2, u_3, u_4\}$$

and

$$E = \{e_1, e_2, e_3\}.$$

Define $F : E \to \tilde{P}(U)$ as follows:

$$F(e_1) = \left\{ \frac{2}{u_1}, \frac{3}{u_2}, \frac{1}{u_3}, \frac{4}{u_4} \right\},$$

$$F(e_2) = \left\{ \frac{4}{u_1}, \frac{4}{u_2}, \frac{5}{u_4} \right\}, \text{ and}$$

$$F(e_3) = \left\{ \frac{2}{u_1}, \frac{1}{u_2}, \frac{1}{u_3} \right\}.$$

Then $\langle F, E, C_F \rangle$ is a soft multiset, characterized by the soft count function C_F given by the parameterized count functions $C_F^{e_1}, C_F^{e_2}, C_F^{e_3} : U \to J$ which are defined as follows:

$$C_F^{e_1}(u_1) = 2,\ C_F^{e_1} = (u_2) = 3,\ C_F^{e_1}(u_3) = 1, C_F^{e_1}(u_4) = 1,$$

$$C_F^{e_2}(u_1) = 4,\ C_F^{e_2} = (u_2) = 4,\ C_F^{e_2}(u_3) = 0, C_F^{e_2}(u_4) = 5, \text{ and}$$

$$C_F^{e_3}(u_1) = 2,\ C_F^{e_3} = (u_2) = 1,\ C_F^{e_3}(u_3) = 1, C_F^{e_3}(u_4) = 0.$$

Babitha and Sunil [6] gave an alternative definition of soft *m*set in the following manner:

Definition 2.2.3 [6]: Soft *m*set
Let *U* be a universe *m*set, *E* a set of parameters and $A \subseteq E$. Then an ordered pair (*F*, *A*) is called a soft *m*set where *F* is a mapping given by $F : A \to PW(U)$ (The set of all whole submsets of *U*).

Example 2.6
Let $U = \left\{ \frac{k_1}{u_1}, \frac{k_2}{u_2}, \frac{k_3}{u_3}, \frac{k_4}{u_4}, \frac{k_5}{u_5}, \frac{k_6}{u_6} \right\}$ be the universe *m*set and $E = \{e_1, e_2, e_3, e_4\}$ the set of parameters with respect *U*.

Let $A=\{e_1,e_2,e_4\}\subset E$ where $F(e_1)=\left\{\frac{k_5}{u_5},\frac{k_6}{u_6}\right\},F(e_2)=\left\{\frac{k_2}{u_2},\frac{k_3}{u_3}\right\}$ and

$F(e_4)=\left\{\frac{k_1}{b_1},\frac{k_4}{b_4}\right\}$.

Then the soft *m*set (*F*, *A*) over *U* is given by $(F,A)=\{F(e_1)\}=\left\{\frac{k_5}{u_5},\frac{k_6}{u_6}\right\}$,

$F(e_2)=\left\{\frac{k_2}{u_2},\frac{k_3}{u_3}\right\},F(e_4)=\left\{\frac{k_1}{b_1},\frac{k_4}{b_4}\right\}$.

Remark 2.1

1. *Definition 2.2.10 and Definition 2.2.9 are equivalent.*
2. *In Definition 2.2.9, U is a crisp universe set while in Definition 2.2.10, U is a universe mset.*
3. *In Definition 2.2.8, since* $\{U_i : i\in I,\text{the index set}\}$ *is a disjoint collection, hence* $\bigcap_{i\in I}U_i=\phi$*, and* $\bigcup_{i\in I}U_i$ *is a crisp universe set.*

Also, $\bigcup_{u\in I}E_{U_i}$ is a multi-parameter set, therefore (*F*, *A*) is a multi-parameterized soft *m*set.

2.3 Multi-Soft Sets

Definition 2.3.1 (Multi-soft set)

Let $U=\{u_1,u_2,u_3,\ldots,u_{1u1}\}$ be a finite set of objects which may be characterized by a finite family of parameter sets $A=\{A_1,A_2,A_3,\ldots,A_{1A1}\}$ where each parameter set $A_i,\ i=1,2,\ldots,|A|$ represents the *i*th class of parameter and the elements of A_i represent a specific property set. A pair (*F*, *A*) over *U*, which consists of a set of soft sets $(F,A_i)\ 1\le i\le|A|$, is called a multi-soft set over U and is defined as

$$(F,A)=\left\{(F,A_1),(F,A_2),\ldots,\left(F,A_{|A|}\right)\right\}.$$

Example 2.7

Let us consider a multi-soft set (*F*, *A*) which describes the attractiveness of houses in an Estate that Mr. *X* (say) is considering to buy.

Suppose that there are ten houses and three parameter sets under consideration. i.e.,

$$U=\{h_1,h_2,h_3,\ldots,h_{10}\},$$

and

$$A=\{A_1,A_2,A_3\}.$$

Let

A_1 be a set of cost parameter given by

A_1 = {expensive, cheap, very expensive, very cheap},

A_2 be a set of location parameter given by

A_2 = { low density, high density}, and

A_3 be a set of the colour parameter given by

$A_3 = \{$ green, blue, red$\}$.

Let the corresponding soft sets be as follows:

$$\left(F, A_1\right) = \left\{\text{expensive} = \left\{h_1, h_2, h_3\right\},\ \text{cheap} = \left\{h_4, h_{10}\right\},\right.$$

$$\left.\text{very expensive} = \left\{h_5, h_8\right\},\ \text{very cheap} = \left\{h_6, h_7, h_9\right\}\right\}.$$

$$\left(F, A_2\right) = \left\{\text{low density} = \left\{h_2, h_4, h_6, h_8\right\},\right.$$

$$\left.\text{high density} = \left\{h_1, h_3, h_5, h_7, h_9, h_{10}\right\}\right\}$$

and

$$\left(F, A_3\right) = \left\{\text{green} = \left\{h_1, h_3, h_4, h_6\right\},\ \text{blue} = \left\{h_2, h_5, h_7, h_9\right\}\right.$$

$$\left.\text{red} = \left\{h_8, h_{10}\right\}\right\}.$$

Then the multi-soft set (F, A) can be viewed as follows:

$$(F, A) = \left\{(F, A_1), \left(F, A_2\right), \left(F, A_3\right)\right\}$$

$$= \begin{cases} \left\{\text{expensive} = \left\{h_1, h_2, h_3\right\},\ \text{cheap} = \left\{h_4, h_{10}\right\}\right. \\ \quad \left.\text{very expensive} = \left\{h_5, h_8\right\},\ \text{very cheap} = \left\{h_6, h_7, h_9\right\}\right\} \\ \\ \left\{\text{low density} = \left\{h_2, h_4, h_6, h_8\right\},\right. \\ \quad \left.\text{high density} = \left\{h_1, h_3, h_5, h_7, h_9, h_{10}\right\}\right\},\ \text{and} \\ \\ \left\{\text{green} = \left\{h_1, h_3, h_4, h_6\right\},\ \text{blue} = \left\{h_2, h_5, h_7, h_9\right\},\right. \\ \quad \left.\text{red} = \left\{h_8, h_{10}\right\}\right\}. \end{cases}$$

Definition 2.3.2[21] (Multi-soft set in Information System)

Let $S = (U, A, V, f)$ be a multi-valued information system. Let $S' = \left(U, a_i, V_{\{0,1\}}, f\right), 1 \le i \le |A|$ be the decomposition of S into $|A|$ binary-valued information systems defined by

$$S' = \left(U, a_i, V_{\{0,1\}}, f\right) = \begin{cases} S^1 = \left(U, a_1, V_{\{0,1\}}, f\right) \Leftrightarrow \left(F, a_1\right) \\ S^2 = \left(U, a_2, V_{\{0,1\}}, f\right) \Leftrightarrow \left(F, a_2\right) \\ \vdots \\ S^{|A|} = \left(U, a_{|A|}, V_{\{0,1\}}, f\right) \Leftrightarrow \left(F, a_{|A|}\right). \end{cases}$$

Then we define

$(F,A)=\left((F,a_1),(F,a_2),\ldots,(F,a_{|A|})\right)$ as a multi-soft set over *U*, representing a multi-valued information system $S=(U,A,V,f)$.

2.4 Fuzzy Soft Set

Maji et al. [24] defined a fuzzy soft set in the following way.

Definition 2.4.1[24]: Fuzzy Soft Set

Let *U* be an initial universe, *E* be a set of parameters and $A\subseteq E.$ Let $\tilde{P}(U)$ denote the set of all fuzzy sets over *U*. A pair (*F*, *A*) is called a fuzzy soft set over *U* where F is a mapping given by $F:A\to\tilde{P}(U)$.

Example 2.8[24]

Suppose $U=\{c_1,c_2,c_3,c_4\}$ is a universal set consisting of four cars, $E=\{e_1,e_2,e_3,e_4,e_5\}$ is a set of parameters with respect to *U* and, $A=\{e_1,e_2,e_3\}\subset E$. Then

$$(F,A)=\left\{F(e_1)=\left\{\frac{c_1}{0.7},\frac{c_2}{0.1},\frac{c_3}{0.2},\frac{c_4}{0.6}\right\},\ F(e_2)=\left\{\frac{c_1}{0.8},\frac{c_2}{0.6},\frac{c_3}{0.1},\frac{c_4}{0.5}\right\},\right.$$

$$\left.F(e_3)=\left\{\frac{c_1}{0.1},\frac{c_2}{0.2},\frac{c_3}{0.7},\frac{c_4}{0.3}\right\}\right\}$$

is a fuzzy soft set over *U*.

2.5 Intuitionistic Fuzzy Soft Set

Definition 2.5.1 [5]: Intuitionistic Fuzzy Set

An intuitionistic fuzzy set *A* in a non-empty set *X* is of the form

$A=\{\langle x,\mu_A(x),\nu_A(x)\rangle : x\in X\}$, where the functions $\mu_A:X\to[0,1]$ and $\nu_A:X\to[0,1]$ define the degree of membership and the degree of non-membership of the element $x\in X$ to the set A respectively such that $0\le\mu_A(x)+\nu_A(x)\le 1,\forall\ x\in X$.

Definition 2.5.2 [25]: Intuitionistic Fuzzy Soft Set

Let *U* be an initial universe, *E* a set of parameters, and let $I\tilde{P}(U)$ denote the collection of all intuitionistic fuzzy subsets of *U* and $A\subseteq E$. A pair (*F*, *A*) is called an intuitionistic fuzzy soft set over *U* where *F* is a mapping $F:A\to I\tilde{P}(U)$.

Example 2.9[25]

Let $U=\{s_1,s_2,s_3,s_4\}$, $E=\{e_1,e_2,e_3,e_4\}$ and $A=\{e_1,e_2,e_3\}\subset E$. Then (*F*, *A*), given by

$$(F,A)=\left\{F(e_1)=\{(s_1,0.8,0.1),(s_2,0.7,0.5),(s_3,0.85,0.1),(s_4,0.5,0.2)\},\right.$$

$$F(e_2)=\{(s_1,0.6,0.3),(s_2,0.65,02),(s_3,0.5,0.2),(s_4,0.65,0.2)\},$$

$$\left.F(e_3)=\{(s_1,0.75,0.2),(s_2,0.5,0.3),(s_3,0.5,0.4),(s_4,0.7,0.2)\}\right\},$$

is an intuitionistic fuzzy soft set.

3. Operations

In this section, the operations of soft sets, soft multisets, multisoft sets, fuzzy soft sets, intuitionistic fuzzy soft sets are discussed and illustrated.

3.1 Soft Set Operations

Definitions 3.1.1 [27]

Let (*F*, *A*) and (*G*, *B*) be two soft sets over a common universe *U*.

(i) *The* ***Union*** *of (F, A) and (G, B), denoted* $(F,A)\tilde{\cup}(G,B)$, *is a soft set (H, C), where* $C = A \cup B$ and $\forall\ e \in C$,

$$H(e) = \begin{cases} F(e), & e \in A - B \\ G(e), & e \in B - A \\ F(e) \cup G(e), & e \in A \cap B. \end{cases}$$

(ii) *The* ***intersection*** *of (F,A) and (G,B), denoted* $(F,A)\tilde{\cap}(G,B)$, *is a soft set (H,C),where C=A∩B and ∀ e∈C, H(e) = F(e) or G(e) (as both are the same set).*

(iii) *The* ***AND-operation*** *of (F, A) and (G, B), denoted (F, A) AND (G, B) or* $(F,A)\wedge(G,B)$, *is a soft set defined by* $(F,A)\wedge(G,B) = (H, A\times B)$, *where* $H(a,b) = F(a)\cap G(b),\ \ \forall\ (a,b)\in A\times B$.

(iv) *The* ***OR-operation*** *of (F, A) and (G, B), denoted (F, A) OR (G, B) or* $(F,A)\vee(G,B)$, *is a soft set defined by* $(F,A)\vee(G,B) = (H, A\times B)$, *where* $H(a,b) = F(a)\cup G(b),\ \ \forall\ (a,b)\in A\times B$.

Pei and Miao [35] pointed out that in Definition 3.1.1 (ii) of [27], *F*(*e*) and *G*(*e*) may not be the same set and thus revised the definition as follows:

Definition 3.1.2 [35]: Intersection redefined

Let (*F*, *A*) and (*G*, *B*) be two soft sets over *U*. The intersection (also called *bi-intersection* in Feng et al. [17]) of (*F*, *A*) and (*G*, *B*), denoted $(F,A)\tilde{\cap}(G,B)$, is a soft set (*H*, *C*) where $C = A\cap B$ and $\forall\ e\in C,\ H(e) = F(e)\cap G(e)$.

Moreover, Ahmad and Kharal [1] pointed out that in the above Definition 3.1.2, $A\cap B$ must be non-empty in order to avoid the degenerate cases. The modified definition is as follows:

Definition 3.1.3 [1]: Intersection redefined

Let (*F*, *A*) and (*G*, *B*) be two soft sets over *U* with $A\cap B \neq \varnothing$. The intersection of (*F*, *A*) and (*G*, *B*), denoted $(F,A)\tilde{\cap}(G,B)$, is a soft set (*H*, *C*), where $C = A\cap B$ and $\forall\ e\in C,\ H(e) = F(e)\cap G(e)$.

In the sequel Ali et al. [2] introduced the following new operations

Definitions 3.1.4 [2]

Let (*F*, *A*) and (*G*, *B*) be two soft sets over *U*.

(i) *The* ***extended intersection*** *of (F, A) and (G, B), denoted* $(F,A)\cap_{\varepsilon}(G,B)$, *is a soft set (H, C) where* $C = A\cup B$ *and* $\forall\ e\in C$,

$$H(e) = \begin{cases} F(e), \text{ if } e \in A - B \\ G(e), \text{ if } e \in B - A \\ F(e) \cap G(e), \text{ if } e \in A \cap B. \end{cases}$$

(ii) *The **restricted intersection** (also called intersection in Pei and Miao [35] and bi-intersection in Feng et al. [17]) of (F,A) and (G,B), denoted (F,A)⋒(G,B), is a soft (H,C) where* $C = A \cap B$ *and* $\forall\ e \in C,\ H(e) = F(e) \cap G(e)$.

(iii) *The **restricted union** of (F, A) and (G, B), denoted* $(F,A) \cup_R (G,B)$, *is a soft set (H, C) where* $C = A \cap B$ and $\forall\ e \in C,\ H(e) = F(e) \cup G(e)$.

(iv) *The **restricted difference** of (F, A) and (G, B), denoted* $(F,A) \smile_R (G,B)$, *is a soft set (H, C) where* $C = A \cap B$ *and* $\forall\ e \in C,\ H(e) = F(e) - G(e)$.

Sezgin and Atagun [44] defined the following operations.

Definition 3.1.5 [44] : Restricted symmetric difference

The restricted symmetric difference of (*F*, *A*) and (*G*, *B*), denoted $(F,A)\tilde{\Delta}(G,B)$, is a soft set defined by $(F,A)\tilde{\Delta}(G,B) = \left((F,A) \cup_R (G,B)\right) \smile_R ((F,A) \Cap (G,B))$, or by

$(F,A)\tilde{\Delta}(G,B) = \left((F,A) \smile_R (G,B)\right) \cup_R \left((G,B) \smile_R (F,A)\right)$.

This can also be defined as follows:

Definition 3.1.6

The restricted symmetric difference of (*F*, *A*) and (*G*, *B*), denoted $(F,A)\tilde{\Delta}(G,B)$ is a soft set (*H*, *C*) where $C = A \cap B$ and $\forall\ e \in C,\ H(e) = F(e)\Delta G(e)$ (the symmetric difference of *F*(*e*) and *G*(*e*)).

Example 3.1

Let $U = \{h_1, h_2, h_3, h_4, h_5, h_6\}$ and, $E = \{e_1, e_2, e_3, e_4, e_5\}$ be the parameter set with respect to *U*, and $A = \{e_1, e_2, e_3\} \subset E$.

Let (*F*, *A*) over *U* be $(F,A) = \left\{\left(e_1, \{h_2, h_4\}\right), \left(e_2, \{h_1, h_3, h_5\}\right), \left(e_3, \{h_3, h_4, h_5\}\right)\right\}$. Let $B = \{e_3, e_4, e_5\}$ and (*G*, *B*) over U be

$(G,B) = \left\{\left(e_3, \{h_1, h_2, h_3\}\right), \left(e_4, \{h_2, h_3, h_6\}\right), \left(e_5, \{h_2, h_3, h_4\}\right)\right\}$.

Then

(i) $(F,A)\tilde{\cup}(G,B) = \left\{\left(e_1, \{h_2, h_4\}\right), \left(e_2, \{h_1, h_3, h_5\}\right), \left(e_3, \{h_1, h_2, h_3, h_4, h_5\}\right)\right.$ $\left.\left(e_4, \{h_2, h_3, h_6\}\right), \left(e_5, \{h_2, h_3, h_4\}\right)\right\}$,

(ii) $(F,A) \cup_R (G,B) = \left\{\left(e_3, \{h_1, h_2, h_3, h_4, h_5\}\right)\right\}$,

(iii) $(F,A) \Cap (G,B) = \left\{\left(e_3, \{h_3\}\right)\right\}$,

(iv) $(F,A) \cap_\varepsilon (G,B) = \left\{\left(e_1, \{h_2, h_4\}\right), \left(e_2, \{h_1, h_3, h_5\}\right), \left(e_3, \{h_3\}\right)\right.$ $\left.\left(e_4, \{h_2, h_3, h_6\}\right), \left(e_5, \{h_2, h_3, h_4\}\right)\right\}$,

(v) $(F,A) \smile_R (G,B) = \left\{\left(e_3, \{h_3, h_5\}\right)\right\}$,

(vi) $(F,A)\tilde{\Delta}(G,B) = \left\{\left(e_3, \{h_1, h_2, h_4, h_5\}\right)\right\}$,

(vii) $(F,A)\wedge(G,B)=\{((e_1,e_3),\{h_2\}),((e_1,e_4),\{h_2\}),((e_1,e_5),\{h_4\})$
$((e_2,e_3),\{h_1,h_3\}),((e_2,e_4),\{h_3\}),((e_2,e_5),\{h_5\})$
$((e_3,e_3),\{h_3\}),((e_3,e_4),\{h_3\}),((e_3,e_5),\{h_3,h_4\})\}$, and

(viii) $(F,A)\vee(G,B)=\{((e_1,e_3),\{h_1,h_2,h_3,h_4\}),$
$((e_1,e_4),\{h_2,h_3,h_4,h_5\}),((e_1,e_5),\{h_2,h_3,h_4\}),$
$((e_2,e_3),\{h_1,h_2,h_3,h_5\}),((e_2,e_4),\{h_1,h_2,h_3,h_5,h_6\}),$
$((e_2,e_5),\{h_1,h_2,h_3,h_4,h_5\}),((e_3,e_3),\{h_1,h_2,h_3,h_4,h_5\}),$
$((e_3,e_4),\{h_2,h_3,h_4,h_5,h_6\}),((e_3,e_5),\{h_2,h_3,h_4,h_5\})\}.$

The above operations of soft sets also hold for soft multisets, multi soft sets, fuzzy soft sets and intuitionistic fuzzy soft sets [see,7,21,24,25].

4 Matrix Representations

We present here matrix representations of the soft set, soft multiset, multi soft set, fuzzy soft set and intuitionistic fuzzy soft set.

4.1 Soft Matrix

Definition 4.1.1[10]: Soft Matrix

Let *U* be an initial universe, *E* be the set of all possible parameters with respect to *U*, and $A\subseteq E$. Let (F, A) be a soft set over *U*.

Let $U=\{u_1,u_2,u_3,\ldots,u_m\}$ *and* $E=\{e_1,e_2,e_3,\ldots,e_n\}$, then the matrix $[a_{ij}]$ representing (*F*, *A*) is called the $m\times n$ soft matrix over *U* and is defined as

$$[a_{ij}]=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix},\quad i=1,2,\ldots,m,\ j=1,2,3,\ldots,n$$

where $a_{ij}=\begin{cases}1, \text{ if } u_i\in F(e_j)\\ 0, \text{ otherwise.}\end{cases}$

Example 4.1

Let $U=\{u_1,u_2,u_3,u_4,u_5\}$ be a universe and $E=\{e_1,e_2,e_3,e_4\}$ be a set of parameters with respect to *U*. Let $A=\{e_1,e_3,e_4\}\subset E$ and (*F*, *A*) be a soft set over *U* given by $(F,A)=\{F(e_1)=\{u_3,u_4\},F(e_4)=\{u_1,u_3,u_5\}\}$. Then the soft matrix $[a_{ij}]$ over *U* representing (*F*, *A*) is given by

$$\left[a_{ij}\right]=\begin{bmatrix}0&0&0&1\\0&0&0&0\\1&0&0&1\\1&0&0&0\\0&0&0&1\end{bmatrix},\ i=1,2,\ldots,5;\ j=1,2,\ldots,4.$$

4.2 Soft Multi Matrix

Definition 4.2.1: Soft Multi Matrix

Let *U* be a universal *m*set and *E* and set of parameters with respect to *U*. Let *U* be given by

$$U=\left\{\frac{k_1}{u_1},\frac{k_2}{u_2},\frac{k_3}{u_3},\ldots,\frac{k_m}{u_m}\right\},$$

$$E=\{e_1,e_2,e_3,\ldots,e_n\},\text{ and } A\subseteq E.$$

Suppose (*F*, *A*) is a soft *m*set over *U*. Then the matrix $\left[a_{ij}\right]$ representing (F, A) is called the $m\times n$ soft multi-matrix over *U*, defined as

$$\left[a_{ij}\right]=\begin{bmatrix}a_{11}&a_{12}&\cdots&a_{1n}\\a_{21}&a_{22}&&a_{2n}\\\vdots&\vdots&&\vdots\\a_{m1}&a_{m2}&\cdots&a_{mn}\end{bmatrix},$$

where

$$a_{ij}=\begin{cases}k_i,\text{ if } u_i\in F\left(e_j\right),\ i=1,2,\ldots,m,\ j=1,2,\ldots,n\\0,\ \text{otherwise.}\end{cases}$$

Example 4.2

Let us consider the soft *m*set (*F*, *A*) constructed in Example (2.7). Then the soft multi-matrix, representing (*F*, *A*) is given by

$$\left[a_{ij}\right]_{5\times4}=\begin{bmatrix}0&0&0&k_1\\0&k_2&0&0\\0&k_3&0&0\\0&0&0&k_4\\k_5&0&0&0\end{bmatrix}.$$

4.3 Multi Soft Matrix

Definition 4.3.1 [23]: Multi Soft Matrix

Let $(F,A)=\left(F,a_i\right):i=1,2,\ldots,|A|$ be a multi soft set representing a multi-valued information system *S* = (*U*, *A*, *V*, *f*). The matrix $M_{ai}=\left[a_{ij}\right],1\le i\le|A|,$ where

$$a_{ij}=\begin{cases}1,\text{ if } |f(u,\alpha)|=1,\ i\le i\le|u|,1\le j\le\left|V_{ai}\right|,u\in U,\alpha\in V_{ai}\\0,\ \text{if } |f(u,\alpha)|=0\end{cases}$$

and dim $\left(M_{ai}\right)=|u|\times\left|V_{ai}\right|$, is called the matrix representation of the soft set $\left(F,a_{ai}\right)$ in the multi soft set (*F*, *A*).

The collection of all matrices representing (*F*, *A*), denoted by M_A, is called the multi soft matrices and is defined by

$$M_A = \{M_{ai} : 1 \le i \le |A|\}.$$

4.4 Fuzzy Soft Matrix

Definition 4.4.1 [12]: Fuzzy Soft Matrix

Let *U* be a universe, *E* a set of parameters, and $A \subseteq E$. Let (*F*, *A*) be a fuzzy soft set over *U*, and let a subset R_A of $U \times E$ be uniquely defined by $R_A = \{(u,e) : e \in A,\ u \in F(e)\}$.

Then the membership function μ_{R_A} of R_A is defined by $\mu_{R_A} : U \times E \to [0,1]$ such that $\mu_{R_A}(u,e) = \mu(u,e) \in [0,1]$ is the membership value of $u \in U$ for each $e \in E$.

Now, if $U = \{u_1, u_2, u_3, \ldots, u_m\}$ then, the matrix $\left[a_{ij}\right]$ representing (*F*, *A*), called the $m \times n$ fuzzy soft matrix, is defined as

$$\left[a_{ij}\right]_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

where

$$a_{ij} = \begin{cases} \mu\left(u_i, e_j\right), & u_i \in F\left(e_j\right) \\ 0, & \text{otherwise.} \end{cases}$$

Example 4.4[12]

Let $U = \{u_1, u_2, u_3, u_4, u_5\}$ be a universe, $E = \{e_1, e_2, e_3, e_4\}$ be a parameter set, and $A = \{e_2, e_3, e_4\}$ .Suppose (*F*, *A*) is a fuzzy soft set over *U* such that

$F\left(e_2\right) = \left\{\frac{u_1}{0.4}, \frac{u_2}{0.5}, \frac{u_3}{1.0}, \frac{u_4}{0.3}, \frac{u_5}{0.3},\right\}, F\left(e_3\right) = \left\{\frac{u_1}{0.3}, \frac{u_2}{0.4}, \frac{u_3}{0.6}, \frac{u_4}{0.5}, \frac{u_5}{1.0}\right\}$ and $F\left(e_4\right) = \left\{\frac{u_1}{0.5}, \frac{u_2}{0.5}, \frac{u_3}{0.4}, \frac{u_4}{0.3}, \frac{u_5}{0.9}\right\}$. Then the fuzzy soft matrix $\left[a_{ij}\right]$ representing (*F*,*A*) is given by

$$\left[a_{ij}\right]_{5\times 4} = \begin{bmatrix} 0 & 0.4 & 0.3 & 0.5 \\ 0 & 0.5 & 0.4 & 0.5 \\ 0 & 1.0 & 0.6 & 0.4 \\ 0 & 0.3 & 0.5 & 0.3 \\ 0 & 0.6 & 1.0 & 0.9 \end{bmatrix}.$$

4.5 Intuitionistic Fuzzy Soft Matrix

Definition 4.5.1 [15]: Intuitionistic Fuzzy Soft Matrix

Let *U* be an initial universe, *E* a set of parameters, and $A \subseteq E$. Let (*F*, *A*) be an intuitionistic fuzzy soft set over *U*. Then a subset R_A of $U \times E$ is uniquely defined by $R_A = \{(u,e) : e \in A, u \in F(e)\}$.

The membership function and the non-membership function are defined by

$\mu_{R_A}: U \times E \to [0,1]$ and $\nu_{R_A}: U \times E \to [0,1]$ where $\mu_{R_A}(u,e) \in [0,1]$ and $\nu_{R_A}(u,e) \in [0,1]$ are the membership value and the non-membership value, respectively, of $u \in U$, for each $e \in E$.

Now, let $U = \{u_1, u_2, \ldots, u_m\}$, $E = \{e_1, e_2, \ldots, e_n\}$ and $(\mu_{ij}, \nu_{ij}) = (\mu_{R_A}(u_i, e_j), \nu_A(u_i, e_j))$. Then

$$\left[(\mu_{ij}, \nu_{ij})\right]_{m\times n} = \begin{bmatrix} (\mu_{11}, \nu_{11}) & (\mu_{12}, \nu_{12}) & \cdots & (\mu_{1n}, \nu_{1n}) \\ (\mu_{21}, \nu_{21}) & (\mu_{22}, \nu_{22}) & \cdots & (\mu_{2n}, \nu_{2n}) \\ \vdots & & \cdots & \vdots \\ (\mu_{m1}, \nu_{m1}) & (\mu_{m2}, \nu_{m2}) & \cdots & (\mu_{mn}, \nu_{mn}) \end{bmatrix}$$

is called an $m\times n$ intuitionistic fuzzy soft matrix of the intuitionistic fuzzy soft set over U

Example 4.5[15]

Suppose $U = \{u_1, u_2, u_3, u_4\}$, $E = \{e_1, e_2, e_3, e_4\}$ and $A = \{e_2, e_3, e_4\}$ such that

$$F(e_2) = \left\{ \frac{u_1}{0.4, 0.5}, \frac{u_2}{0.5, 03}, \frac{u_3}{1.0, 0}, \frac{u_4}{0.5, 0.6} \right\},$$

$$F(e_3) = \left\{ \frac{u_1}{0.3, 0.5}, \frac{u_2}{0.4, 0.6}, \frac{u_3}{0.6, 0.2}, \frac{u_4}{0.5, 0.5} \right\}, \text{ and}$$

$$F(e_4) = \left\{ \frac{u_1}{0.5, 0.2}, \frac{u_2}{0.5, 0.5}, \frac{u_3}{0.4, 0.6}, \frac{u_4}{0.3, 0.6} \right\}.$$

Then the intuitionistic fuzzy soft set (F, A) over U is given by

$$(F, A) = \{F(e_2), F(e_3), F(e_4)\}, \text{ and}$$

the corresponding intuitionistic fuzzy soft matrix $\left[(\mu_{ij}, \nu_{ij})\right]$ is given by

$$\left[(\mu_{ij}, \nu_{ij})\right]_{4\times 4} = \begin{bmatrix} 0 & (0.4, 0.5) & (0.3, 0.5) & (0.5, 0.2) \\ 0 & (0.5, 0.3) & (0.4, 0.6) & (0.5, 0.5) \\ 0 & (1.0, 0) & (0.6, 0.2) & (0.4, 0.6) \\ 0 & (0.3, 0.6) & (0.5, 0.5) & (0.3, 0.6) \end{bmatrix}.$$

6. Applications of Soft Set Theory

A number of researchers [7,9,10,11,12,13,26,29,31,37,39,42, 43,48] have applied the concepts of soft set theory and its hybrids, to real-life problems involving uncertainties. Such problems involve decision-making, medical diagnosis, forecasting, among others. In this section, some typical applications using different approaches, especially in decision-making, will be discussed.

6.1 Soft Set Uni-Int decision making method [9]

Cagman and Enginoglu [9] constructed an Uni-Int (Union-Intersection) soft decision-making method which reduces a large set of alternatives to smaller subsets according to the choice parameters of decision makers. We illustrate this method by taking the following example:

Suppose there are 48 candidates who applied for a position in a company, and there are two decision makers who want to select a smaller set of candidates to be interviewed according to their choice parameters.

Let the set of candidates be U= $\{u_1, u_2, ..., u_{48}\}$ and the set of parameters be E= $\{e_1, e_2, ..., e_7\}$ where each e_i, i= 1,2,...,7 stands for e_1= Experience, e_2= Computer knowledge, e_3=Training, e_4= Young Age, e_5=High Education, e_6=Marital Status and e_7=Good Health. In order to get a set of short-listed candidates, we apply the Uni-Int soft decision making method in steps as follows:

Step 1: Assume that the decision makers' choice of parameters are A= $\{e_1, e_2, e_4, e_7\} \subset E$, and B= $\{e_1, e_2, e_5\} \subset E$.

Step 2: Assume that the following soft sets F_A *and* F_B are obtained according to their choice parameters:

$$
\begin{aligned}
F_A = \{&(e_1, \{u_4, u_7, u_{13}, u_{21}, u_{28}, u_{31}, u_{32}, u_{36}, u_{39}, u_{41}, u_{43}, u_{44}, u_{48}\}),\\
&(e_2, \{u_1, u_3, u_{13}, u_{18}, u_{19}, u_{21}, u_{22}, u_{24}, u_{28}, u_{32}, u_{36}, u_{42}, u_{44}, u_{46}\}),\\
&(e_4, \{u_2, u_3, u_{13}, u_{15}, u_{18}, u_{23}, u_{25}, u_{28}, u_{30}, u_{33}, u_{36}, u_{38}, u_{42}, u_{43}\}),\\
&(e_7, \{u_1, u_5, u_{12}, u_{13}, u_{17}, u_{20}, u_{24}, u_{28}, u_{29}, u_{34}, u_{36}, u_{41}, u_{45}, u_{47}\})\},
\end{aligned}
$$

$$
\begin{aligned}
F_B = \{&(e_1, \{u_3, u_4, u_5, u_8, u_{14}, u_{21}, u_{22}, u_{26}, u_{27}, u_{34}, u_{35}, u_{37}, u_{40}, u_{42}, u_{46}\}),\\
&(e_1, \{u_1, u_4, u_7, u_{10}, u_{11}, u_{13}, u_{15}, u_{21}, u_{29}, u_{30}, u_{32}, u_{36}, u_{42}, u_{43}, u_{45}\}),\\
&(e_5, \{u_2, u_4, u_8, u_9, u_{12}, u_{13}, u_{14}, u_{16}, u_{17}, u_{21}, u_{23}, u_{28}, u_{36}, u_{42}, u_{44}\})\}.
\end{aligned}
$$

Step 3: *Compute the* $\wedge$ *- product* $F_A \wedge F_B$ *:*

$$
\begin{aligned}
F_A \wedge F_B = \{&((e_1, e_1), \{u_4, u_{21}\}), ((e_1, e_2), \{u_4, u_7, u_{13}, u_{21}, u_{32}, u_{36}, u_{43}\}),\\
&((e_1, e_5), \{u_4, u_{13}, u_{21}, u_{28}, u_{36}, u_{44}\}), ((e_2, e_1), \{u_3, u_{21}, u_{22}, u_{42}, u_{46}\}),\\
&((e_2, e_2), \{u_1, u_{13}, u_{21}, u_{32}, u_{36}, u_{42}\}), ((e_2, e_5), \{u_{13}, u_{21}, u_{28}, u_{36}, u_{42}, u_{44}\}),\\
&((e_4, e_1), \{u_3, u_{42}\}), ((e_4, e_2), \{u_{13}, u_{15}, u_{30}, u_{36}, u_{42}, u_{43}\}),\\
&((e_4, e_5), \{u_2, u_{13}, u_{23}, u_{28}, u_{36}, u_{42}\}), ((e_7, e_1), \{u_5, u_{34}\}),\\
&((e_7, e_2), \{u_1, u_{13}, u_{29}, u_{36}, u_{45}\}), ((e_7, e_5), \{u_{12}, u_{13}, u_{17}, u_{28}, u_{36}\})\}.
\end{aligned}
$$

Step 4: Compute the decision set Uni-Int $(F_A \wedge F_B)$ as follows:

Uni-Int $(F_A \wedge F_B)$ = $\text{Uni}_x\text{-Int}_y$ $(F_A \wedge F_B)$ $\cup$ $\text{Uni}_y\text{-Int}_x$ $(F_A \wedge F_B)$.

Now, $\text{Uni}_x\text{-Int}_y$ $(F_A \wedge F_B) = \bigcup_{x\in A} \left\{ \bigcap_{y\in B} \left(f_{A\wedge B}(x,y) \right) \right\}$

$$
= \bigcup \begin{cases}
\bigcap\left\{\{u_4,u_{21}\},\{u_4,u_7,u_{13},u_{21},u_{32},u_{36},u_{43}\},\{u_4,u_{13},u_{21},u_{28},u_{36},u_{44}\}\right\}\\
\bigcap\left\{\{u_3,u_{21},u_{22},u_{42},u_{46}\},\{u_1,u_{13},u_{21},u_{32},u_{36},u_{42}\},\{u_{13},u_{15},u_{30},u_{36},u_{42},u_{44}\}\right\}\\
\bigcap\left\{\{u_3,u_{42}\},\{u_{13},u_{15},u_{30},u_{36},u_{42},u_{43}\},\{u_2,u_{13},u_{23},u_{29},u_{36},u_{42}\}\right\}\\
\bigcap\left\{\{u_5,u_{34}\},\{u_1,u_{13},u_{29},u_{36},u_{45}\},\{u_{12},u_{13},u_{17},u_{28},u_{36}\}\right\}
\end{cases}
$$

$$
= \bigcup\left\{\{u_4,u_{21}\},\{u_{42}\},\{u_{42}\},\varnothing\right\} = \left\{u_4,u_{21},u_{42}\right\}, \textit{and}
$$

and $\text{Uni}_y\text{-Int}_x$ $(F_A \wedge F_B) = \bigcup_{y\in B} \left\{ \bigcap_{x\in A} \left(f_{A\wedge B}(x,y) \right) \right\}$

$$
= \bigcup \begin{cases}
\bigcap\left\{\{u_4,u_{21}\},\{u_3,u_{21},u_{22},u_{42},u_{46}\},\{u_3,u_{42}\},\{u_5,u_{34}\}\right\}\\
\bigcap\left\{\{u_4,u_7,u_{13},u_{21},u_{32},u_{36},u_{43}\},\{u_1,u_{13},u_{21},u_{32},u_{36},u_{42}\},\{u_{13},u_{15},u_{30},u_{36},u_{42},u_{43}\},\{u_1,u_{13},u_{29},u_{36},u_{45}\}\right\}\\
\bigcap\left\{\{u_4,u_{13},u_{21},u_{28},u_{36},u_{44}\},\{u_{13},u_{21},u_{28},u_{36},u_{42},u_{44}\},\{u_2,u_{13},u_{23},u_{28},u_{36},u_{42}\},\{u_{12},u_{13},u_{17},u_{28},u_{36}\}\right\}
\end{cases}
$$

$$
= \bigcup\left\{\varnothing,\{u_{13},u_{36}\},\{u_{13},u_{36}\}\right\} = \left\{u_{13},u_{36}\right\}.
$$

Thus, *Uni-Int* $(F_A \wedge F_B) = \{u_4, u_{21}, u_{42}\} \cup \{u_{13}, u_{36}\} = \{u_4, u_{13}, u_{21}, u_{36}, u_{42}\}$,which is the list of objectically short-listed set of candidates to be finally interviewed.

6.2 Soft Matrix Max- Min decision-making method [10]

Cagman and Enginoglu[10] constructed a soft max-min decision-making method(SMmDMM) which selects optimum alternatives from a set of alternatives according to decision maker choice of parameters. We demonstrate this method by way of taking an example of decision making problem.

Suppose a man and his wife want to purchase a car from a dealer who has a set of four cars $U=\{c_1, c_2, c_3, c_4\}$, which are characterized by a set of parameters $E=\{e_1, e_2, e_3, e_4\}$,where e_i, i= 1, 2, 3, 4, stand for cheap (e_1), fuel efficient (e_2) modern (e_3), durable (e_4). The SMmDMM runs in step as follows:

Step 1: Assume that the man and his wife choose the sets of parameters $A=\{e_2, e_3, e_4\}$ and $B=\{e_1, e_3, e_4\}$ respectively.

Step 2: Assume that the following soft matrices are obtained according to the parameters:

$$[a_{ij}]=\begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \quad \text{and} \quad [b_{il}]=\begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

Step 3: Find the $\wedge$ product $[c_{ip}]$ of $[a_{ij}]$ and $[b_{il}]$ (since the choices of both the man and his wife are to be considered conjunctively) as follows:

$$[a_{ij}] \wedge [b_{il}] = [c_{ip}] = \begin{bmatrix} 0&0&0&0&1&0&0&1&1&0&0&1&1&0&0&1 \\ 0&0&0&0&0&0&0&0&1&0&1&1&1&0&1&1 \\ 0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&1&0&0&1&1&0&0&0&0 \end{bmatrix}$$

Step 4: Find a max-min one- column decision soft matrix Mm ($[c_{ip}]$) of $[c_{ip}]$,defined by

$$[d_{i1}] = \mathrm{Min}([c_{ip}]) = \left[\max_{k \in I}\{t_{ik}\}\right], \textit{where } t_{ik} = \begin{cases} \min\limits_{p \in I_{ik}}\{c_{ip}\}, & I_{ik} \neq \varnothing \\ o & , I_{ik} = \varnothing \end{cases}$$

and $I_{ik} = \{p : \exists i, c_{ip} \neq 0, (k-1)n < p \leq kn\}$ for all $k \in I = \{1,2,..., n\}$.

$$\text{Thus } [d_{i1}] = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

Step 5: Finally, the optimum set of U with respect to $[d_{i1}]$ is determined:

$$\mathrm{Opt}_{[d_{i1}]}(U) = \{c_2\}$$

Hence c_2 is identified as the car of optimum choice on which both the man and his wife opted to agree to purchase.

6.3 Soft Multiset approach in decision-making [7]

Babitha and Sunil [7], extended the soft set reduction approach adopted in [26] to soft multiset case in the following way: Suppose a retail shop keeper wants to select a particular type of bags satisfying his demand.

Let U={10/b_i, 15/b_2, 7/b_3, 8/b_4, 18/b_5, 11/b_6, 10/b_7} be a multiset of bags and A={leather (e_1), cheap (e_2), big (e_3) discount (e_4)} be a set of parameters.

Let a soft multiset (F,A), describing the different types of bags under consideration, be given by (F,A)={(e_1,{10/b_1, 15/b_2}), (e_2,{15/b_2, 7/b_3, 8/b_4}), (e_3,{10/b_1, 15/b_2, 7/b_3, 8/b_4}), (e_4,{18/b_5, 11/b_6, 10/b_7})}.

Suppose that the shop keeper wants to buy a set of bags according to his choice parameter set B={e_1, e_2, e_4} along with weights w_1,=0.6, w_2,=0.4, w_4,=0.4 imposed on the parameters e_1, e_2, e_4 respectively. Let the quantity of bags demanded at any instant and their availability be the same. Applying the algorithm described in [26], the steps are as follows:

Step1: The weighted reduct soft set (F, B) is computed in table 6 below.

Table 6: Reduct soft set (F, B)

U	$e_1 \times w_1$	$e_2 \times w_2$	$e_4 \times e_4$	choice value
b_1	10	0	0	6.0
b_2	15	15	0	14.0
b_3	0	7	0	2.4
b_4	0	8	0	3.2
b_5	0	0	18	16.2
b_6	0	0	11	9.9
b_7	0	0	10	9.0

Step2: FromTable5, the highest choice value is identified which is 16.2

Step3: Finally, corresponding to the row with choice value 16.2, 18 bags of category b_5 can be bought by shop keeper.

6.4 Fuzzy Soft Set in decision-making [11]

Cagman and Enginoglu [11] constructed an aggregate fuzzy soft set decision-making process which chooses the best alternative according to the choice parameters of a decision maker. The method is described below by taking an example.

Suppose a company wants to fill a position and there are eight candidates, that is, U= {u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8}. If the interviewing committee considers a set of parameters E= {x_1, x_2, x_3, x_4, x_5} where x_i, i = 1, 2, 3, 4, 5, stand for "expensive ", "computer knowledge" "young age" "good speaking" "friendly", respectively, and agree to choose a subset A={x_2,x_3,x_4} of E for consideration. The goal is to select the best candidate using the aggregate fuzzy soft set decision method described in the following steps :

Step 1: Let the fuzzy soft set Γ_A over U, according to the chosen parameter set A={x_2, x_3, x_4} , viz.,

$$\Gamma_{A=} \{(x, \gamma_A(x)) : x \in E, \gamma_A(x) \in F(U), \text{the fuzzy set over } U\},$$

be given by

$$\Gamma_A=\{(x_2, \{0.3/u_2, 0.5/u_3, 0.1/u_4, 0.8/u_5, 0.7/u_7\}),$$
$$(x_3, \{0.4/u_1, 0.4/u_2, 0.9/u_3, 0.3/u_4\})\},$$
$$(x_4, \{0.2/u_1, 0.5/u_2, 0.1/u_5, 0.7/u_7, 0.1/u_8\})\}.$$

Thus, the fuzzy soft matrix $M\Gamma_A$ of Γ_A is

$$M\Gamma_A=\begin{bmatrix}0 & 0 & 0.4 & 0.2 & 0\\ 0 & 0.3 & 0.4 & 0.5 & 0\\ 0 & 0.5 & 0.9 & 0 & 0\\ 0 & 0.1 & 0.3 & 0 & 0\\ 0 & 0.8 & 0 & 0.1 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0.7 & 0 & 0.7 & 0\\ 0 & 0 & 0 & 0.1 & 0\end{bmatrix}.$$

Step 2: Now, the cardinal fuzzy set $c\Gamma_A$ of Γ_A viz.,

$c\Gamma_A= \{\mu_c\Gamma_A{}^{(x)} / x: x\epsilon E, \mu c\Gamma_A{}^{(x)}\epsilon [0, 1]$

where $\mu\, {}_c\Gamma_A(x)= \frac{1}{|U|}\sum_{u\in U, \chi\in E} \mu_{\gamma A(x)}(u)$, is given by

$c\Gamma_A$ = 1 /8{0 .3+0.5+0.1+0.8+0.7, 0.4+0.4+0.9+0.3, 0.2+0.5+0.1+0.7+0.1} = {0.3, 0.25, 0.2} .

That is, $c\Gamma_A$ ={0.3/x_2, 0.25/x_3, 0.2/x_4} and, the row- matrix $Mc\Gamma_A$ of $c\Gamma_A$ is given by $Mc\Gamma_A$ =[0 0.3 0.25 0.2 0]

Step 3: Now, the aggregate fuzzy set Γ^*_A of Γ_A viz.,

$\Gamma^*_A= \{ \mu\Gamma^*_A (u)/u : u\epsilon U, \ \ \mu\Gamma^*_A (U) \epsilon [0, 1]\}$

where $\mu\Gamma^*_A (u)= \frac{1}{|E|}\sum_{,\chi e\in E} \mu c\Gamma_A(x)\mu_{\gamma A(x)}(u)$

$$= \frac{1}{|E|}\ \{M\Gamma_A \times M_C\Gamma_A{}^T\}$$

$$= \frac{1}{5}\begin{bmatrix}0 & 0 & .4 & .2 & 0\\ 0 & .3 & .4 & .5 & 0\\ 0 & .5 & .9 & 0 & 0\\ 0 & .1 & .3 & 0 & 0\\ 0 & .8 & 0 & 0.1 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & .7 & 0 & .7 & 0\\ 0 & 0 & 0 & 1.0 & 0\end{bmatrix} \begin{bmatrix}0\\ .3\\ .25\\ 1.2\\ 0\end{bmatrix} = \begin{bmatrix}0 & .028\\ 0 & .058\\ 0 & .075\\ 0 & .021\\ 0 & .052\\ 0 & .000\\ 0 & .070\\ 0 & .004\end{bmatrix}.$$

Thus, Γ^*_A = {0.028/u_1, 0.058/u_2, 0.075/u_3, 0.021/u_4, 0.052/u_5, 0.00/u_6, 0.070/u_7, 0.004/u_8}.

Step 4: Now, from above the largest membership grade, max $\mu\Gamma^*_A (u)$= 0.075.

Hence, the candidate u_3 may be selected for the job.

6.5 Fuzzy Soft set in medical diagnosis using fuzzy soft complement [49]

Tridiv and Dusmanta [49] extended Sanchez's approach in [43] adopted for medical diagnosis by using the notion of the complement of a fuzzy soft set. The method is described below by taking an example.

Let P={p_1, p_2, p_3} be a set of three patients in a hospital with a set of symptoms S={temperature (s_1), headaches (s_2), cough (s_3), stomach problem (s_4)}, and related to a set of diseases D={diarrhea (d_1), malaria (d_2)}.

Let (F,D) be a fuzzy soft set over S where $F: D \longrightarrow P(S)$ and let (G, S) be another fuzzy soft set over P where $G: S \rightarrow P(P)$.In order to diagnose which patient is suffering from what disease (s), the use of algorithm constructed in [49] is demonstrated as follows:

Step 1: Compute the fuzzy soft sets (F,D) and $(F,D)^C$ as follows:

(F,D)={ $F(d_1)$={(s_1, 0.85, 0), (s_2, 0.25 ,0), (s_3, 0.55, 0), (s_4, 0.30, 0)},
$F(d_2)$= {(s_3, 0.75, 0), (s_4, 0.50, 0, (s_3, 0.45, 0), (s_4, 0.45, 0)}}.
$(F,D)^C$={$F^C(d_1)$= {(s_1, 1.0, 0.85), (s_2, 1.0, 0.25), (s_3, 1.0, 0.55) (s_4,1.0,0.30)}.
$F^C(d_2)$={(s_1, 1.0, 0. 75), (s_2, 1.0, 0.50), (s_3, 1.0, 0.45), (s_4, 1.0, 0.45)}}.

Step 2: Compute the fuzzy soft matrices R_1 and R_2 corresponding to (F, D) and $(F, D)^C$, respectively :

$$R_1=\begin{bmatrix} d_1 & d_2 \\ (0.85, 0) & (0.75, 0) \\ (0.25, 0) & (0.50, 0) \\ (0.55, 0) & (0.45, 0) \\ (0.30, 0) & (0.45, 0) \end{bmatrix}, \quad R_2=\begin{bmatrix} d_1 & d_2 \\ (1.0, 0.85) & (1.0, 075) \\ (1.0, 0.25) & (1.0, 0.50) \\ (1.0, 055) & (1.0, 0.45) \\ (1.0,0.30) & (1.0, 0.45) \end{bmatrix}$$

Step 3: Compute the fuzzy soft sets (G,S) and $(G, S)^C$ as follows :

(G,S)= {G(s_1)= {p_1, 0.75, 0), (p_2, 0.40, 0), (p_3, 0.70, 0)},
G (s_2)= {p_1, 0.40, 0), (p_2, 0.50, 0), (p_3, 0.40, 0)},
G (s_3)= {p_1, 0.90, 0), (p_2, 0.30, 0), (p_3, 0.60, 0)},
G (s_4)= {p_1, 0.75, 0), (p_2, 0.40, 0), (p_3, 0.30, 0)}}.
$(G,S)^C$={$G^C(s_1)$={(p_1, 1.0, 0.75), (p_2, 1.0, 0), (p_3, 1.0, 0.70)},
$G^C(s_2)$={(p_1, 1.0, 0.40), (p_2, 1.0, 0.50), (p_3, 1.0, 0.60)},
$G^C(s_3)$={(p_1, 1.0, 0.90), (p_2, 1.0, 0.30), (p_3, 1.0, 0.60)},
$G^C(s_3)$={(p_1, 1.0, 0.75), (p_2, 1.0, 0.40), (p_3, 1.0, 0.30)}}.

Step 4: Compute the corresponding fuzzy soft matrices Q_1 andQ_2 as follows :

$$Q_1 = \begin{bmatrix} s_1 & s_2 & s_3 & s_4 \\ (0.75, 0) & (0.40, 0) & (0.90, 0) & (0.75, 0) \\ (0.40, 0) & (0.50, 0) & (0.30, 0) & (0.40, 0) \\ (0,70, 0) & (0.40, 0) & (0.60, 0) & (0.30, 0) \end{bmatrix}.$$

$$Q_2=\begin{array}{c} \begin{array}{cccc} s_1 & s_2 & s_3 & s_4 \end{array} \\ \begin{bmatrix} (1.0, 0.75) & (1.0, 0.40) & (1.0, 0.90) & (1.0, 0.75) \\ (1.0, 0.40) & (1.0, 0.50) & (1.0, 0.30) & (1.0, 0.40) \\ (1.0, 0.70) & (1.0, 0.40) & (1.0, 0.60) & (1.0, 0.30) \end{bmatrix} \end{array}.$$

Step 5: Compute the products $T_1 = Q_1.R_1$, $T_2=Q_1.R_2$, $T_3= Q_2. R_1$, and $T_4=Q_2. R_2$ as follows:

$$T_1=\begin{bmatrix} d_1 & d_2 \\ (0.75,\ 0) & (0.75,\ 0) \\ (0.40,\ 0) & (0.50,\ 0) \\ (0.70,\ 0) & (0.70,\ 0) \end{bmatrix}, \quad T_2 = \begin{bmatrix} d_1 & d_2 \\ (0.90,\ 0.25) & (0.90,\ 0.45) \\ (0.50,\ 0.25) & (0.50,\ 0.45) \\ (0.70,\ \ 0) & (0.70,\ \ 0) \end{bmatrix},$$

$$T_3 = \begin{array}{c} \begin{matrix} \quad d_1 \quad & \quad d_2 \quad \end{matrix} \\ \begin{bmatrix} (0.85,\ 0.40) & (0.75,\ 0.40) \\ (0.85,\ 0.30) & (0.75,\ 0.30) \\ (0.85,\ 0.30) & (0.75,\ 0.03) \end{bmatrix} \end{array}, \text{ and } T_4 = \begin{array}{c} \begin{matrix} \quad d_1 \quad & \quad d_2 \quad \end{matrix} \\ \begin{bmatrix} (0.85,\ 0.40) & (0.75,\ 0.40) \\ (0.85,\ 0.30) & (0.75,\ 0.30) \\ (0.85,\ 0.30) & (0.75,\ 0.03) \end{bmatrix} \end{array}.$$

Step 6: Compute the corresponding *membership value* matrices $MV(T_1)$, $MV(T_2)$, $MV(T_3)$, and $MV(T_4)$ as follows:

$$MV(T_1) = \begin{array}{c} \begin{matrix} d_1 & d_2 \end{matrix} \\ \begin{bmatrix} 0.75 & 0.75 \\ 0.40 & 0.50 \\ 0.70 & 0.70 \end{bmatrix} \end{array}, \qquad MV(T_2) = \begin{array}{c} \begin{matrix} d_1 & d_2 \end{matrix} \\ \begin{bmatrix} 0.65 & 0.45 \\ 0.25 & 0.05 \\ 0.70 & 0.70 \end{bmatrix} \end{array},$$

$$MV(T_3) = \begin{array}{c} \begin{matrix} d_1 & d_2 \end{matrix} \\ \begin{bmatrix} 0.45 & 0.35 \\ 0.55 & 0.45 \\ 0.55 & 0.45 \end{bmatrix} \end{array}, and \quad MV(T_4) = \begin{array}{c} \begin{matrix} d_1 & d_2 \end{matrix} \\ \begin{bmatrix} 0.60 & 0.50 \\ 0.60 & 0.55 \\ 0.70 & 0.55 \end{bmatrix} \end{array}.$$

Step 7: Compute the *diagnosis score* matrices DS_2 and DS_1 defined as follows :

$$DS_1 = MV(T_1) - MV(T_3) = \begin{array}{c} \begin{matrix} d_1 & d_2 \end{matrix} \\ \begin{bmatrix} 0.30 & 0.40 \\ -0.15 & 0.05 \\ 0.15 & 0.15 \end{bmatrix} \end{array}, \text{ and}$$

$$DS_2 = MV(T_2) - MV(T_4) = \begin{array}{c} \begin{matrix} d_1 & d_2 \end{matrix} \\ \begin{bmatrix} 0.05 & 0.05 \\ -0.35 & -0.50 \\ 0 & 0.15 \end{bmatrix} \end{array}.$$

Step 8: Find the *difference* table D between DS_1 and DS_2 as:

$D = DS_1 - DS_2 =$

	d_1	d_2
p_1	0.25	0.45
p_2	0.20	0.55
p_3	0.15	0

This leads to the conclusion that patient p_1 and p_2 both are suffering from malaria, while patient p_3 is suffering from diarrhea.

6.6 Intuitionistic fuzzy Soft matrix in decision-making [39]

Rajarajeswari and Dhanalakshmi[39] defined different types of intuitionistic fuzzy soft matrices together with some operations and applied them in solving decision-making problems. The method is described below by taking an example.

Let $U=\{c_1, c_2, c_3, c_4\}$ be a set of candidates appearing in an interview for appointment as a manager in a company, and $E = \{e_1, e_2, e_{3,} e_4\}$ be a set of parameters characterizing a candidate, where e_i (i=1, 2, 3) stand for "confident", " ability to take risk", "academically sound", respectively. Let X and Y be the decision experts to conduct the selection procedure,

Step 1: Let the intuitionistic fuzzy soft sets (F,E) and (G,E), representing the evaluation of the candidates by experts X and Y respectively, be as follows : $(F,E)=\{ F(e_1)=\{(c_1, 0.7, 0.1), (c_2, 0.5, 0.5), (c_3, 0.1, 0.8), (c_4, 0.4, 0.6)\}$,

$F(e_2)=\{(c_1, 0.5, 0.4), (c_2, 0.4, 0.6), (c_4, 0.5, 0.4), (c_4, 0.7, 0.3)\}$,

$F(e_3)=\{(c, 0.5, 0.4), (c_2, 0.7, 0.2), (c_3, 0.6, 0.3), (c_4, 0.5, 0.4)\}$.

$(G,E)= \{G(e_1)=\{(c_1, 0.6, 0.2), (c_2, 0.6, 0.4), (c_3, 0.2, 0.7), (c_{4,} 0.6, 0.4)\}$,

$G(e_2)=\{(c_1, 0.6 , 0.3), (c_2, 0.5, 0.5), (c_3, 0.6, 0.4), (c_4, 0.8, 0.1)\}$,

$G(e_3)=\{(c_1, 0.5, 0.5), (c_2, 0.8, 0.1), (c_3, 0.7, 0.7), (c_4, 0.5, 0.4)\}$.

Step 2: Compute the intuitionistic fuzzy soft matrices A and B and their complements A^c and B^c, corresponding to (F, E) and (G, E), respectively as follows :

$$A=\begin{array}{c} \begin{matrix} e_1 \qquad & e_2 \qquad & e_3 \end{matrix} \\ \begin{bmatrix} (0.7, 0.1) & (0.6, 0.3) & (0.5, 0.4) \\ (0.5, 0.5) & (0.4, 0.6) & (0.7, 0.2) \\ (0.1, 0.8) & (0.5, 0.4) & (0.6, 0.3) \\ (0.4, 0.6) & (0.7, 0.3) & (0.5, 0.4) \end{bmatrix} \end{array}, \quad B=\begin{array}{c} \begin{matrix} e_1 \qquad & e_2 \qquad & e_3 \end{matrix} \\ \begin{bmatrix} (0.6, 0.2) & (0.6, 0.3) & (0.5, 0.5) \\ (0.6, 0.4) & (0.5, 0.5) & (0.8, 0.1) \\ (0.2, 0.7) & (0.6, 0.4) & (0.7, 0.1) \\ (0.6, 0.4) & (0.8, 0.1) & (0.5, 0.4) \end{bmatrix} \end{array},$$

$$A^C=\begin{array}{c} \begin{matrix} e_1 \qquad & e_2 \qquad & e_3 \end{matrix} \\ \begin{bmatrix} (0.1, 0.7) & (0.3, 0.6) & (0.4, 0.5) \\ (0.5, 0.5) & (0.6, 0.4) & (0.2, 0.7) \\ (0.8, 0.1) & (0.4, 0.5) & (0.3, 0.6) \\ (0.6, 0.4) & (0.3, 0.7) & (0.4, 0.5) \end{bmatrix} \end{array}, \quad B^{C=}\begin{array}{c} \begin{matrix} e_1 \qquad & e_2 \qquad & e_3 \end{matrix} \\ \begin{bmatrix} (0.2, 0.6) & (0.3, 0.6) & (0.5, 0.5) \\ (0.4, 0.6) & (0.5, 0.5) & (0.1, 0.8) \\ (0.7, 0.2) & (0.4, 0.6) & (0.1, 0.7) \\ (0.4, 0.6) & (0.1, 0.8) & (0.4, 0.5) \end{bmatrix} \end{array}.$$

Step 3: Compute the sums $A + B$ and $A^C + B^C$ as follows:

$$A+B=\begin{bmatrix} (\max\{0.7, 0.6\}, \min\{0.1, 0.2\}) & (\max\{0.6, 0.6\},\min\{0.3, 0.3\}) & (\max\{0.5, 0.5\}, \min\{0.4, 0.5\}) \\ (\max\{0.5, 0.6\}, \min\{0.5, 0.4\}) & \max\{0.4, 0.5\}, \min\{0.6, 0.5\}) & (\max\{0.7, 0.8\}, \min\{0.2, 0.1\}) \\ (\max\{0.1, 0.2\}, \min\{0.8, 0.7\}) & \max\{0.5, 0.6\}, \min\{0.4, 0.4\}) & (\max(0.6, 0.7\}, \min\{0.6, 0.7\}) \\ (\max\{0.4, 0.6\}, \min\{0.4, 0.6\}) & \max\{0.7, 0.8\}, \min\{0.3, 0.1\}) & (\max(0.5, 0.5\}, \min\{0.4, 0.4\}) \end{bmatrix}$$

$$\text{i.e., } A+B=\begin{array}{c} \begin{matrix} e_1 \qquad & e_2 \qquad & e_3 \end{matrix} \\ \begin{bmatrix} (0.7, 0.1) & (0.6, 0.3) & (0.5, 0.4) \\ (0.6, 0.4) & (0.6, 0.3) & (0.8, 0.1) \\ (0.2, 0.7) & (0.6, 0.4) & (0.7, 0.1) \\ (0.6, 0.4) & (0.8, 0.1) & (0.5, 0.4) \end{bmatrix} \end{array}, and$$

$$A^C+B^C = \begin{array}{c} \begin{bmatrix} e_1 & e_2 & e_3 \\ (0.2,\ 0.6) & (0.3,\ 0.6) & (0.5,\ 0.5) \\ (0.5,\ 0.5) & (0.6,\ 0.4) & (0.2,\ 0.7) \\ (0.8,\ 0.1) & (0.4,\ 0.5) & (0.3,\ 0.6) \\ (0.6,\ 0.4) & (0.3,\ 0.7) & (0.4,\ 0.5) \end{bmatrix} \end{array}.$$

Step 4: Compute the *value* matrices $V(A+B)$ and $V(A^C+B^C)$ as follows:

$$V(A+B)\begin{bmatrix} e_1 & e_2 & e_3 \\ 0.7-0.1 & 0.6-0.3 & 5-0.4 \\ 0.6-04. & 0.5-0.5 & 8-0.1 \\ 0.2-0.7 & 0.6-0.4 & 7-0.1 \\ 0.6-0.4 & 0.8-0.1 & 4-0.4 \end{bmatrix} = \begin{bmatrix} e_1 & e_2 & e_3 \\ 0.6 & 0.3 & 0.2 \\ 0.2 & 0.0 & 0.7 \\ -.5 & 0.2 & 0.6 \\ 0.2 & 0.7 & 0.1 \end{bmatrix}$$

$$V(A^C+B^C)= \begin{bmatrix} e_1 & e_2 & e_3 \\ -.6 & -.3 & 0.0 \\ 0.0 & 0.1 & -.5 \\ 0.7 & -.1 & -.3 \\ 0.2 & -.4 & -.1 \end{bmatrix}$$

Step 5: Compute the matrix $S_{(A+B),(A^C+B^C)}$ and the total score S_i as follows :

$$S_{(A+B),\ (A^C+B^C)}= V(A+B)-V(A^C+B^C) = \begin{bmatrix} e_1 & e_2 & e_3 \\ 1.0 & 0.6 & 0.1 \\ 0.2 & -.1 & 1.2 \\ -1.2 & 0.3 & 0.9 \\ 0.0 & 1.1 & 0.1 \end{bmatrix}, \quad \text{and}$$

$$S_i= \begin{bmatrix} 1.0 + 0.6 + 0.1 \\ 0.2 - 0.1 + 1.2 \\ -1.2 + 0.3 + 0.9 \\ 0.0 + 1.1 + 0.2 \end{bmatrix} = \begin{bmatrix} 1.7 \\ 1.3 \\ 0.0 \\ 1.3 \end{bmatrix}.$$

This leads to the conclusion that the candidate $c_1$is the best choice, since it has the maximum value.

Conclusion

A comprehensive study of the fundamentals and applications of soft set theory has been carried out. Besides, several hybrids of soft sets are introduced to attract further research. Since the subject is relatively new, the panoply of basic notions introduced here may appear a bit staggering. Most of the related references are included.

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COMPUTATIONAL MODELLING FOR THE FORMATION OF GEOMETRIC SERIES USING ANNAMALAI COMPUTING METHOD

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Abstract
This paper presents a mathematical model for the formation as well as computation of geometric series in a novel way. Using Annamalai computing method a simple mathematical model is established for analysis and manipulation of geometric series and summability. This new model could be used in the research fields of physics, engineering, biology, economics, computer science, queueing theory, and finance. In this paper, a novel computational model had also been developed such that $a\sum_{i=k}^{\infty} y^i = \frac{ay^k}{1-y}$ and $\sum_{i=0}^{\infty}\sum_{j=i}^{\infty} ay^j = \frac{a}{(1-y)^2}$, $(0 < y < 1)$. This could be very interesting and informative for current students and researchers.

Keywords: Annamalai computing method, Computational geometric series

1. Introduction

Geometric series played a vital role in differential and integral calculus at the earlier stage of development and still continues as an important part of the study in mathematics, science, management, and technology. The finite and infinite geometric sequence, series, and summability have important applications in engineering, economics, computer science, medicine, biology, physics, queueing theory, and finance [3, 4, 5, 8].
It is eventually understood that the summations of finite geometric series were

$$\sum_{i\backslash=0}^{n-1} ax^i = \frac{a(x^n - 1)}{(x-1)} \text{ and } \sum_{i=1}^{n-1} ax^i = \frac{a(x^n - x)}{x-1}, \quad x \neq 1$$

and the summations of infinite geometric series were

$$\sum_{i=0}^{\infty} ax^i = \frac{a}{1-x} \text{ and } \sum_{i=1}^{\infty} ax^i = \frac{ax}{1-x} \; (0 < x < 1)$$

Geometric series can be used to convert the decimal to a fraction.
For examples,

(i) $0.9999999...... = \frac{9}{10} + \frac{9}{10^2} + \frac{9}{10^3} + ... = \frac{ax}{1-x} = 1$ where $a = 9$ and $x = \frac{1}{10}$

(ii) $9.9999999...... = 9 + \frac{9}{10} + \frac{9}{10^2} + \frac{9}{10^3} + ... = \frac{a}{1-x} = 10$ where $a = 9$ and $x = \frac{1}{10}$

(iii) $0.777777......... = \frac{7}{10} + \frac{7}{10^2} + \frac{7}{10^3} + ... = \frac{ax}{1-x} = \frac{7}{9}$ where $a = 7$ and $x = \frac{1}{10}$

2. Annamalai Computing Geometric Series

Annamalai computing method [1] provided a novel approach for computation of geometric series in a new way.

$$\sum_{i=-m}^{n-1} ax^i = \frac{a\left(x^n - x^{-m}\right)}{x-1} \Leftrightarrow ax^n = ax^n,\ \left(x \neq 1\right)$$

Proof

RHS $\Rightarrow\ ax^n = ax^n$

$\Rightarrow ax^n = a(x-1)x^{n-1} + ax^{n-1}$

$\Rightarrow ax^n = a(x-1)x^{n-1} + a(x-1)x^{n-2} + ... + a(x-1)x^i + ... + a(x-1)x^{-m} + ax^{-m}$

$$ax^n = ax^n \Rightarrow \sum_{i=-m}^{n-1} ax^i = \frac{a\left(x^n - x^{-m}\right)}{x-1} \qquad \text{---- (1)}$$

LHS $\Rightarrow \sum_{i=-m}^{n-1} ar^i = \dfrac{a\left(x^n - x^{-m}\right)}{x-1}$

$\Rightarrow ax^n = a(x-1)x^{n-1} + a(x-1)x^{n-2} + ... + a(x-1)x^i + ... + a(x-1)x^{-m} + ax^{-m}$

$\Rightarrow\ ax^n = ax^n$

$$\sum_{i=-m}^{n-1} ax^i = \frac{a\left(x^n - x^{-m}\right)}{x-1} \Rightarrow ax^n = ax^n \qquad \text{---- (2)}$$

From (1) and (2) we get:

$$\sum_{i=-m}^{n-1} ax^i = \frac{a\left(x^n - x^{-m}\right)}{x-1} \Leftrightarrow ax^n = ax^n,\ \left(x \neq 1\right)$$

From the above result we can further find the following sumability :

$$(i) \quad a\sum_{i=k}^{n-1} x^i = \frac{a\left(x^n - 1 + 1 - x^k\right)}{x-1} = \frac{a\left(x^n - 1\right)}{x-1} - \frac{a\left(x^k - 1\right)}{x-1} = a\left(\sum_{i=0}^{n-1} x^i - \sum_{i=0}^{k-1} x^i\right)$$

$$(ii) \quad \sum_{i=-m}^{n-1} ax^i = \sum_{i=-m}^{-1} ax^i + \sum_{i=0}^{n-1} ax^i = \sum_{i=1}^{m} \frac{a}{x^i} + \sum_{i=o}^{n-1} ax^i = a\left(\frac{\frac{1}{x} - \frac{1}{x^{m+1}}}{1 - \frac{1}{x}} + \frac{x^n - 1}{x-1}\right) = \frac{a\left(x^n - x^{-m}\right)}{x-1}$$

3. Computational Modelling

The equality $ax = ax$ [7] was used to design the computational modelling,

$$ax = ax \Leftrightarrow ax = (x-1)a + a \Leftrightarrow ax = (x-1)\frac{a}{x^0} + (x-1)\frac{a}{x} + (x-1)\frac{a}{x^2} + ... + (x-1)\frac{a}{x^n} + \frac{a}{x^n}$$

$$ax = ax \Leftrightarrow \sum_{1=0}^{n} \frac{a}{x^i} = \frac{\left(ax - \frac{a}{x^n}\right)}{x-1} = \frac{ax\left(1 - \frac{1}{x^{n+1}}\right)}{x\left(1 - \frac{1}{x}\right)} = \frac{a\left(1 - y^{n+1}\right)}{1-y} \text{where } y = \frac{1}{x}$$

Now $\dfrac{a}{y} = \dfrac{a}{y} \Leftrightarrow \sum_{i=0}^{n-1} ay^i = \dfrac{a\left(y^n - 1\right)}{y-1}$ where it is understood that $\dfrac{a}{y} = \dfrac{a}{y} \Rightarrow ay = ay$

We know that if $0 < y < 1$, then $\sum_{i=0}^{n-1} ay^i = \dfrac{a\left(1 - y^n\right)}{1-y}$ and $\sum_{i=0}^{\alpha} ay^i = \dfrac{a}{1-y}$

Similarly, using Annamalai computing geometric series $a\sum_{i=k}^{n-1} y^i = \frac{a(y^n - y^k)}{y-1}$ $(y \neq 1)$,

we can derive $a\sum_{i=k}^{n-1} y^i = \frac{a(y^k - y^n)}{1-y}$ and $a\sum_{i=k}^{\alpha} y^i = \frac{ay^k}{1-y}$ $(0 < y < 1)$

where $k > 0$ is an integer constant.

From the above result it can further found the following sum ability:

$$(i)\quad \sum_{i=0}^{\infty} ay^i - \sum_{i=k}^{\infty} ay^i = \frac{a}{1-y} - \frac{ay^k}{1-y} = \frac{a(1-y^k)}{1-y} = \sum_{i=0}^{k-1} ay^i$$

$$(ii)\quad \sum_{i=0}^{\infty}\sum_{j=i}^{\infty} ay^j = \sum_{j=0}^{\infty} ay^j + \sum_{j=1}^{\infty} ay^j + \sum_{j=2}^{\infty} ay^j + \ldots.. = \frac{a}{1-y} + \frac{ay}{1-y} + \frac{ay^2}{i-y} + \ldots.. = \frac{a}{(1-y)^2}$$

Conclusion

In the research study, a novel technique has been introduced to form the generalized geometric series and computing it. Also, a novel computational model was also developed such that $a\sum_{i=k}^{\infty} y^i = \frac{ay^k}{1-y}$ and $\sum_{i=0}^{\infty}\sum_{j=i}^{\infty} ay^j = \frac{a}{(1-y)^2}$, $(0 < y < 1)$. This new model can be used in the research fields of physics, engineering, biology, economics, computer science, queueing theory, and finance.

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SOME FIXED POINT THEOREMS FOR MULTI-VALUED MAPS ON *S*-METRIC SPACE

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Abstract
In this paper we introduced a multi-valued contraction in S-metric space and proved some fixed point theorems for multi-valued maps on *S*-metric space. Our results extend and generalize the results of Nadler [8], Sedghi et al. [12] and others. Some examples are also given in support of our results.

2010 Mathematics Subject Classification: 47H10, 54H25.

Keywords: fixed point, contraction maps, multi-valued maps, *S*-metric space.

1. Introduction

Metric spaces are very important in various area of Mathematics such as analysis, topology, applied mathematics etc. So, various generalizations of metric space have been studied and many fixed point theorems were obtained by several authors. For example Gahler [4], Dhage [1], and Mustafa-Sim [7] introduced the concept of 2-metric space, *D*-metric space and *G*-metric space respectively as a generalization of metric space.

Recently Sedghi et al. [12] introduced an *S*-metric space, which is different from other spaces and gave some basic properties of *S*-metric space. Many authors have investigated the fixed point property for single-valued maps on *S*-metric space (see for instance [2]-[3], [5]-[6], [9]-[11] and references therein). In this paper, we defined a multi-valued contraction on *S*-metric space and proved some fixed point theorems for multi-valued maps on *S*-metric space.

2. Preliminaries

Throughout the paper, we shall assume that (X,d) be a metric space and *N* will denote the set of natural numbers. We follow the following notations of Nadler [8].

$CL(X)=\{A: A$ is a nonempty closed subset of $X\}$;

$CB(X)=\{A: A$ is a nonempty closed and bounded subset of $X\}$;

$C(X)=\{A: A$ is a nonempty compact subset of $X\}$.

Definition 2.1. [12]. Let X be a non-empty set. An *S*-metric on X is a function $S: X\times X\times X\to[0,\infty)$ that satisfies the following conditions, for each $x,y,z,a\in X$,

(S_1) $S(x,y,z)\geq 0$,

(S_2) $S(x,y,z)=0$ if and only if $x=y=z$,

(S_3) $S(x,y,z)\leq S(x,x,a)+S(y,y,a)+S(z,z,a)$.

The pair (X,S) is called an *S*-metric space.

Some immediate examples of such *S*-metric spaces are given in [12] as follows:

Example 2.1. Let $X=\mathbb{R}^n$ and $\|.\|$ a norm on X, then $S(x,y,z)=\|y+z-2x\|+\|y-z\|$ is an *S*-metric on X.

Example 2.2. Let $X = \mathbb{R}^n$ and $\|.\|$ a norm on X, then $S(x,y,z) = \|x-z\| + \|y-z\|$ is an S-metric on X.

Example 2.3. Let X be a nonempty set, d is ordinary metric on X, then $S(x,y,z) = d(x,z) + d(y,z)$ is an S-metric on X.

Lemma 2.1. [12]. In an S-metric space, we have $S(x,x,y) = S(y,y,x)$.

Definition 2.2. [12]. Let (X,S) be an S-metric space. For $r > 0$ and $x \in X$, the open ball $B_S(x,r)$ and closed ball $B_S[x,r]$ with a center x and a radius r is defined as follow:

$$B_S(x,r) = \{y \in X : S(y,y,x) < r\},$$

$$B_S[x,r] = \{y \in X : S(y,y,x) \le r\}.$$

Definition 2.3. [9]. Let (X,S) be an S-metric space and A be a nonempty subset of X. The diameter of A is defined by

$$diam\{A\} = \sup\{S(x,x,y) : x, y \in A\}. \tag{2.1}$$

If A is S-bounded, then $diam\{A\} < \infty$.

Definition 2.4. [12]. Let (X,S) be an S-metric space.

(1) If for every $x \in A$ there exist $r > 0$ such that $B_S(x,r) \subset A$, then the subset A is called an open subset of X.

(2) A subset A of X is said to be S-bounded if there exists $r > 0$ such that $S(x,x,y) < r$ for all $x, y \in A$.

(3) A sequence $\{x_n\}$ in X converges to x if and only if $S(x_n, x_n, x) \to 0$ as $n \to \infty$. That is for each $\varepsilon > 0$ there exists $n_0 \in N$ such that for all $n \ge n_0$, $S(x_n, x_n, x) < \varepsilon$ and we denote this by $\lim_{n\to\infty} x_n = x$.

(4) A sequence $\{x_n\}$ in X is called Cauchy sequence if for each $\varepsilon > 0$, there exists $n_0 \in N$ such that for all $n \ge n_0$, $S(x_n, x_n, x) < \varepsilon$.

(5) The S-metric space (X,S) is said to be complete if every Cauchy sequence is convergent.

Lemma 2.2. [12]. Let (X,S) be an S-metric space. If there exist sequences $\{x_n\}$ and $\{y_n\}$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$ then $\lim_{n\to\infty} S(x_n, x_n, y_n) = S(x,x,y)$.

The relationship between a metric and an S-metric was shown in [6] as follow:

Lemma 2.3. Let (X,d) be a metric space. Then the following properties are satisfied:

(1) $S_d(x,y,z) = d(x,z) + d(y,z)$ for all $x, y, z \in X$ is an S-metric on X.

(2) $x_n \to x$ in (X,d) if and only if $x_n \to x$ in (X, S_d).

(3) $\{x_n\}$ is Cauchy in (X,d) if and only if $\{x_n\}$ is Cauchy in (X, S_d).

(4) (X,d) is complete if and only if (X, S_d) is complete.

The metric S_d is called as the S-metric generated by d (see [9]).

Definition 2.5. [10]. Let (X,S) be an S-metric space, $x \in X$ and $A, B \subset X$, then the distance of the point x to the set A is defined as

$$S(x,x,A) = \inf\{S(x,x,y) : y \in A\}.$$

It is clear by the definition of $S(x,x,A)$, that $S(x,x,A) = 0 \Leftrightarrow x \in \bar{A}$.

Now we define Hausdorff S-distance on $CB(X)$ as follow:

$$H_S(A,A,B) = \max\left\{\sup_{x\in A} S(x,x,B),\ \sup_{y\in B} S(y,y,A)\right\},$$

where we recall $S(x,x,B) = \inf\{S(x,x,y):\ y\in B\}$. Then H_S is called Hausdorff S-distance on $CB(X)$ induced by S-metric.

3. Fixed Point Theorems

In this section we start with the following lemmas. The proof of these lemmas are simple consequence of definition of the Hausdorff S-distance $H_S(A,A,B)$.

Lemma 3.1. Let (X,S) be an S-metric space and $A,B\in CB(X)$. Then for each $a\in A$, we have

$$S(a,a,B) \le H_S(A,A,B).$$

Lemma 3.2. If $A,B\in CB(X)$ and $a\in A$, then for each $\varepsilon > 0$ there exists $b\in B$ such that

$$S(a,a,b) \le H_S(A,A,B) + \varepsilon.$$

Now, first we define a multi-valued contraction on S-metric space and then state our main results.

Definition 3.1. Let (X,S) be an S-metric space. A function $T: X\to CB(X)$ is said to be a multi-valued contraction on X if there exists a constant α, with $0\le\alpha<1$, such that

$$H_S(Tx,Tx,Ty) \le \alpha S(x,x,y) \text{ for all } x,y\in X. \tag{3.1}$$

Theorem 3.1. Let (X,S) be a complete S-metric space. If $T: X\to CB(X)$ is a multi-valued contraction on X then T has a fixed point.

Proof. Let x_0 be an arbitrary point in X and choose $x_1\in Tx_0$. If $x_1 = x_0$ then x_0 is a fixed point of T and the proof is complete. Assume that $x_1\ne x_0$. Then by Lemma 3.2, there exists $x_2\in Tx_1$ such that

$$S(x_1,x_1,x_2) \le H_S(Tx_0,Tx_0,Tx_1) + \alpha.$$

Also we get there exist $x_3\in Tx_2$ such that

$$S(x_2,x_2,x_3) \le H_S(Tx_1,Tx_1,Tx_2) + \alpha^2.$$

Continuing in this fashion, we construct a sequence $\{x_n\}$ in X such that

$$x_{n+1}\in Tx_n$$

and

$$S(x_n,x_n,x_{n+1}) \le H_S(Tx_{n-1},Tx_{n-1},Tx_n) + \alpha^n, \text{ for all } n\ge 1.$$

Then from (3.1)

$$\begin{aligned}
S(x_n,x_n,x_{n+1}) &\le H_S(Tx_{n-1},Tx_{n-1},Tx_n) + \alpha^n\\
&\le \alpha\left(H_S\left(Tx_{n-2},Tx_{n-2},Tx_{n-1}\right) + \alpha^{n-1}\right) + \alpha^n\\
&\le \alpha^2 S(x_{n-2},x_{n-2},x_{n-1}) + 2\alpha^n\\
&\ \ \vdots\\
&\le \alpha^n S(x_0,x_0,x_1) + n\alpha^n, \text{ for all } n\ge 1.
\end{aligned}$$

Hence for all $m,n\ge 1$, we have

$$S(x_n, x_n, x_m) \le 2\sum_{r=n}^{m-2} S(x_r, x_r, x_{r+1}) + S(x_{m-1}, x_{m-1}, x_m)$$

$$\le 2\sum_{r=n}^{m-2} \alpha^r S(x_0, x_0, x_1) + 2\sum_{r=n}^{m-2} r\alpha^r + \alpha^{m-1} S(x_0, x_0, x_1) + (m-1)\alpha^{m-1}$$
$$\le 2\sum_{r=n}^{m-1} \alpha^r S(x_0, x_0, x_1) + 2\sum_{r=n}^{m-1} r\alpha^r .$$

It follows that the sequence $\{x_n\}$ is a Cauchy. Since (X, S) is a complete S-metric space, the sequence $\{x_n\}$ converges to same point $x_0 \in X$. Therefore the sequence $\{Tx_n\}$ converges to Tx_0 and since $x_{n+1} \in Tx_n$ for all $n \ge 1$, it follows that $x_0 \in Tx_0$. This completes the proof of the theorem.

Example 3.1. Let $X = [0, \infty)$ be endowed with S-metric $S(x, y, z) = |x - z| + |x + z - 2y|$ for all $x, y \in X$. Define $T : X \to CB(X)$ by $Tx = [0, x/2]$ for all $x \in X$.
Then for all $x, y \in X$, we have

$$S(x, x, y) = |x - y| + |x + y - 2x| = 2|x - y|$$

and

$$H_S(Tx, Tx, Ty) = |x - y| .$$

Thus

$$H_S(Tx, Tx, Ty) \le \alpha S(x, x, y) \text{ for } \alpha = 1/2$$

that is, T is a multi-valued contraction on X with $\alpha = 1/2$ and T has a fixed point at $x = 0$ in X.

Theorem 3.2. Let (X, S) be a compact S-metric space with $T : X \to CB(X)$ satisfying

$$H_S(Tx, Tx, Ty) < S(x, x, y) \text{ for all } x, y \in X \text{ and } x \ne y. \tag{3.2}$$

Then T has a fixed point in X.

Proof. Let us suppose that T does not have any fixed point in X that is, $x \notin Tx$ for all $x \in X$. Then the function $f : X \to [0, \infty)$ given by $f(x) = S(x, x, Tx) = \inf\{S(x, x, z) : z \in Tx\}$ for all $x \in X$, is continuous on compact set X therefore it attains its minimum value at some point (say) a in X. Thus

$$S(a, a, Ta) \le S(x, x, Tx) \text{ for all } x \in X.$$

Since $S(a, a, Ta) = \inf\{S(a, a, z) : z \in Ta\}$ therefore for particular $z = z_0 \in Ta$, we have $S(a, a, Ta) = S(a, a, z_0)$ and

$$S(a, a, z_0) \le S(x, x, Tx) \text{ for all } x \in X.$$

Now using (3.2) and Lemma 3.1, we have

$$S(a, a, z_0) \le S(z_0, z_0, Tz_0) \le H_S(Ta, Ta, Tz_0) < S(a, a, z_0),$$

which gives $S(a, a, z_0) = 0$ and $z_0 = a \in Ta$. Hence T has a fixed point on a compact S-metric space X.

Example 3.2. Let $X=[0,1]$ be endowed with usual S-metric space on X defined in [11] as follows

$$S(x,y,z)=|x-y|+|y-z| \text{ for all } x,y\in X.$$

We define $T:X\to CB(X)$ by $Tx=\left[0,\dfrac{1}{1+x}\right]$ for all $x\in X$. Then for all $x,y\in X$ and $x\neq y$, we have

$$S(x,x,y)=|x-y|$$

and

$$H_S(Tx,Tx,Ty)=\left|\frac{1}{1+x},-\frac{1}{1+y}\right|=\frac{|x-y|}{(1+x)(1+y)}<|x-y|.$$

Hence T satisfies condition (3.2) of Theorem 3.2 and the set of fixed points of T is $\left[0,\dfrac{\sqrt{5}-1}{2}\right]$.When x,y are sufficient close to 0, then $\dfrac{H_S(Tx,Tx,Ty)}{S(x,x,y)}=\dfrac{1}{(1+x)(1+y)}$ obtained its value arbitrary closed to 1. Therefore *T* is not a multi-valued contraction on [0,1] with usual *S*-metric.

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SHEFFER AND BRENKE POLYNOMIALS ASSOCIATED WITH GENERALIZED BELL NUMBERS

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Abstract
In a recent paper, the first two authors have studied the integer number sequences generated by the higher order Bell polynomials. In this article we introduce new sets of Sheffer polynomial sequences based on the higher order Bell numbers, and we study their basic properties. Furthermore, we introduce new sequences of associated Sheffer and Brenke polynomial sequences.

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1. Introduction

The Bell polynomials [1], first appear as a mathematical tool for representing the nth derivative of a composite function.

Being related to partitions, the Bell polynomials are often used in Combinatorial Analysis [28]. They have been also applied in many different frameworks, such as: the Blissard problem (see [28, p. 46]); the representation of Lucas polynomials of the first and second kind [9, 14]; the construction of representation formulas for the Newton sum rules for the zeros of polynomials [19, 20]; the recurrence relations for a class of Freud-type polynomials [4], the representation formulas for the symmetric functions of a countable set of numbers, generalizing the classical algebraic Newton-Girard formulas. Consequently, as it was recognized [10], it is possible to find reduction formulas for the *orthogonal invariants* of a strictly Positive Compact Operator (shortly PCO), deriving in a simple way the so called Robert formulas [29].

Some generalized forms of Bell polynomials already appeared in literature (see e.g. [27, 17]).
Generalizations of the Bell polynomials suitable for the differentiation of multivariable composite functions have been also defined (see [26, 5]).
Recently the Adomian polynomials [21], have been used in order to derive relations between the Bell polynomials [22].

Higher order Bell polynomials (i.e. Bell polynomials for representing the derivatives of several nested functions) were already defined by the first two authors [23], and the introduction of higher order multivariable Bell polynomials was also achieved [5], and this subject has been considered in a forthcoming article by Chinese mathematicians [16], who are also able to show an application to white noise distribution theory.

In a recent article [25] the first two authors have studied the integer number sequences generated by the higher order Bell polynomials. In this article we introduce new sets of Sheffer polynomial sequences based on the higher order Bell numbers, and we study, in some particular case, their basic properties. See also [36].
Furthermore, we introduce new sequences of associated Sheffer and Brenke polynomial sequence, associated with the same number sequences.

2. Recalling the Bell polynomials

We recall that the Bell polynomials are a classical mathematical tool for representing the *n*th derivative of a composite function. In fact by considering the composite function $\Phi(t) := f(g(t))$ of functions $x = g(t)$ and $y = f(x)$ defined in suitable intervals of the real axis and n times differentiable with respect to the relevant independent variables and by using the following notation

$$\Phi_h := D_t^h \Phi(t), f_h := D_x^h f(x) \Big|_{x = g(t)}, g_h := D_t^h g(t) \tag{2.1}$$

and

$$([f, g]_n) := (f_1, g_1; f_2, g_2; \dots; f_n, g_n) \tag{2.2}$$

They are defined as follows:

$$Y_n([f, g]_n) := \Phi_n \tag{2.3}$$

Inductively, by using the notation
$[g]_n := (g_1, g_2, \dots, g_n)$,

we can write

$$Y_n([f, g]_n) = \sum_{k=1}^n A_{n,k}([g]_n) f_k, \tag{2.4}$$

where, the coefficient $A_{n,k}$, for any $k = 1, \dots, n$, is a polynomial in $g_1, g_2, \dots, g_n$, homogeneous of degree k and *isobaric* of weight n (i.e., it is a linear combination of monomials ${g_1}^{k_1} {g_2}^{k_2} \dots {g_n}^{k_n}$ whose weight is constantly given by $k_1 + 2k_2 + \dots + nk_n = n$).

Proposition 2.1. *The Bell polynomials satisfy the recurrence relation*

$$\begin{cases} Y_0([f, g]_0) := f_1 \\ Y_{n+1}([f, g]_{n+1}) = \sum_{k=0}^n \binom{n}{k} Y_{n-k}([f_1, g]_{n-k}) g_{k+1} \end{cases} \tag{2.5}$$

where, $([f_1, g]_{n-k}) := (f_2, g_1; f_3, g_2; \dots; f_{n-k+1}, g_{n-k})$.

An explicit expression for the Bell polynomials is given by the Fa`a di Bruno formula [15, 30, 32], however, as this formula makes use of partitions, it is not useful for computing higher order Bell polynomials, whereas this can be done in a easy way by means of the following recursion formula for the coefficients $A_{n,k}$ in equation (2.4) which are known as partial Bell polynomials.

Theorem 2.2. *We have, for all integer* n,

$$A_{n+1,1} = g_{n+1}, A_{n+1,n+1} = g_1^{n+1}. \tag{2.6}$$

Furthermore, for all $k = 1, 2, \ldots, n-1$, *the* $A_{n,k}$ *coefficients can be computed by the recurrence relation*

$$A_{n+1,k+1}([g]_{n+1}) = \sum_{h=0}^{n-k} \binom{n}{h} A_{n-h,k}([g]_{n-h}) g_{h+1}. \tag{2.7}$$

Definition 2.1. *The complete Bell polynomials, considered in literature, are defined by*

$$B_n([g]_n) = Y_n(1, g_1; 1, g_2; \ldots; 1, g_n) = \sum_{k=1}^{n} A_{n,k}([g]_n), \tag{2.8}$$

and the Bell numbers by

$$b_n = Y_n(1,1; 1,1; \ldots; 1,1) = \sum_{k=1}^{n} A_{n,k}(1,1, \ldots, 1)\,. \tag{2.9}$$

3. Bell polynomials of order r

In [5] the following extension of the classical Bell polynomials was achieved.

Consider $\Phi(t) := f(\varphi^{(1)}(\varphi^{(2)}(\cdots(\varphi^{(r)}(t)))))$, i.e., the composition of functions $x^{(r)} = \varphi^{(r)}(t), \ldots, x^{(2)} = \varphi^{(2)}(x^{(3)})$, $x^{(1)} = \varphi^{(1)}(x^{(2)})$ $y = f(x^{(1)})$ defined in suitable intervals of the real axis, and suppose that the functions $\varphi^{(r)}, \ldots, \varphi^{(2)}, \varphi^{(1)}, f$ are n times differentiable with respect to the relevant independent variables so that, by using the chain rule, $\Phi(t)$ can be differentiated n times with respect to t.

We use the following notation

$\Phi_h := D_t^h \Phi(t)$,

$f_h := D_{x^{(1)}}^h f \Big|_{x^{(1)} = \varphi^{(1)}\left(\ldots\left(\varphi^{(r)}(t)\right)\right)}$,

$\varphi_h^{(1)} := D_{x^{(2)}}^h \varphi^{(1)} \Big|_{x^{(2)} = \varphi^{(2)}\left(\ldots\left(\varphi^{(r)}(t)\right)\right)}$,

$\varphi_h^{(r)} := D_t^h \varphi^{(r)}(t)$, and

$$\left([f, \varphi^{(1)}, \ldots, \varphi^{(r)}]_n\right) := \left(f_1, \varphi_1^{(1)}, \ldots, \varphi_1^{(r)}; \ldots; f_n, \varphi_n^{(1)}, \ldots, \varphi_n^{(r)}\right). \tag{3.1}$$

Then the nth derivative of the function Φ allows us to define the one-dimensional Bell polynomials of order r, $Y_n^{[r]}$ as follows:

$$Y_n^{[r]}\left([f, \varphi^{(1)}, \ldots, \varphi^{(r)}]_n\right) := \Phi_n. \tag{3.2}$$

For $r = 1$ we obtain the ordinary Bell polynomials

$$Y_n^{[1]}\left([f, \varphi^{(1)}]_n\right) = Y_n\left([f, \varphi^{(1)}]_n\right).$$

Note that we are considering here the one-dimensional case, while in [5] even the multi-dimensional Bell polynomials were introduced.

The first polynomials have the following explicit expressions

$$Y_1^{[r]}\left([f, \varphi^{(1)}, \ldots, \varphi^{(r)}]_1\right) = f_1 \varphi_1^{(1)} \ldots \varphi_1^{(r)},$$

$$Y_2^{[r]}\left([f, \varphi^{(1)}, \ldots, \varphi^{(r)}]_2\right) = f_2 \left(\varphi_1^{(1)} \ldots \varphi_1^{(r)}\right)^2 + f_1 \varphi_2^{(1)} \left(\varphi_1^{(2)} \ldots \varphi_1^{(r)}\right)^2$$
$$+ f_1 \varphi_1^{(1)} \varphi_2^{(2)} \left(\varphi_1^{(3)} \ldots \varphi_1^{(r)}\right)^2 + f_1 \varphi_1^{(1)} \varphi_1^{(2)} \ldots \varphi_1^{(r-1)} \varphi_2^{(r)}. \tag{3.3}$$

In general, we have

$$Y_n^{[r]}\left(\left[f,\varphi^{(1)},\dots,\varphi^{(r)}\right]_n\right)=\sum_{k=1}^{n}A_{n,k}^{[r]}\left(\left[\varphi^{(1)},\dots,\varphi^{(r)}\right]_n\right)f_k. \tag{3.4}$$

A generalized form of the Fa`a di Bruno formula, a recurrence relation and further results relevant to this extension can be found in [5].

The recurrence relation (2.6) - (2.7) can be generalized as follows:

Theorem 3.1. *For all integer* n, *we have*

$$A_{n+1,1}^{[r]}=Y_{n+1}^{[r-1]}\left(\left[\varphi^{(1)},\dots,\varphi^{(r)}\right]_{n+1}\right), A_{n+1,n+1}^{[r]}=\left(Y_1^{[r-1]}\left(\left[\varphi^{(1)},\dots,\varphi^{(r)}\right]_1\right)\right)^{n+1}= \left(\varphi_1^{(1)},\dots,\varphi_1^{(r)}\right)^{n+1}. \tag{3.5}$$

Furthermore, for all $k = 1,2,\dots,n-1$, *the* rth *order partial Bell polynomials* $A_{n,k}^{[r]}$ *satisfy the recursion*

$$A_{n+1,n+1}^{[r]}\left(\left[\varphi^{(1)},\dots,\varphi^{(r)}\right]_{n+1}\right)= \sum_{h=0}^{n-k}\binom{n}{h}A_{n-h,k}^{[r]}\left(\left[\varphi^{(1)},\dots,\varphi^{(r)}\right]_{n-h}\right)Y_{h+1}^{[r-1]}\left(\left[\varphi^{(1)},\dots,\varphi^{(r)}\right]_{h+1}\right). \tag{3.6}$$

Definition 3.1. *The complete Bell polynomials of order* r,$B_n^{[r]}$, *are defined by the equation*

$$\begin{aligned} B_n^{[r]}\left(\left[\varphi^{(1)},\dots,\varphi^{(r)}\right]_n\right) &= Y_n^{[r]}\left(1,\varphi_1^{(1)},\dots,\varphi_1^{(r)};\dots;1,\varphi_n^{(1)},\dots,\varphi_n^{(r)}\right)\\ &=\sum_{k=1}^{n}A_{n,k}^{[r]}\left(\left[\varphi^{(1)},\dots,\varphi^{(r)}\right]_n\right). \end{aligned}$$

and the rth *order Bell numbers by*

$$b_n^{[r]}=Y_n^{[r]}(1,1,1;\dots;1,1,1)=\sum_{k=1}^{n}A_{n,k}^{[r]}(1,1;\dots;1,1).$$

4. Higher order Bell numbers, for $r = 2,3,4,5$

It is worth to note that the sequences of higher order Bell numbers which are presented here appear in the Encyclopedia of Integer Sequences [34] under the $A144150$, arising from a problem of Combinatorial Analysis and even as the McLaurin coefficients of the functions [2, 18]

$$\begin{gathered} exp(exp(exp(x)-1)-1),\\ exp(exp(exp(exp(x)-1)-1)-1),\\ exp(exp(exp(exp(exp(x)-1)-1)-1)-1),\\ exp(exp(exp(exp(exp(exp(x)-1)-1)-1)-1)-1), \end{gathered} \tag{4.1}$$

for the cases $r = 2, r = 3, r = 4, r = 5$, respectively, and so on for the subsequent values of r. Whereas in our approach they assume a more general meaning, as they are independent of the functions $f,\varphi^{(1)},\dots,\varphi^{(r)}$.

Let $b_n^{[1]} := b_n$ (namely, the classical Bell numbers). According to the above reference we have found, using the recurrence relation (3.5) - (3.6) and by means of the computer algebra program

Mathematica©, the following sequences for the higher order Bell numbers $b_n^{[2]}, b_n^{[3]}, b_n^{[4]}, b_n^{[5]}$, $(n = 1,2,3, \dots, 10)$:

n	$b_n^{[1]}$	$b_n^{[2]}$	$b_n^{[3]}$	$b_n^{[4]}$	$b_n^{[5]}$
1	1	1	1	1	1
2	2	3	4	5	6
3	5	12	22	35	51
4	15	60	154	315	561
5	52	358	1304	3455	7556
6	203	2471	12915	44590	120196
7	877	19302	146115	660665	2201856
8	4140	167894	1855570	11035095	45592666
9	21147	1606137	26097835	204904830	1051951026
10	115975	16733779	402215465	4183174520	26740775306

Table 1: Bell and higher order Bell numbers for $n = 1, 2, \dots, 10$.

Of course, the above table can be extended up to the desired order, as the Mathematica© program runs efficiently.

5. Sheffer polynomials

The Sheffer polynomials $\{s_n(x)\}$ are introduced [33] by means of the exponential generating function [35] of the type:

$$A(t)exp\big(xH(t)\big) = \sum_{n=0}^{\infty} s_n(x)\frac{t^n}{n!} \tag{5.1}$$

where, $A(t) = \sum_{n=0}^{\infty} a_n \frac{t^n}{n!} (a_0 \neq 0)$, and $H(t) = \sum_{n=0}^{\infty} h_n \frac{t^n}{n!} (h_0 = 0)$. (5.2)

According to a different characterization (see [31, p. 18]), the same polynomial sequence can be defined by means of the pair $(g(t), f(t))$, where $g(t)$ is an invertible series and $f(t)$ is a delta series:

$$g(t) = \sum_{n=0}^{\infty} g_n \frac{t^n}{n!} (g_0 \neq 0), \text{ and } f(t) = \sum_{n=0}^{\infty} f_n \frac{t^n}{n!} (f_0 = 0, f_1 \neq 0). \tag{5.3}$$

Denoting by $f^{-1}(t)$ the compositional inverse of $f(t)$ (i.e. such that $f\,(f^{-1}(t)) = f^{-1}(f(t)) = t$), the exponential generating function of the sequence $\{s_n(x)\}$ is given by

$$\frac{1}{g[f^{-1}(t)]} exp(xf^{-1}(t)) = \sum_{n=0}^{\infty} s_n(x)\frac{t^n}{n!}, \tag{5.4}$$

so that $A(t) = \frac{1}{g[f^{-1}(t)]}, H(t) = f^{-1}(t)$. (5.5)

When $g(t) \equiv 1$, the Sheffer sequence corresponding to the pair $(1, f(t))$ is called the associated Sheffer sequence $\{\sigma_n(x)\}$ for $f(t)$, whose exponential generating function is

$$exp(xf^{-1}(t)) = \sum_{n=0}^{\infty} \sigma_n(x)\frac{t^n}{n!}. \tag{5.6}$$

A list of known Sheffer polynomial sequences and associated ones can be found in [7]. In particular, assuming

$$A(t) = exp(\beta t), H(t) = 1 - exp(t), \tag{5.7}$$

the corresponding pair $(g\ (t),\ f\ (t))$ is given by
$$g\ (t)\ =\ (1\ -\ t)^{-\beta}\ ,\ f\ (t)\ =\ log\ (1\ -\ t)\ , \tag{5.8}$$

and the exponential generating function of the Sheffer sequence is
$$exp\big(\beta t + x(1-e^t)\big) = \sum_{n=0}^{\infty} \alpha_n^{(\beta)}(x)\frac{t^n}{n!} \tag{5.9}$$

or, in equivalent form:

$$exp\big(\beta t + x(e^t-1)\big) = \sum_{n=0}^{\infty} \alpha_n^{(\beta)}(-x)\frac{t^n}{n!}. \tag{5.10}$$
The polynomials $\{\alpha_n^{(\beta)}(x)\}$ are called in [7] "Actuarial polynomials".

6. New Sheffer polynomial sequences

In this section, we introduce new Sheffer sequences generalizing the above mentioned Actuarial polynomials. This can be done by assuming the pair

$$A(t)\ =\ exp(\beta t)\ , H_1(t)\ =\ f_1^{-1}(t)\ =\ exp(exp(t)\ -\ 1) - 1, \tag{6.1}$$

so that the coefficients of the Taylor expansion of $H_1(t)$ are given by the Bell numbers $b_1\ = b_n^{[1]}$:

$$H_1(t)\ = \sum_{n=0}^{\infty} b_n^{(1)}\frac{t^n}{n!}, \tag{6.2}$$
and consequently
$$f_1(t) =\ log\ (log(t\ +\ 1) +\ 1), g_1(t) =\ [log(t\ +\ 1) +\ 1]^{-\beta}. \tag{6.3}$$

Therefore, the exponential generating function of the corresponding Sheffer sequence $\left\{\alpha_n^{([1],\beta)}(x)\right\}$ is given by
$$exp(\beta t + xH_1(t)) = \sum_{n=0}^{\infty} \alpha_n^{([1],\beta)}(x)\frac{t^n}{n!} \tag{6.4}$$

The above method can be iterated by letting
$$A(t)\ =\ exp\ (\beta t)\ ,$$
$$H_2\ (t)\ =\ f_2^{-1}(t)\ =\ exp(exp(exp(t)\ -\ 1)\ -\ 1)\ -\ 1\ , \tag{6.5}$$

so that the coefficients of the Taylor expansion of $H_2(t)$ are given by the 2nd order Bell numbers $b_n^{[2]}$:
$$H_2(t)\ \ = \sum_{n=0}^{\infty} b_n^{(2)}\frac{t^n}{n!} \tag{6.6}$$
and consequently
$$f_2(t) =\ log\ (log\ (log(t\ +\ 1) +\ 1), g_2(t) =\ [log(log(t\ +\ 1) +\ 1]^{-\beta} \tag{6.7}$$
Therefore, the exponential generating function of the corresponding Sheffer sequence $\left\{\alpha_n^{([2],\beta)}(x)\right\}$ is given by:
$$exp(\beta t + xH_2(t)) = \sum_{n=0}^{\infty} \alpha_n^{([2],\beta)}(x)\frac{t^n}{n!}\ . \tag{6.8}$$

In general, by letting
$A(t) = exp(\beta t),$
$H_r\,(t) = exp(\dots exp(exp(t) - 1) - 1)\dots - 1$, ($r$ times exp) (6.9)
so that the coefficients of the Taylor expansion of $H_r(t)$ are given by the Bell numbers $b_n = b_n^{[r]}$:
$$H_r(t) = \sum_{n=0}^{\infty} b_n^{(r)} \frac{t^n}{n!} \tag{6.10}$$
and consequently
$f_r(t) = log\,(log(\dots\,(log(t + 1) + 1)\dots) + 1)$, ($r$ times log)
$g_r(t) = [log(\dots(log(t + 1) + 1)\dots) + 1]^{-\beta}$, $\big((r - 1)$ times $log\big)$. (6.11)
Therefore, the exponential generating function of the corresponding Sheffer sequence $\left\{\alpha_n^{([r],\beta)}(x)\right\}$ is given by

$$exp(\beta t + xH_r(t)) = \sum_{n=0}^{\infty} \alpha_n^{([r],\beta)}(x) \frac{t^n}{n!}. \tag{6.12}$$

7. The associated Sheffer polynomials

The associated Sheffer polynomials relevant to the pair $(1, f(t))$, where
$f(t) = log(t + 1)\,, f^{-1}(t) = H(t) = e^t - 1\,,$ (7.1)
have been introduced by E.T. Bell [1] and are known as “Exponential polynomials” $\Phi_n(x)$. Their exponential generating function is given by

$$exp\big(x(\exp(t) - 1)\big) = \sum_{n=0}^{\infty} \Phi_n(x) \frac{t^n}{n!}. \tag{7.2}$$

8. New sets of associated Sheffer polynomials

Results similar to that obtained in Section 6 can be achieved by introducing the associated Sheffer polynomials relevant to the pair $(1, f_1(t))$, where $f_1(t) = log(log(t + 1) + 1)\,, H_1(t) = f_1^{-1}(t) = exp(exp(t) - 1) - 1\,,$ (8.1)
so that the exponential generating function of the associated Sheffer polynomials is given by
$$exp(xH_1(t)\,) = \sum_{n=0}^{\infty} (\mathrm{x})_{\mathrm{n}}^{[1]} \frac{t^n}{n!}, \tag{8.2}$$

and in general, assuming:

$$f_r(t) = log\,(log\,(\dots(log(t + 1) + 1)\dots) + 1)\,, (r\ times\ log)\,, \tag{8.3}$$

$$H_r(t) = f_r^{-1}(t) = exp(\dots exp(exp(t) - 1) - 1) \cdots - 1\,, (r\ times\ exp)\,, \tag{8.4}$$
we find the associated Sheffer polynomials defined by the exponential generating function
$$exp(xH_r(t)\,) = \sum_{n=0}^{\infty} (\mathrm{x})_{\mathrm{n}}^{[\mathrm{r}]} \frac{t^n}{n!}. \tag{8.5}$$

9. The Sheffer polynomials $\left\{\boldsymbol{\alpha}_{\boldsymbol{n}}^{([r],\boldsymbol{\beta})}(\boldsymbol{x})\right\}$

9.1 Explicit expression

The explicit expression of these polynomials can be easily done only by using the Taylor expansion of the associated Sheffer polynomials $(\mathrm{x})_{\mathrm{n}}^{[\mathrm{r}]}$, namely

Theorem 9.1. *We have, for all integer n,*

$$\left\{\alpha_n^{([r],\beta)}(x)\right\} = \sum_{k=0}^{n} \binom{n}{k} \beta^{n-k}.\,(x)_k^{[r]} \tag{9.1}$$

Proof. - It is a trivial consequence of equation (6.12) by using the Cauchy product of $exp(\beta t)$ times the series expansion (8.5) of $exp\ (xH_r(t))$.

Note that in order to find the explicit expression of the above Sheffer polynomials $(x)_k^{[r]}$ it is necessary to evaluate the coefficients of the Taylor expansion of the function $exp\ (xH_r(t))$ (as a function of t, because x is just a parameter), and this can be done by using the Bell polynomials of order r [23], recalled in Sect. 3, as the considered function is a higher order composite function of the type $exp(xy_1(y_2(\ldots(y_r(t))\ldots)$
where $y_r(t) = exp(t) - 1, y_{r-1} = exp(y_r(t)) - 1, \ldots, y_1 = exp(y_2(t)) - 1$.

For completeness, in order to depict the polynomials $\left\{\alpha_n^{([r],\beta)}(x)\right\}$, here we report the explicit expression of the associated Sheffer polynomials $(x)_k^{[r]}$for $r = 1$ up to order $n = 6$:

$(x)_0^{[1]} = 1;$
$(x)_1^{[1]} = x;$
$(x)_2^{[1]} = x(x + 2);$
$(x)_3^{[1]} = x(x^2 + 6x + 5);$
$(x)_4^{[1]} = x(x^3 + 12x^2 + 32x + 15);$
$(x)_5^{[1]} = x(x^4 + 20x^3 + 110x^2 + 175x + 52);$
$(x)_6^{[1]} = x(x^5 + 30x^4 + 280x^3 + 945x^2 + 1012x + 203),$

$and\ r = 2\ up\ to\ order\ n = 6$:
$(x)_0^{[2]} = 1;$
$(x)_1^{[2]} = x;$
$(x)_2^{[2]} = x(x + 3);$
$(x)_3^{[2]} = x(x^2 + 9x + 12);$
$(x)_4^{[2]} = x(x^3 + 18x^2 + 75x + 60);$
$(x)_5^{[2]} = x(x^4 + 30x^3 + 255x^2 + 660x + 358);$
$(x)_6^{[2]} = x(x^5 + 45x^4 + 645x^3 + 3465x^2 + 6288x + 2471).$

Now, using the expression of the associated Sheffer polynomials above, we can depict the Sheffer polynomials $\left\{\alpha_n^{([r],\beta)}(x)\right\}$ for any β for the cases $r = 1, r = 2$. As an example, we report in Figures 1, 2, 3, 4, the graphs of the $\left\{\alpha_n^{([1],\beta)}(x)\right\}$ for different values of β.

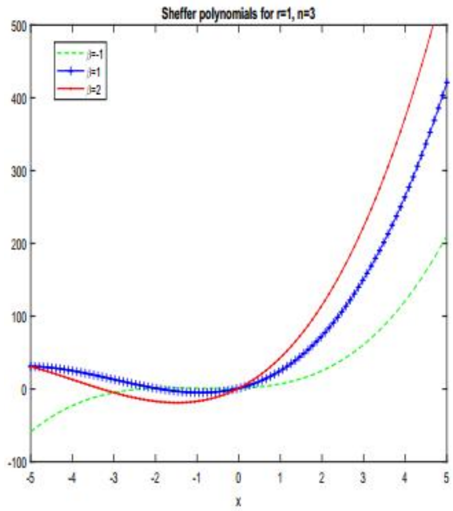

Figure 1: Graph of the Sheffer polynomials $\left\{\alpha_n^{([1],\beta)}(x)\right\}$ of degree $n = 3$, for $\beta = -1, 1, 2$.

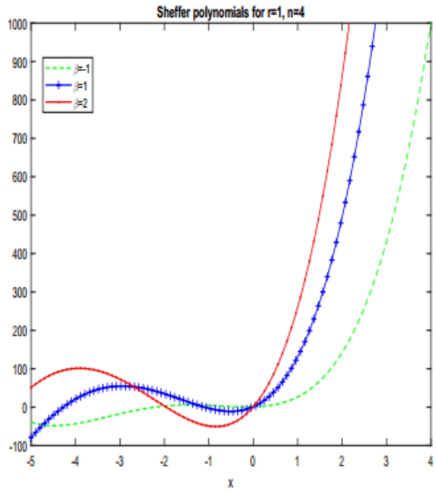

Figure 2: Graph of the Sheffer polynomials $\left\{\alpha_n^{([1],\beta)}(x)\right\}$ of degree $n = 4$, for $\beta = -1, 1, 2$.

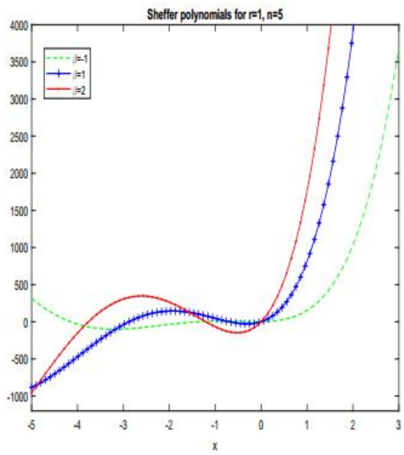

Figure 3: Graph of the Sheffer polynomials $\left\{\alpha_n^{([1],\beta)}(x)\right\}$ of degree $n = 5$, for $\beta = -1, 1, 2$.

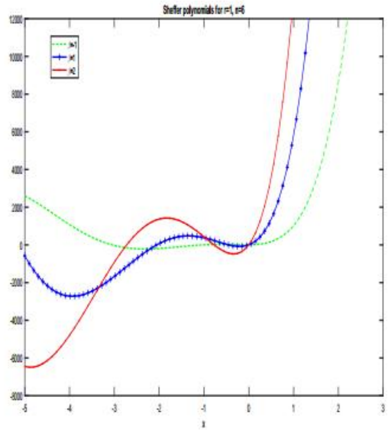

Figure 4: Graph of the Sheffer polynomials $\left\{\alpha_n^{([1],\beta)}(x)\right\}$ of degree $n = 6$, for $\beta = -1, 1, 2$.

10. Brenke polynomials

The Brenke polynomials $\{\Upsilon_n(x)\}$ are introduced [8] by means of the exponential generating function of the type:

$$A(t)\psi(xt) = \sum_{n=0}^{\infty} \Upsilon_n(x)\frac{t^n}{n!}, \tag{10.1}$$

where

$$A(t) = \sum_{n=0}^{\infty} a_n \frac{t^n}{n!}, (a_0 \neq 0),$$

$$\psi(t) = \sum_{n=0}^{\infty} \gamma_n \frac{t^n}{n!}, (\gamma_0 \neq 0). \tag{10.2}$$

They are a particular case of a more general family of polynomials, namely, the BoasBuck polynomials [6, 7], which are defined by the exponential generating function

$$A(t)\psi(xH(t) = \sum_{n=0}^{\infty} p_n(x)\frac{t^n}{n!}, \tag{10.3}$$

$where$

$$A(t) = \sum_{n=0}^{\infty} a_n \frac{t^n}{n!}, (a_0 \neq 0),$$

$$\psi(t) = \sum_{n=0}^{\infty} \gamma_n \frac{t^n}{n!}, (\gamma_n \neq 0 \quad \forall n), \tag{10.4}$$

with $\psi(t)$ not a polynomial, and lastly

$$H(t) = \sum_{n=0}^{\infty} h_n \frac{t^n}{n!}, (h_1 \neq 0). \tag{10.5}$$

The Boas-Buck polynomials include also the Humbert polynomials (see [35]), and, as a particular case, the Gegenbauer polynomials.

11. New Brenke polynomial sequences

In this section we introduce new Brenke sequences.

We start by assuming

$A(t) = e^t$,

$$\psi_1(t) = exp(exp(t) - 1) , \tag{11.1}$$

so that the coefficients of the Taylor expansion of $\psi(t)$ are given by the Bell numbers $b_n = b_n^{[1]}$:

$$\psi_1(t) = \sum_{n=0}^{\infty} b_n^{[1]} \frac{t^n}{n!}. \tag{11.2}$$

Therefore, the exponential generating function of the corresponding Brenke sequence $\left\{\Upsilon_n^{[1]}(x)\right\}$ is given by

$$A(t)\psi_1(xt) = e^t \sum_{n=0}^{\infty} b_n^{[1]}(x) \frac{(xt)^n}{n!} = \sum_{n=0}^{\infty} \Upsilon_n^{[1]}(x) \frac{t^n}{n!}. \tag{11.3}$$

The above method can be iterated by letting

$$A(t) = e^t,$$
$$\psi_2(t) = exp(exp(exp(t) - 1) - 1), \tag{11.4}$$

so that the coefficients of the Taylor expansion of $\psi(t)$ are given by the 2nd order Bell numbers $b_n^{[2]}$:

$$\psi_2(t) = \sum_{n=0}^{\infty} b_n^{[2]} \frac{t^n}{n!} \tag{11.5}$$

Therefore, the exponential generating function of the corresponding Brenke sequence $\left\{\Upsilon_n^{[2]}(x)\right\}$ is given by

$$A(t)\psi_2(xt) = e^t \sum_{n=0}^{\infty} b_n^{[2]}(x) \frac{(xt)^n}{n!} = \sum_{n=0}^{\infty} \Upsilon_n^{[2]}(x) \frac{t^n}{n!}. \tag{11.6}$$

In general, by letting

$$A(t) = e^t,$$
$$\psi_r(t) = exp(\ldots exp(exp(t) - 1) \cdots - 1), (r\ times\ exp) \tag{11.7}$$

so that the coefficients of the Taylor expansion of $\psi_r(t)$ are given by the rth order Bell numbers $b_n^{[r]}$:

$$\psi_r(t) = \sum_{n=0}^{\infty} b_n^{[r]} \frac{t^n}{n!}, \tag{11.8}$$

we find the corresponding Brenke sequence $\left\{\Upsilon_n^{[r]}(x)\right\}$ whose exponential generating function is given by

$$A(t)\psi_r(xt) = e^t \sum_{n=0}^{\infty} b_n^{[r]}(x) \frac{(xt)^n}{n!} = \sum_{n=0}^{\infty} \Upsilon_n^{[r]}(x) \frac{t^n}{n!}. \tag{11.9}$$

12. Main properties of the polynomials $\Upsilon_n^{[r]}(x)$

First of all, we derive the explicit expression of the $\Upsilon_n^{[r]}(x)$.

12.1 Explicit expression

We assume, as it is usual (see [34]), for every integer $r, b_0^{[r]} = 0$.

Theorem 12.1. *We have, for every integer n,*

$$\Upsilon_n^{[r]}(x) = \sum_{k=0}^{n} \binom{n}{k} b_k^{[r]} x^k \tag{12.1}$$

Proof. - It is a trivial consequence of equation (11.9) by using the Cauchy product of $exp(t)$ times the series expansion of $\psi_r(t)$.

For completeness, we report some graphs of the Brenke polynomials in the next Figures 5, 6, 7.

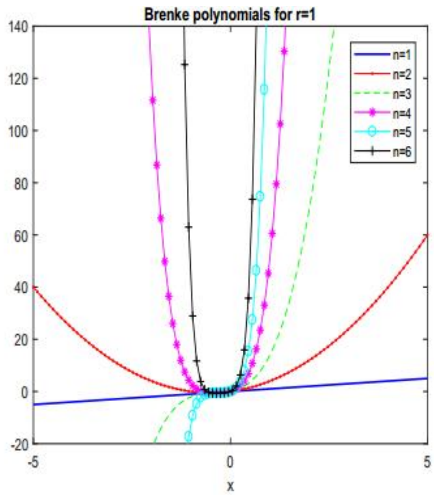

Figure 5: Graph of the Brenke polynomials $\Upsilon_n^{[1]}(x)$ up to $n = 6$.

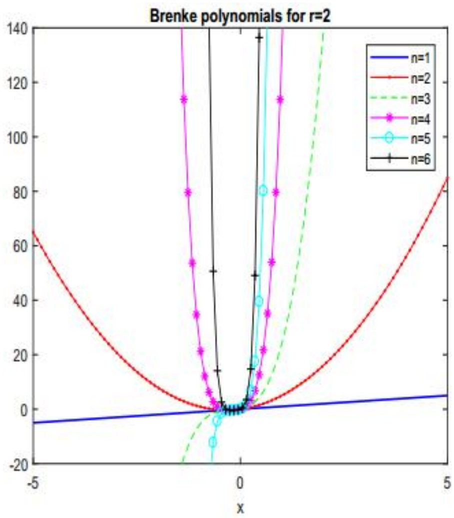

Figure 6: Graph of the Brenke polynomials $\Upsilon_n^{[2]}(x)$ up to $n = 6$.

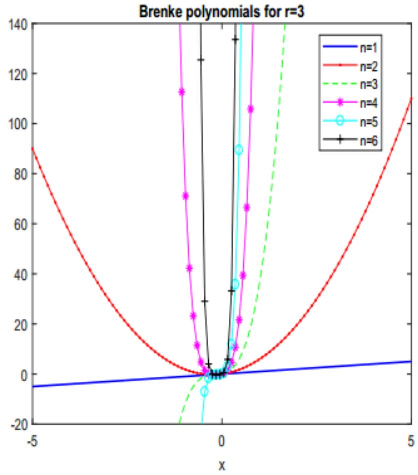

Figure 7: Graph of the Brenke polynomials $\Upsilon_n^{[3]}(x)$ up to $n = 6$.

.12.2. Shift operators

We recall that a polynomial set $\{p_n(x)\}$ is called quasi-monomial if and only if there exist two operators $\acute{P}$ and $\acute{M}$ such that

$$\acute{P}\,(p_n\,(x)) = np_{n-1}\,(x),\ \acute{M}(p_n\,(x)) = p_{n+1}(x),\ (n = 1,2,\dots). \tag{12.2}$$

$\acute{P}$ is called the *derivative* operator and $\acute{M}$ the *multiplication* operator, as they act in the same way of classical operators on monomials.

This definition traces back to a paper by G. Dattoli [11], and was used in many articles and was the source of several application.

Y. Ben Cheikh [3] proved that every polynomial set is quasi-monomial under the action of suitable derivative and multiplication operators. In particular, in the same article (Corollary 3.2), the following result is proved

Theorem 12.2. *Let $\big(p_n(x)\big)$ a Boas-Buck polynomial set generated by the generating function*

$$A(t)\psi(xH(t) = \sum_{n=0}^{\infty} p_n(x)\frac{t^n}{n!}, \tag{12.3}$$

where

$$A(t) = \sum_{n=0}^{\infty} \tilde{a}_n t^n\,, (\tilde{a}_0 \neq 0),$$

$$\psi(t) = \sum_{n=0}^{\infty} \tilde{\gamma}_n t^n\,, (\tilde{\gamma}_n \neq 0 \quad \forall n), \tag{12.4}$$

with $\psi(t)$ not a polynomial, and lastly

$$H(t) = \sum_{n=0}^{\infty} \tilde{h}_n t^{n+1}\,, \big(\tilde{h}_0 \neq 0\big). \tag{12.5}$$

Let $\sigma \in \Lambda^{(-)}$ the lowering operator defined by

$$\sigma(1) = 0\,, \sigma(x^n) = \frac{\tilde{\gamma}_{n-1}}{\tilde{\gamma}_n} x^{n-1}, (n = 1,2,\dots). \tag{12.6}$$

Put

$$\sigma^{-1}(x^n) = \frac{\tilde{\gamma}_{n+1}}{\tilde{\gamma}_n} x^{n+1}, (n = 0,1,2,\dots). \tag{12.7}$$

Denoting, as before, by $f(t)$ the compositional inverse of $H(t)$, the Boas-Buck polynomial set $\{p_n(x)\}$ is quasi-monomial under the action of the operators

$$\acute{P} = f(\sigma), \acute{M} = \frac{A'(\sigma)}{A(\sigma)} + xD_x H'(\sigma)\sigma^{-1}, \tag{12.8}$$

where prime *denotes the ordinary derivatives with respect to t.*

For the Brenke polynomials we are considering here, as $A(t) = e^t$ and $H(t) = t$ (so that $f(t) = t$), taking into account the coefficients of ψ, this result reduces to

Theorem 12. 3. *Let $(\Upsilon_n^{[r]}(x))$ the polynomial set defined by the generating function*

$$e^t \sum_{n=0}^{\infty} b_n^{[r]}(x) \frac{(xt)^n}{n!} = \sum_{n=0}^{\infty} \Upsilon_n^{[r]}(x) \frac{t^n}{n!}. \tag{12.9}$$

Let $\sigma \in \Lambda^{(-)}$ the lowering operator defined by

$$\sigma(1) = 0, \sigma(x^n) = n \frac{b_{n-1}^{[r]}}{b_n^{[r]}} x^{n-1}, (n = 1,2,\dots). \tag{12.10}$$

Put

$$\sigma^{-1}(x^n) = \frac{1}{n+1} \frac{b_{n+1}^{[r]}}{b_n^{[r]}} x^{n+1}, (n = 0,1,2,\dots). \tag{12.11}$$

The Brenke polynomial set $(\Upsilon_n^{[r]}(x))$ is quasi-monomial under the action of the operators

$$\acute{P} = \sigma, \ \acute{M} = Id + xD_x\sigma^{-1}, \tag{12.12}$$

where Id denotes the identity operator.

12.3. Differential equation

According to the results of monomiality principle [11], the quasi-monomial polynomials $\{p_n(x)\}$ satisfy the differential equation

$$\acute{M}\acute{P}p_n(x) = np_n(x) \tag{12.13}$$

In the present case, we have

Theorem 12.4. *The Brenke polynomials $(\Upsilon_n^{[r]}(x))$ satisfy the operational-differential equation*

$$(xD_x + \sigma)\Upsilon_n^{[r]}(x) = n\Upsilon_n^{[r]}(x), \tag{12.14}$$

where σ is the operator defined by equation (12.10).

Conclusion

We have shown that the higher order Bell numbers enter in a natural way in the definition of new sets of Sheffer, associated Sheffer and Brenke polynomials. This could be used in order to understand the combinatorial character of the higher order Bell numbers.

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A corrigendum on "VON MISES DISTRIBUTION: PROPERTIES AND CHARACTERIZATIONS",

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http://www.vijnanaparishadofindia.org/ Jñānābha/volume-45-2015.

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In the above-said original research paper, there was an error. We should have also the following statement before Table 2.1. The rest of the materials remain same. The authors are apologetic for this error. Also, the authors would like to state that this does not change the contents and conclusions of the paper in any way. Further, the authors would like to deeply regret and apologize for any inconvenience caused by the above-said error, which occurred inadvertently. Finally, the authors hope that that the readers will understand the reasons for publishing and posting this corrigendum of the above-said article online by the publishers of *Jñānābha*.

1. **Corrections in Table 2.1**

The values given in Table 2.1 are for the Mises distribution, $vM(\theta, k)$, when $0 < \theta < 2\pi$ and $k \geq 0$. However, the values given in Table 2.1 can be used for standard Von Missed distribution also by subtracting $\pi\ (\approx 3.14159)$ from each value given in Table 2.1.

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